The algebra of the box–spline

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A theorems Combinatorics Splines Algebra Approximation Arithmetic Residues Z
1. Introduction
2. Combinatorics
3. Splines
4. Algebra
5. Approximation theory
6. Arithmetic
7. Residues
The basic input of this theory is a (real, sometimes integer) $n \times m$ matrix $A$. We always think of $A$ as a LIST of vectors in $V = \mathbb{R}^n$, its columns:

$$A := (a_1, \ldots, a_m)$$

Constrain

We assume that 0 is NOT in the convex hull of its columns.

From $A$ we make several constructions, algebraic, combinatorial, analytic etc..
IMPORTANT EXAMPLE

A is the list of **POSITIVE ROOTS** of a root system, e.g. $B_2$:

$$A = \begin{vmatrix} -1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{vmatrix}$$

We also identify vectors with linear forms as:

$-x + y, \ x, \ x + y, \ y$
IMPORTANT EXAMPLE

$A$ is the list of **POSITIVE ROOTS** of a root system, the associated cone $C(A)$ has three big cells.
The book: 
C. De Boor, K. Höllig, S. Riemenschneider, 

Box splines 


For a partial survey see 

The algebra of the box spline 

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SEVERAL NAMES OF CONTRIBUTORS

From numerical analysis
A.A. Akopyan; Ben-Artzi, Asher; C.K. Chui, C. De Boor, W. Dahmen, H. Diamond, N. Dyn, K. Höllig, C. Micchelli, Jia, Rong Qing, A. Ron, A.A. Saakyan

From algebraic geometry
Orlik–Solomon on cohomology, Baldoni, Brion, Szenes, Vergne and of Jeffrey–Kirwan, on partition functions.
In fact a lot of work originated from the seminal paper of Khovanskiĭ, Pukhlikov, interpret the counting formulas as Riemann–Roch formulas for toric varieties

From enumerative combinatorics
A.I. Barvinok, Matthias Beck, Sinai Robins, Richard Stanley
First a system of linear equations:

\[ \sum_{i=1}^{m} a_i x_i = b, \quad \text{or} \quad Ax = b, \quad A := (a_1, \ldots, a_m) \quad (1) \]

The \textit{columns} \(a_i, b\) are vectors with \(n\) coordinates

\[(a_{j,i}, b_j, \quad j = 1, \ldots, n).\]
As in Linear Programming Theory we deduce and want to study the

**VARIABLE POLYTOPES:**

\[ \Pi_A(b) := \{ x \mid Ax = b, x_i \geq 0, \forall i \} \]

\[ \Pi_A^1(b) := \{ x \mid Ax = b, 1 \geq x_i \geq 0, \forall i \} \]

which are **convex and bounded** for every \( b \).
The polytopes $\Pi_{A}^{1}(b)$ are sections of a hypercube, for instance the simple case of a cube $A$ a list of 3 numbers:
Geometric pictures
The object of study

Basic functions

- Set $T_A(x)$, $B_A(x)$ to be the volume of $\Pi_A(x)$, $\Pi_A^1(x)$.
- If $A$, $b$ have integer coordinates

Arithmetic case, $A$, $x$ with integer coefficients

In this case set $P_A(x)$ to be the number of solutions of the system in which the coordinates $x_i$ are non negative integers.
Up to a multiplicative normalization constant:
$T_A(x)$ is the Multivariate–spline
$B_A(x)$ the Box–spline
$P_A(x)$ is called the \textit{partition function}

We are interested in
Computing the three functions $T_A(x)$, $B_A(x)$, $P_A(x)$ and describe their qualitative properties.

Applications of these functions to arithmetic, numerical analysis, Lie theory.
EXAMPLE, the $C^1$ function $T_A$, case ZP

In the literature of numerical analysis the Box spline associated to the root system $B_2$ is called the Zwart–Powell or ZP element:

$$ A = \begin{vmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 1 \end{vmatrix} $$
EXAMPLE, the $C^1$ function $T_A$, case $B_2$ or ZP

$2T_A$ is 0 outside the cone and on the three cells:

- $(x+y)^2/2$ for the triangle
- $(x+y)^2/2 - x^2$ for the pentagon

0 for the triangle

$y^2$ for the triangle
EXAMPLE, the $C^1$ function $T_A$, case $B_2$ or ZP

$2T_A$ is 0 outside the cone and on the three cells:

\[
\begin{align*}
\frac{(x+y)^2}{2} \\
^{triangle}
\end{align*}
\]

\[
\begin{align*}
\frac{(x+y)^2}{2} - x^2 \\
^{pentagon}
\end{align*}
\]

\[
\begin{align*}
0 \\
^{triangle}
\end{align*}
\]

\[
\begin{align*}
y^2
\end{align*}
\]
The box–spline for type $B_2$, ZP element
The computation of the box–spline has some geometric, combinatorial and algebraic flavor. It appears as a piecewise polynomial function on a compact polyhedron.

From simple data we get soon a complicated picture!
THE PARTITION FUNCTION

When $A$, $b$ have integer elements it is natural to think of an expression like:
\[ b = t_1a_1 + \cdots + t_ma_m \]
with $t_i$ not negative integers as a partition of $b$ with the vectors $a_i$, in $t_1 + t_2 + \cdots + t_m$ parts, hence the name partition function for the number $P_A(b)$, thought of as a function of the vector $b$.

SIMPLE EXAMPLE

$m = 2, \ n = 1, \ A = \{2, 3\}$

Parts are 2 and 3

In how many ways can you write a number $b$ as:
\[ b = 2x + 3y, \quad x, y \in \mathbb{N} \]
THE PARTITION FUNCTION

When \( A, b \) have integer elements it is natural to think of an expression like:
\[
b = t_1a_1 + \cdots + t_m a_m \text{ with } t_i \text{ not negative integers as a:}
\]

**partition of \( b \) with the vectors \( a_i \),**
in \( t_1 + t_2 + \cdots + t_m \) parts, hence the name **partition function** for the number \( P_A(b) \), thought of as a function of the vector \( b \).

**SIMPLE EXAMPLE**

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m = 2, \quad n = 1, \quad A = \{2, 3\}
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**Parts are 2 and 3**

In how many ways can you write a number \( b \) as:

\[
b = 2x + 3y, \quad x, y \in \mathbb{N}
\]
ANSWER (Quasi polynomial!)

It depends on the class of $n$ modulo 6.

\[
\begin{align*}
    n \equiv 0 & : \quad \frac{n}{6} + 1 \\
    n \equiv 1 & : \quad \frac{n}{6} - \frac{1}{6} \\
    n \equiv 2 & : \quad \frac{n}{6} + \frac{2}{3} \\
    n \equiv 3 & : \quad \frac{n}{6} + \frac{1}{2} \\
    n \equiv 4 & : \quad \frac{n}{6} + \frac{1}{3} \\
    n \equiv 5 & : \quad \frac{n}{6} + \frac{1}{6}
\end{align*}
\]
What is a quasi-polynomial?

Two equivalent definitions

FIRST
A function on a lattice $\Lambda$ which is a polynomial on each coset of some sublattice $M$ of finite index

SECOND
The restriction to a lattice $\Lambda$ of a function which is a sum of products of a polynomial with an exponential function, periodic on some sublattice $M$ of finite index
The classical method of study of the partition function associated to a list of numbers \( a_1, \ldots, a_i, \ldots, a_m \) is to expand in partial fractions the generating function:

\[
\prod_{i=1}^{m} \frac{1}{1 - x^{a_i}}
\]

This is done by using the decomposition \( 1 - x^n = \prod_{k=0}^{n-1} (1 - \zeta_n^k x) \) where \( \zeta_n = e^{\frac{2\pi i}{n}} \).
OTHER EXAMPLE, THE HAT FUNCTION

\[ A = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} \] the corresponding partition function \( P_A \) is piecewise polynomial with top degree coinciding with \( T_A \)
IMPORTANT EXAMPLE  Lie theory

In Lie theory the *Kostant partition function* counts in how many ways can you decompose a weight as a sum of positive roots.
This is used in many computations.
THE THEOREMS

There are several general formulas to compute the previous functions which are obtained by a mixture of techniques.
Assume $A$ spans $\mathbb{R}^n$.

Let $h$ or $m(X)$ be the minimum number of columns that one can remove from $A$ so that the remaining columns do not span $\mathbb{R}^n$.

**The basic function $T_A(x)$ is a spline**

1. $T_A(x)$ has support on the cone $C(A)$
2. $T_A(x)$ is of class $m(X) - 2$
3. $T_A(x)$ coincides with a homogeneous polynomial of degree $m - n$ on each big cell.
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3. $T_A(x)$ coincides with a homogeneous polynomial of degree $m - n$ on each big cell.
SUMMARIZING

In order to compute $T_A(x)$ we need to

- Determine the decomposition of $C(A)$ into cells
- Compute on each big cell the homogeneous polynomial of degree $m-n$ coinciding with $T_A(x)$
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1. Determine the decomposition of $C(A)$ into cells

2. Compute on each big cell the homogeneous polynomial of degree $m - n$ coinciding with $T_A(x)$. 
GENERAL FORMULA FOR $T_X$

One can find explicit polynomials $p_{b,A}(x)$, indexed by a combinatorial object called *unbroken bases* and characterized by certain explicit differential equations so that: Given a point $x$ in the closure of a big cell $c$ we have

Jeffry-Kirwan residue formula

$$T_A(x) = \sum_{b \mid c \subset C(b)} \left| \det(b) \right|^{-1} p_{b,A}(-x).$$
From $T_A$ one computes $B_A$

For a given subset $S$ of $A$ define $a_S := \sum_{a \in S} a$

the basic formula is:

$$B_A(x) = \sum_{S \subseteq A} (-1)^{|S|} T_A(x - a_S).$$

So $T_A$ is the fundamental object.

Notice that the local pieces of $B_A$ are no more homogeneous polynomials.
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There is a parallel theory, as a result we can compute a set of polynomials $q_{b,\phi}(-x)$ indexed by pairs, a character $\phi$ of finite order and a unbroken basis in $A_\phi = \{ a \in A \mid \phi(e^a) = 1 \}$.

The analogue of the Jeffrey–Kirwan formula is:

**Theorem**

Given a point $x$ in the closure of a big cell $c$ we have

$$P_A(x) = \sum_{\phi \in P(A)} e^{\phi} \sum_{b \in NB_{A_\phi} \mid c \subset C(b)} q_{b,\phi}(-x)$$
There is a parallel theory, as a result we can compute a set of polynomials $q_{b,\phi}(-x)$ indexed by pairs, a character $\phi$ of finite order and a unbroken basis in $A_\phi = \{a \in A \mid \phi(e^a) = 1\}$.

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From volumes to partition functions

One can deduce the partition function from some combinatorics and multivariate splines (with parameters):

**Theorem**

Given a point $x$ in the closure of a big cell $c$ we have

$$P_A = \sum_{\phi \in P(A)} \hat{Q}_\phi T_{A_\phi, \phi}$$

$$Q_\phi = \prod_{a \notin A_\phi} \frac{1}{1 - e^{-a}} \prod_{a \in A_\phi} \frac{a - \langle \phi | a \rangle}{1 - e^{-a + \langle \phi | a \rangle}}$$
FIRST

FIRST STEP

SOME COMBINATORICS
one can explain the ideas relating unbroken bases with cells and all bases with the structure of the polyhedron where the box–spline has its support.

Let us illustrate these ideas through examples.
Unbroken bases

From the theory of matroids to cells.

Let \( c := a_{i_1}, \ldots, a_{i_k} \in A, \ i_1 < i_2 \cdots < i_k \), be a sublist of linearly independent elements.

**Definition**

We say that \( a_i \) **breaks** \( c \) if there is an index \( 1 \leq e \leq k \) such that:

- \( i \leq i_e \).
- \( a_i \) is linearly dependent on \( a_{i_e}, \ldots, a_{i_k} \).

In matroid theory one says that \( a_i \) is *externally active*. 
From the theory of matroids to cells.

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EXAMPLE of the elements breaking a basis

Example

Take as $A$ the list of positive roots for type $A_3$.

$$A = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}.$$ 

Let us draw in Red one particular basis and in Green the elements which break it.

We have 16 bases

10 broken and 6 unbroken

FIRST THE 10 broken
EXAMPLE of the elements breaking a basis

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\( \alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3 \)

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\( \alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3 \)
We have 6 unbroken, all contain necessarily $\alpha_1$:

- $\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3$.
- $\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3$.
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- $\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3$.
Let us apply the theory to cells in the same example:

We do everything on a transversal section, where the cone looks like a bounded convex polytope and then project.

We want to decompose the cone \( C(A) \) into big cells and define its singular and regular points.
PROJECTING A POLYHEDRON TO FORM A CONE
EXAMPLE Type $A_3$ in section (big cells):

We have 7 big cells.
Let us visualize the simplices generated by the 6 unbroken:

\[ \alpha_1 \rightarrow \alpha_2 \rightarrow \alpha_3 \]
\[ \alpha_2 + \alpha_3 \]

\[ \alpha_1 \]

\[ \alpha_2 \]
\[ \alpha_1 + \alpha_2 + \alpha_3 \]
\[ \alpha_1 + \alpha_2 \]

Diagram:

- \( \alpha_1 \) is connected to \( \alpha_2 \) and \( \alpha_3 \).
- \( \alpha_1 \) and \( \alpha_3 \) are connected.

\( \alpha_1 \) and \( \alpha_3 \) are connected by a direct line, indicating addition or some form of mathematical relationship.
\[ \alpha_1 + \alpha_2 \quad \Rightarrow \quad \alpha_2 + \alpha_3 \]
Decomposition into big cells and unbroken bases

Let us visualize the decomposition into big cells, obtained overlapping the cones generated by unbroken bases.
A remarkable fact
A remarkable fact
A remarkable fact
A remarkable fact
A remarkable fact
A remarkable fact
A remarkable fact
The overlapping theorem

You have visually seen a theorem we proved in general:

by overlapping the cones generated by the unbroken bases one obtains the entire decomposition into big cells!!
A list of vectors $a_i$ can be thought of as a list of linear equations defining a set of linear hyperplanes.

These intersect in a complicated pattern giving rise to a second interesting combinatorial geometric object, the set of all their intersections called:

HYPERPLANE ARRANGEMENT
PROJECTIVE PICTURE OF THE ARRANGEMENT $A_3$

Same example but as linear functions, or hyperplane arrangement:

We have drawn in the projective plane of 4-tuples of real numbers with sum 0, the 6 lines

$$x_i - x_j = 0, \ 1 \leq i < j \leq 4$$

the 7 intersection points are also subspaces of the arrangement!
PROJECTIVE PICTURE OF THE ARRANGEMENT $A_3$

Same example but as linear functions, or hyperplane arrangement:

We have drawn in the projective plane of 4–tuples of real numbers with sum 0, the 6 lines

$$x_i - x_j = 0, \quad 1 \leq i < j \leq 4$$

the 7 intersection points are also subspaces of the arrangement!
Describe the previous pictures for root systems. For type $A_n$ the unbroken bases are known and can be indexed by certain binary graphs or by permutations of $n$ elements. The decomposition into cells is unknown.
The box spline $B_A(x)$ is supported in the compact polytope:

\[ \text{THE BOX } B(A) \]

The Box $B(A)$

that is the compact convex polytope

\[ B(A) := \left\{ \sum_{i=1}^{m} t_i a_i \right\}, \quad 0 \leq t_i \leq 1, \quad \forall i. \]
The box spline $B_A(x)$ is supported in the compact polytope:

$$\text{THE BOX } B(A)$$

that is the compact convex polytope

$$B(A) := \left\{ \sum_{i=1}^{m} t_i a_i \right\}, 0 \leq t_i \leq 1, \forall i.$$
The box $B(A)$ has a nice combinatorial structure, and can be paved by a set of parallelepipeds indexed by:

all the bases which one can extract from $A$!

Example

In the next example

$$A = \begin{vmatrix} 1 & 0 & 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 1 & 2 \end{vmatrix}$$

we have 15 bases and 15 parallelograms.
The box $B(A)$ has a nice combinatorial structure, and can be paved by a set of parallelepipeds indexed by: all the bases which one can extract from $A$!

Example

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EXAMPLE  

*paving the box*

\[
A = \begin{vmatrix} 
1 & 0 & 1 & -1 & 2 & 1 \\
0 & 1 & 1 & 1 & 1 & 2 
\end{vmatrix}
\]
Step–wise paving of the box

\[ \begin{vmatrix} 1 & 0 & 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 1 & 2 \end{vmatrix} \]

START WITH

\[ A = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \]
Step–wise paving of the box

\[ A = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} \]
Step-wise paving of the box

$$A = \begin{vmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{vmatrix}$$
Step–wise paving of the box

\[ A = \begin{vmatrix} 1 & 0 & 1 & -1 & 2 \\ 0 & 1 & 1 & 1 & 1 \end{vmatrix} \]
Step–wise paving of the box

\[ A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 1 & 2 \end{bmatrix} \]
SECOND STEP

THE SPLINES
The volume of the variable polytope $\Pi_A(x)$ equals, up to the constant $\sqrt{\det AA^t}$ to the

**multivariate spline**

that is the function $T_A(x)$ characterized by the formula:

$$\int_{\mathbb{R}^n} f(x)T_A(x)\,dx = \int_{\mathbb{R}^m_+} f(\sum_{i=1}^m t_i a_i)\,dt,$$

where $f(x)$ is any continuous function with compact support.
The volume of the variable polytope $\Pi_A(x)$ equals, up to the constant $\sqrt{\det AA^t}$ to the

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WARNING

We have given a \textbf{weak} definition for $T_A(x)$

\textbf{In general}

$T_A(x)$ is a tempered distribution, supported on the cone $C(A)$!

Only when $A$ has maximal rank $T_A(x)$ is a function.
We have given a weak definition for $T_A(x)$

**In general**

$T_A(x)$ is a tempered distribution, supported on the cone $C(A)$!

Only when $A$ has maximal rank $T_A(x)$ is a function.
We have given a weak definition for $T_A(x)$.

**In general**

$T_A(x)$ is a tempered distribution, supported on the cone $C(A)$!

Only when $A$ has maximal rank $T_A(x)$ is a function.
While the function $T_A(x)$ is the basic object, the more interesting object for numerical analysis is the **box spline** that is the function $B_A(x)$ characterized by the formula:

$$\int_{\mathbb{R}^n} f(x)B_A(x)dx = \int_{[0,1]^m} f(\sum_{i=1}^m t_i a_i)dt,$$

where $f(x)$ is any continuous function.
two box splines of class $C^0$, $h = 2$ and $C^1$, $h = 3$

$$A = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$
Hat function

or Courant element

$$A = \begin{vmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{vmatrix}$$
Zwart-Powell element
EXAMPLE OF A two dimensional box spline

Non continuous, $A = \begin{vmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix} (h = 1)$
3 reasons WHY the BOX SPLINE?

\[ \int_{\mathbb{R}^n} B_A(x) \, dx = 1 \]

In the case of integral vectors, we have

PARTITION OF UNITY

The translates \( B_A(x - \lambda) \), \( \lambda \) runs over the integral vectors form a partition of 1.

recursive definition

\[ B_{[A,v]}(x) = \int_0^1 B_A(x - tv) \, dt \]
3 reasons WHY the BOX SPLINE?

\[ \int_{\mathbb{R}^n} B_A(x) \, dx = 1 \]

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\[ \int_{\mathbb{R}^n} B_A(x) \, dx = 1 \]

\[ B_{[A,v]}(x) = \int_0^1 B_A(x - tv) \, dt \]

in the case of integral vectors, we have

PARTITION OF UNITY

The translates \( B_A(x - \lambda) \), \( \lambda \) runs over the integral vectors form a partition of 1.
A = \{1, 1\} we have for the box spline

Now let us add to it its translates!
EXAMPLE OF THE HAT FUNCTION

\[ A = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} \]

From the function \( T_A \)
EXAMPLE OF THE HAT FUNCTION

we get the box spline *hat function* summing over the 6 translates of $T_A$
Recall the box–spline for type $B_2$
The box spline, when the $a_i$ are integral vectors, can be effectively used in the **finite element method** to approximate functions. This follows from the fact that it satisfies the STRANG-FIX condition:

The transform

$$f \mapsto \sum_{i \in \mathbb{Z}^s} B_x(x - i)f(i)$$

transforms polynomials of degree $\leq h - 1$ into polynomials (of the same degree).
THIRD STEP

FROM ANALYSIS TO ALGEBRA
Here comes the algebra

How to compute $T_A$? or the partition function $P_A$?

We use the Laplace transform which will change the analytic problem to one in

$\text{algebra}$
**LAPLACE TRANSFORM** from $\mathbb{R}^s = V$ to $U = V^*$. 

$$Lf(u) := \int_V e^{-\langle u | v \rangle} f(v) dv.$$ 

**basic properties**

$p \in U$, $w \in V$, write $p$, $D_w$ for the linear function $\langle p | v \rangle$ and the directional derivative on $V$

$$L(D_wf)(u) = wLf(u), \quad L(pf)(u) = -D_pLf(u),$$

$$L(e^pf)(u) = Lf(u - p), \quad L(f(v + w))(u) = e^wLf(u).$$
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Consider any $a \in V$ as a **LINEAR FUNCTION ON** $U$ we have:

An easy computation gives the Laplace transforms:

\[
LB_A = \prod_{a \in A} \frac{1 - e^{-a}}{a}
\]

and

\[
LT_A = \prod_{a \in A} \frac{1}{a}
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and

$$LT_A = \prod_{a \in A} \frac{1}{a}$$
MORE PARTIAL FRACTIONS

We need to rewrite $LT_A = \prod_{a \in A} \frac{1}{a}$ for this we need to develop a theory of partial fractions in several variables, in this case for the algebra $S[V][\prod_{a \in X} a^{-1}]$

WE DO THIS BY NON COMMUTATIVE ALGEBRA!!!!
BASIC NON COMMUTATIVE ALGEBRAS

Algebraic Fourier transform

Weyl algebras

Set $W(V)$, $W(U)$ be the two Weyl algebras of differential operators with polynomial coefficients on $V$ and $U$.

Fourier transform

There is an algebraic Fourier isomorphism between them, so any $W(V)$ module $M$ becomes a $W(U)$ module $\hat{M}$.
**D-modules in Fourier duality:**

1. The $D$–module $\mathcal{D}_A := W(V) T_A$ generated, in the space of tempered distributions, by $T_A$ under the action of the algebra $W(V)$ of differential operators on $V$ with polynomial coefficients.

2. The algebra $R_A := S[V][\prod_{a \in A} a^{-1}]$ obtained from the polynomials on $U$ by inverting the element $d_A := \prod_{a \in A} a$

$$R_A = W(U) d_A^{-1}$$
Two modules Fourier isomorphic

1. The $D$–module $\mathcal{D}_A := W(\mathcal{V}) T_A$ generated, in the space of tempered distributions, by $T_A$ under the action of the algebra $W(\mathcal{V})$ of differential operators on $\mathcal{V}$ with polynomial coefficients.

2. The algebra $R_A := S[V][\prod_{a \in A} a^{-1}]$ obtained from the polynomials on $\mathcal{U}$ by inverting the element $d_A := \prod_{a \in A} a$

$$R_A = W(\mathcal{U})d_A^{-1}$$
$R_A$ and it is the coordinate ring of the open set $A_A$ complement of the union of the hyperplanes of $U$ of equations $a = 0$, $a \in A$.

It is a cyclic module under $W(U)$ generated by $d_A^{-1}$. 
The module $R_A$

Localization

It is well known that once we invert an element in a polynomial algebra we get a holonomic module over the algebra of differential operators. **$R_A$ is holonomic!**

In particular $R_A$ has a finite composition series and it is cyclic.

We want to describe a composition series of $R_A$. 
The module $R_A$

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We want to describe a composition series of $R_A$. 
The building blocks

The irreducible module $N_W$

For each subspace $W$ of $U$ we have an irreducible module $N_W$ generated by the $\delta$ function of $W$.

As $W$ runs over the subspaces of the hyperplane arrangement given by the equations $a_i = 0, \ a_i \in A$

the $N_W$ run over all the composition factors of $R_A$ (with multiplicities).
Take coordinates $x_1, \ldots, x_n$

\[ W = \{ x_1 = x_2 = \ldots = x_k = 0 \} \]

$N_W$ is generated by an element $\delta_W$ satisfying:

\[ x_i \delta_W = 0, \quad i \leq k, \quad \frac{\partial}{\partial x_i} \delta, \quad i > k. \]

$N_W$ is free of rank 1 generated by $\delta_W$ over:

\[ \mathbb{C}[x_1, x_2, \ldots, x_k, \frac{\partial}{\partial x_{k+1}}, \ldots, \frac{\partial}{\partial x_n}]. \]
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\[ \mathbb{C}[x_1, x_2, \ldots, x_k, \frac{\partial}{\partial x_{k+1}}, \ldots, \frac{\partial}{\partial x_n}]. \]
The filtration of $R_A$ by polar order

**Definition**

$R_A$ is filtered by the $W(U)$–submodules:

**filtration degree $\leq k$**

$R_{A,k}$ the span of all the fractions $f \prod_{a \in A} a^{-h_a}$, $h_a \geq 0$ for which the set of vectors $a$, with $h_a > 0$, spans a space of dimension $\leq k$.

We have $R_{A,s} = R_A$. 

The filtration of $R_A$ by polar order

For all $k$ we have that $R_{A,k}/R_{A,k-1}$ is semisimple.

- The isotypic components of $R_{A,k}/R_{A,k-1}$ are of type $N_W$ as $W$ runs over the subspaces of the arrangement of codimension $k$.

$R_{A,s}/R_{A,s-1}$ is a free module over

$$S[U] = \mathbb{C}[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_s}]$$

- It is important to choose a basis.
The basis theorem

**Theorem**

A basis for $\mathbb{R}_{A,s}/\mathbb{R}_{A,s-1}$ over $S[U]$ is given by

the classes of the elements $\prod_{a \in b} a^{-1}$ as $b$ runs over the set of unbroken bases.
The expansion of $d_{A}^{-1} = \prod_{a \in A} a^{-1}$

Denote by $\mathcal{NB}$ the unbroken bases extracted from $A$.

We have a more precise THEOREM.

$$\frac{1}{\prod_{a \in A} a} = \sum_{b \in \mathcal{NB}} p_b \prod_{a \in b} a^{-1}, \quad p_b \in S[U] = \mathbb{C}[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_s}]$$
EXAMPLE Courant element

Example

USE COORDINATES $x, y$ SET

$$A = [x + y, x, y] = [x, y] \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$\frac{1}{(x + y)x y} = \frac{1}{x(x + y)^2} + \frac{1}{y(x + y)^2} =$$

$$- \frac{\partial}{\partial y} \left( \frac{1}{x(x + y)} \right) - \frac{\partial}{\partial x} \left( \frac{1}{y(x + y)} \right)$$
EXAMPLE ZP element

Example

\[ A = [x + y, x, y, -x + y] = [x, y] \begin{vmatrix} 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & 1 \end{vmatrix} \]

\[
\frac{1}{(x + y)(-x + y)} = \frac{1}{(x + y)^3 x} + \frac{4}{(x + y)^3 (-x + y)} - \frac{1}{(x + y)^3 y}
\]

\[
\frac{1}{2} \left[ \frac{\partial^2}{\partial y^2} \left( \frac{1}{(x + y)x} \right) + \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right)^2 \left( \frac{1}{(x + y)(-x + y)} \right) - \frac{\partial^2}{\partial x^2} \left( \frac{1}{(x + y)y} \right) \right]
\]
We now need the basic inversion

Let \( X = \{a_1, \ldots, a_n\} \) be a basis, \( d := |\det(a_1, \ldots, a_s)| \)
\( \chi_{C(X)} \) the characteristic function of the positive quadrant \( C(X) \)
generated by \( X \).

**Basic example of inversion**

\[
L(d^{-1}\chi_{C(X)}) = \prod_{i=1}^{n} a_i^{-1}
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**Basic example of inversion**

$$L(d^{-1}\chi_{C(X)}) = \prod_{i=1}^{n} a_i^{-1}$$
We are ready to invert!

We want to invert

\[ d_A^{-1} = \sum_{b \in N'B} p_{b,A}(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_s}) \prod_{a \in b} a^{-1} \]

From the basic example and the properties!

We get

\[ \sum_{b \in N'B} p_{b,A}(-x_1, \ldots, -x_s) d_b^{-1} \chi C(b) \]
We are ready to invert!

We want to invert

\[ d_A^{-1} = \sum_{b \in NB} p_{b,A} \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_s} \right) \prod_{a \in b} a^{-1} \]

From the basic example and the properties!

We get

\[ \sum_{b \in NB} p_{b,A}(-x_1, \ldots, -x_s) d_b^{-1} \chi C(b) \]
EXAMPLE ZP element

Inverting

\[
\frac{1}{2}\left[\frac{\partial^2}{\partial^2 y} \left( \frac{1}{x (x + y)} \right) + \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right)^2 \left( \frac{1}{(x + y)(-x + y)} \right) - \frac{\partial^2}{\partial^2 x} \left( \frac{1}{y (x + y)} \right) \right]
\]

we get

\[
\frac{1}{2}\left[ y^2 \chi_C((1,0),(1,1)) + \frac{(x + y)^2}{2} \chi_C((1,1),(-1,1)) - x^2 \chi_C((0,1),(1,1)) \right]
\]
The theory of Dahmen–Micchelli

We need a basic definition of combinatorial nature

**Definition**

We say that a sublist \( Y \subset A \) is a **cocircuit**, if the elements in \( A - Y \) do not span \( V \).

**The basic differential operators**

For such \( Y \) set \( D_Y := \prod_{a \in Y} D_a \), a differential operator with constant coefficients.

\( (D_a \) is directional derivative, first order operator)
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**The basic differential operators**

For such $Y$ set $D_Y := \prod_{a \in Y} D_a$, a differential operator with constant coefficients.

($D_a$ is directional derivative, first order operator)
For a given unbroken circuit basis $b$, consider the element

$$D_b := \prod_{a \notin b} D_a.$$  

Characterization by differential equations

The polynomials $p_{b,A}$ are characterized by the differential equations

$$D_Y p = 0, \quad \forall Y, \quad a \text{ cocircuit in } A$$

$$D_b p_{c,A}(x_1, \ldots, x_s) = \begin{cases} 
1 & \text{if } b = c \\
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The space $D(A)$

A remarkable space of polynomials

$$D(A) := \{ p \mid D_Y p = 0, \ \forall Y, \ \text{a cocircuit in } A \}$$
The theorem of Dhamen Micchelli

\[ \dim D(A) \text{ equals the total number of bases extracted from } A \]

We have a more precise theorem

The polynomials \( p_{b,A} \) form a basis for the top degree part \((m - n)\) of \( D(A) \).

The graded dimension of \( D(A) \) is given by

\[
H_A(q) = \sum_{b \in B(A)} q^{m - n(b)}.
\]

\( n(b) \) is the number of elements of \( A \) breaking \( b \).
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For $A_3$ we get

The graded dimension is:

$$6q^3 + 6q^2 + 3q + 1$$

Remark that for all polynomials in three variables it is:

$$\ldots + 10q^3 + 6q^2 + 3q + 1$$
The theorem of Dahmen–Micchelli can be formulated both in terms of commutative as well as non–commutative algebra.
APPROXIMATION THEORY

APPROXIMATION POWER

THE STRANG-FIX CONDITIONS
THE STRANG-FIX CONDITIONS

The interest of the space of polynomials \( D(A) \) comes in approximation theory from the problem of studying the approximation of a function \( f(x) \) by the finite element method:

**discrete convolution**

\[
f(x) \mapsto \sum_{i \in \mathbb{Z}^n} B_A(x - i)f(i)
\]

Or at order \( n \in \mathbb{N} \):

\[
f(x) \mapsto \sum_{i \in \mathbb{Z}^n} B_A(nx - i/n)f(i/n)
\]
More generally we want to find weights $c_i$ and approximate $f$ by $f(x) \mapsto \sum_{i \in \mathbb{Z}^n} B_A(nx - i/n)c_i$ and determine a constant $k \in \mathbb{N}$ so that (on some bounded region):

$$|f(x) - \sum_{i \in \mathbb{Z}^s} B_A(nx - i/n)c_i| \leq Cn^{-k}$$

The maximum $k$ is the approximation power of $B_A$. 
THE CARDINAL SPLINE SPACE

Given a spline $M(x)$ on $\mathbb{R}^s$, with compact support one may define the **THE CARDINAL SPLINE SPACE** $S_M$ to be the space of all (infinite) linear combinations:

$$S_M := \left\{ \sum_{i \in \mathbb{Z}^s} M(x - i) c_i \right\}.$$ 

The approximation power of $M(x)$ is related to two questions:

1. For which polynomials $f(x)$ we have that $\sum_{i \in \mathbb{Z}^s} M(x - i) f(i)$ is a polynomial?
2. Which polynomials lie in the cardinal spline space?
The Strang–Fix conditions is a general statement:

*The approximation power of a function $M$ is the maximum $r$ such that the space of all polynomials of degree $\leq r$ is contained in the cardinal space $S_M$.**
The case of $B_A$

For the cardinal spline space in the case $B_A(x)$ with $A$ integral we have:

1. $D(A)$ is characterized as the space of polynomials $f(x)$ which reproduce, i.e. map to polynomials under the discrete convolution.

2. $D(A)$ is also characterized as the space of polynomials lying in the cardinal spline space.

**Strang-Fix conditions**

The power of approximation by discrete convolution is measured by the maximum degree of the space of polynomials which reproduce under discrete convolution.
SUPERFUNCTIONS

Consider the following algorithm applied to a function $g$:

$$g_h := \sum_{i \in \Lambda} F(x/h - i)g(hi)$$

There are functions $F$ in the cardinal spline space such that this transformation is the identity on polynomials of degree $< m(X)$, these are the super–functions. For such functions the previous algorithm satisfies the requirements of the Strang–Fix approximation
Theorem

We have, under the explicit algorithm previously constructed that, for any domain $G$:

$$\| f_h - f \|_{L^\infty(G)} = O(h^m(X)).$$

For every multi-index $\alpha \in \mathbb{N}^s$ with $|\alpha| \leq m(X) - 1$, we have:

$$\| \partial^\alpha f_h - \partial^\alpha f \|_{L^\infty(G)} = \| \partial^\alpha (f_h - f) \|_{L^\infty(G)} = O(h^{m(X)}-|\alpha|).$$
THE DISCRETE CASE

PARTITION FUNCTIONS

THE DISCRETE CASE
THE DISCRETE CASE

We think of the partition function

\[ P_A(b) = \#\{t_1, \ldots, t_m \in \mathbb{N} \mid \sum_{i=1}^{m} t_i a_i = b\} \]

as a distribution

\[ \sum_{\lambda \in \Lambda} P_A(\lambda) \delta_\lambda. \]

with Laplace transform

\[ \sum_{\lambda \in \Lambda} P_A(\lambda) e^{-\lambda} = \prod_{a \in A} \frac{1}{1 - e^{-a}}. \]
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\[ \sum_{\lambda \in \Lambda} P_A(\lambda) e^{-\lambda} = \prod_{a \in A} \frac{1}{(1 - e^{-a})}. \]
One can see that the leading part of the formula for the partition function is given by the multivariate spline $T_A$ and also that there are formulas using differential operators to pass from the functions $T_A$ to the partition functions.
OTHER FOURIER ISOMORPHIC ALGEBRAS

The periodic Weyl algebras $\tilde{W}(U)$ and $W^\#(\Lambda)$

$$\tilde{W}(U) = \mathbb{C}[e^{\pm t_1}, \ldots, e^{\pm t_n}, \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_n}]$$

algebra of difference operators

$$W^\#(\Lambda) = \mathbb{C}[x_1, \ldots, x_n, \nabla_{\pm x_1}, \ldots, \nabla_{\pm x_n}]$$

$$\nabla_{x_i}(x_j) = \begin{cases} x_j & \text{if } i \neq j \\ x_i - 1 & \text{otherwise} \end{cases}.$$
Two modules Fourier isomorphic

1. The $\mathcal{W}^\#$-module $\mathcal{D}_A^\# := \mathcal{W}^\#(\Lambda) \mathcal{P}_A$ generated, in the space of tempered distributions, by the partition distribution $\mathcal{P}_A$.

2. The $\mathcal{W}(U)$ module $S_A = \mathbb{C}[\Lambda][\prod_{a \in A}(1 - e^{-a})^{-1}]$ is the algebra obtained from the character ring $\mathbb{C}[\Lambda]$ by inverting $u_A := \prod_{a \in A}(1 - e^{-a})$. 
The toric arrangement

$T$ the torus of character group $\Lambda$

$S_A = \mathbb{C}[\Lambda][u_A^{-1}]$ is the coordinate ring of the open set $T_A \subset T$ complement of the union of the subgroups of $T$ of equations $e^a = 1, \ a \in A$. 
The toric arrangement

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The toric arrangement

The toric arrangement is the finite set consisting of all the connected components of the subvarieties obtained by intersecting the subgroups of $T$ of equations $e^a = 1$, $a \in A$.

EXAMPLE $s = 1$, $T = \mathbb{C}^*$, $A = \{5, 3\}$
The arrangement consists of the connected components of the variety $x^5 = 1$ or $x^3 = 1$, i.e. of the five, fifth roots of 1 and the three third roots of 1.
Points of the arrangement

The elements of the toric arrangement are ordered by reverse inclusion, particular importance is given to the points of the arrangement, \( P(A) \) which are the zero-dimensional, i.e. points, elements of the arrangement.

A very special case is when \( P(A) \) reduces to the point 1, this is the unimodular case.
Each point $p \in P(A)$ determines a sublist:

$$A_p := \{ a \in A \mid e^a(p) = 1 \}.$$
EXAMPLE ZP

\[ A = \begin{vmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 1 \end{vmatrix} \]

The subgroups are

\[ xy = 1, \ x = 1, \ y = 1, \ x^{-1}y = 1. \]

We have two points in \( P(A) \)

\[ (1, 1), \ (-1, -1). \]

\[ A_{(1,1)} = A, \quad A_{(-1,-1)} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}. \]
THE FILTRATION

We have as for hyperplanes a filtration by polar orders on $S_A$.

Each graded piece is semisimple.

The isotypic components appearing in grade $k$ correspond to the connected components of the toric arrangement of codimension $k$.

For the top part $S_{A,n}/S_{A,n-1}$ we have a sum over the points of the arrangement $P(A)$.

The isotypic component associated to a point $e^\phi$ decomposes as direct sum of irreducibles indexed by the unbroken bases in $A_\phi := \{a \in A \mid e^{(a)\phi} = 1\}$. 
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Local structure of $\mathcal{P}_A$

LINEAR COMBINATION OF POLYNOMIALS TIME PERIODIC EXPONENTIALS.

Such a function is called a QUASI POLYNOMIAL.
Basic equations

As for the case of the multivariate spline:

The quasi polynomials appearing in the formula for $P_A$ satisfy

special difference equations
For $a \in \Lambda$ and $f$ a function on $\Lambda$ we define the difference operator:

$$\nabla_a f(x) = f(x) - f(x - a), \quad \nabla_a = 1 - \tau_a.$$ 

**Example**

As special functions we have the characters, eigenvectors of difference operators.
Parallel to the study of $D(A)$, one can study the

**system of difference equations**

$$\nabla_Y f = 0, \text{ where } \nabla_Y := \prod_{v \in Y} \nabla_v$$

as $Y \in \mathcal{E}(A)$ runs over the cocircuits.

Let us denote the space of solutions by:

$$\nabla(A) := \{ f : \Lambda \rightarrow \mathbb{C}, \mid \nabla_Y(f) = 0, \forall Y \in \mathcal{E}(A) \}.$$
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SECOND THEOREM OF DAHMEN–MICCHELLI

Weighted dimension

The dimension of $\nabla(A)$ is the Volume of the box $B(A)!$ we have:

$$\delta(A) := \sum_{b \in B(A)} |\det(b)|.$$  

This formula has a strict connection with the paving of the box.
Example

Let us take

\[
A = \begin{vmatrix}
0 & 1 & 1 & -1 \\
1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 \\
\end{vmatrix}
\]

See that

\[
\delta(A) = 1 + 1 + 1 + 1 + 1 + 2 = 7
\]

is the number of points in which the box \(B(A)\), shifted generically a little, intersects the lattice!
logarithm isomorphism

There is a formal machinery which allows us to interpret, locally around a point, difference equations as restriction to the lattice of differential equations, we call it the logarithm isomorphism.

We have this for any module over the periodic Weyl algebra $\mathbb{C}[\frac{\partial}{\partial x_i}, e^{x_i}]$ as soon as for algebraic reasons (nilpotency) we can deduce from the action of $e^{x_i}$ also an action of $x_i$. 
Finally Dahmen–Micchelli prove that the partition function on a given big cell coincides with the quasi polynomial in $\nabla(A)$ which takes values 1 at 0 and 0 at the points of the lattice lying in the $x_0 - B(X)$ where $x_0$ is in the big cell very close to 0.
ALGEBRAIC GEOMETRY

WONDERFUL MODELS

RESIDUES
There is an approach to compute the partition function based on residues

Start from the case of numbers. Fix positive numbers \( h := (h_1, \ldots, h_m) \).

Given an integer \( n \),
the number of ways \( n = \sum_i k_i h_i \) is the coefficient of \( x^{-1} \) in
\[
P_h(x) = \prod_i \frac{x^{-n} - x h_i}{1 - x h_i};
\]

Thus it is a residue.

We can use the residue theorem by passing to the other poles which are roots on one.
In order to define multidimensional residues we need **divisors with normal crossings** or a function which in some coordinates \( x_i \) has a pole on the hyperplanes \( x_i = 0, \ i = 1, \ldots, s \). The residue is the coefficient of \( \prod_i x_i^{-1} \).
Definition

Given a subset $A \subset X$ the list $\overline{A} := X \cap \langle A \rangle$ will be called the completion of $A$. In particular $A$ is called complete if $A = \overline{A}$.

The space of vectors $\phi \in U$ such that $\langle a | \phi \rangle = 0$ for every $a \in A$ will be denoted by $A^\perp$. Notice that clearly $\overline{A}$ equals to the list of vectors $a \in X$ which vanish on $A^\perp$.

From this we see that we get a bijection between the complete subsets of $X$ and subspaces of the arrangement defined by $X$. 
A central notion in what follows is given by

**Definition**

Given a complete set $A \subset X$, a **decomposition** is a decomposition $A = A_1 \cup A_2$ in non empty sets, such that:

$$\langle A \rangle = \langle A_1 \rangle \oplus \langle A_2 \rangle.$$  

Clearly the two sets $A_1$, $A_2$ are necessarily complete.

**We shall say that:**  

a complete set $A$ is irreducible if it does not have a non trivial decomposition.
Theorem

Every set $A$ can be decomposed as $A = A_1 \cup A_2 \cup \cdots \cup A_k$ with the $A_i$ irreducible and:

$$\langle A \rangle = \langle A_1 \rangle \oplus \langle A_2 \rangle \oplus \cdots \oplus \langle A_k \rangle.$$

This decomposition is unique up to order.

$A = A_1 \cup A_2 \cup \cdots \cup A_k$ is called the decomposition into irreducibles of $A$. 
Example

An interesting example is that of the configuration space of $s$-ples of point in a line (or the root system $A_{s-1}$). In this case $X = \{z_i - z_j | 1 \leq i < j \leq s\}$.

In this case, irreducible sets are in bijection with subsets of $\{1, \ldots, s\}$ with least 2 elements. If $S$ is such a subset the corresponding irreducible is $I_S = \{z_j - z_i | \{i, j\} \subset S\}$.

Given a complete set $C$, the irreducible decomposition of $C$ corresponds to a family of disjoint subsets $S_1, \ldots, S_k$ of $\{1, \ldots, s\}$ each with at least 2 elements.
A family $S$ of irreducibles $A_i$ is called *nested* if, given elements $A_{i_1}, \ldots, A_{i_h} \in S$ mutually incomparable we have that $C := A_1 \cup A_2 \cup \cdots \cup A_i$ is complete and $C := A_1 \cup A_2 \cup \cdots \cup A_i$ is its decomposition into irreducibles.
Consider the hyperplane arrangement $\mathcal{H}_X$ and the open set

$$A_X = U/(\cup_{H \in \mathcal{H}_X} H)$$

complement of the union of the given hyperplanes. Let us denote by $\mathcal{I}$ the family of irreducible subsets in $X$. We construct a minimal smooth variety $Z_X$ containing $A_X$ as an open set with complement a normal crossings divisor, plus a proper map $\pi : Z_X \to U$ extending the identity of $A_X$. 
For any irreducible subset $A \in \mathcal{I}$ take the vector space $V/A^\perp$ and the projective space $\mathbb{P}(V/A^\perp)$.
Notice that, since $A^\perp \cap \mathcal{A}_X = \emptyset$ we have a natural projection $\pi_A : \mathcal{A}_X \to \mathbb{P}(V/A^\perp)$. If we denote by $j : \mathcal{A}_X \to U$ the inclusion we get a map

**The model**

$$i := j \times (\times_{a \in \mathcal{I}} \pi_A) : \mathcal{A}_X \to U \times (\times_{a \in \mathcal{I}} \mathbb{P}(U/A^\perp))$$

**Definition**

The model $Z_X$ is the closure of $i(\mathcal{A}_X)$ in $U \times (\times_{a \in \mathcal{I}} \mathbb{P}(U/A^\perp))$. 
There is a very efficient approach to computations by residue at points at infinity in the wonderful compactification of the associated hyperplane arrangement. Divisors at infinity correspond to irreducible subsets. Points at infinity correspond to maximal nested sets. A basis of the corresponding residues corresponds to special points indexed by unbroken bases.
The non linear coordinates

Given a MNS $S$ choose a basis $b := b_1, \ldots, b_s$ from $X$ so that if $A_i$ is the minimal element of $S$ containing $b_i$ we have all the $A_i$ distinct.

Construct new coordinates $z_A$, $A \in S$ using the monomial expressions:

$$b_A := \prod_{B \in S, A \subseteq B} z_B.$$  \hspace{1cm} (2)

The residue at the point 0 for these coordinates is denoted by $res_b$.

There is a similar theory for toric arrangements. Now we need to build such data for each point of the arrangement getting residues $res_{b, \phi}$. 
The residue

The polynomials building the multivariate spline and the partition functions are

residues:

**Spline**

\[ p_{b,x}(-y) = \det(b) \text{res}_b \left( \frac{e^{\langle y| x \rangle}}{\prod_{a \in X} \langle x | a \rangle} \right). \]  

**Partition function**

\[ q_{b,x,\phi}(-y) = \det(b) \text{res}_{b,\phi} \left( \frac{e^{\langle y| z \rangle}}{\prod_{a \in X} (1 - e^{-a(z) - \langle \phi | a \rangle})} \right). \]
SUMMARIZING
Given a list of vectors $A$.

- We have the two functions $T_A, B_A$.
  - $T_A$ is supported on the cone $C(A)$ and coincides on each big cell with a homogeneous polynomial in the space $D(A)$ defined by the differential equations $D_Y f = 0$, $Y \in \mathcal{E}(A)$.
  - The space $D(A)$ has as dimension the number $d(A)$ of bases extracted from $A$.
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- $P_A$ is supported on the intersection of the lattice with the cone $C(A)$ and coincides on each big cell with a quasi polynomial in the space $\nabla(A)$ defined by the difference equations $\nabla_Y f = 0$, $Y \in \mathcal{E}(A)$.
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We have shown that there are interesting constructions in commutative and non commutative algebra associated to the study of these functions.

A particularly interesting case is when we take for $A$ the list of positive roots of a root system, or multiples of this list. In this case one has applications to Clebsh–Gordan coefficients.
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