The algebra of the box–spline

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1. Introduction
2. Combinatorics
3. Splines
4. Algebra
5. Approximation theory
6. Arithmetic
7. Residues

BACK TO MATHEMATICS BASIC INPUT

The basic input of this theory is a

(real, sometimes integer) $n \times m$ matrix A.
We always think of A as a LIST of vectors in $V = \mathbb{R}^n$, its columns:

$$A := (a_1, \dots, a_m)$$

Constrain

We assume that 0 is NOT in the convex hull of its columns.

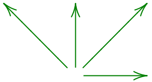
From A we make several constructions, algebraic, combinatorial, analytic etc..

IMPORTANT EXAMPLE

A is the list of POSITIVE ROOTS of a root system,
e.g. B_2:

$$A = \begin{vmatrix} -1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{vmatrix}$$

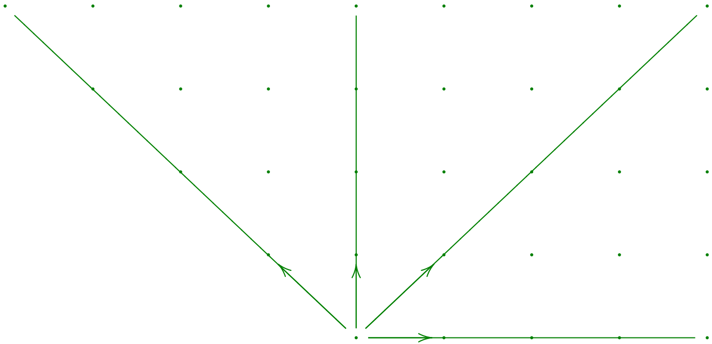
We also identify vectors with linear forms as:



$$-x + y, \; x, \; x + y, \; y$$

IMPORTANT EXAMPLE

A is the list of POSITIVE ROOTS of a root system, the associated cone $C(A)$ has three big cells



BASIC REFERENCE

The book:

C. De Boor, K. Höllig, S. Riemenschneider,

Box splines

Applied Mathematical Sciences 98 (1993).

For a partial survey see

The algebra of the box spline

De Concini C., Procesi C.

math.NA/0602019

SEVERAL NAMES OF CONTRIBUTORS

From numerical analysis

A.A. Akopyan; Ben-Artzi, Asher; C.K. Chui, C. De Boor, W. Dahmen, H. Diamond, N. Dyn, K. Höllig, C. Micchelli, Jia, Rong Qing, A. Ron, A.A. Saakyan

From algebraic geometry

Orlik–Solomon on cohomology, Baldoni, Brion, Szenes, Vergne and of Jeffrey–Kirwan, on partition functions.

In fact a lot of work originated from the seminal paper of Khovanskiĭ, Pukhlikov, interpret the counting formulas as Riemann–Roch formulas for toric varieties

From enumerative combinatorics

A.I. Barvinok, Matthias Beck, Sinai Robins, Richard Stanley

CONVEX POLYTOPES FIRST OUTPUT

FROM THE MATRIX A WE PRODUCE:

First a system of linear equations:

$$\sum_{i=1}^{m} a_i x_i = b, \quad \text{or} \quad Ax = b, \quad A := (a_1, \dots, a_m) \tag{1}$$

The *columns* a_i, b are vectors with n coordinates

$$(a_{j,i}, b_j,\ j = 1, \dots, n).$$

Variable polytopes

As in Linear Programming Theory we deduce and want to study the

VARIABLE POLYTOPES:

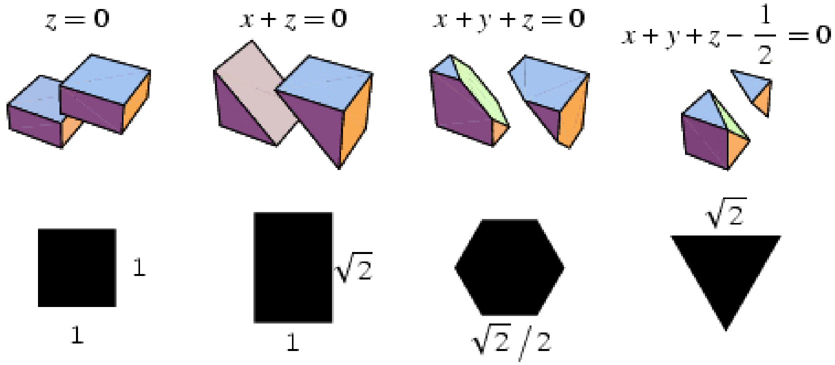
$$\Pi_A(b) := \{x \mid Ax = b, x_i \geq 0, \ \forall i\}$$

$$\Pi_A^1(b) := \{x \mid Ax = b, 1 \geq x_i \geq 0, \ \forall i\}$$

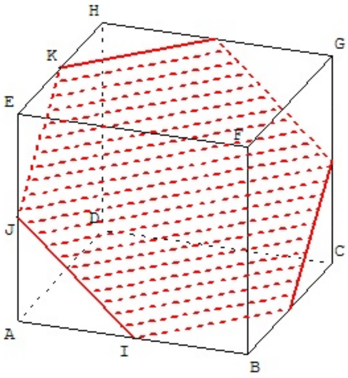
which are convex and bounded for every b.

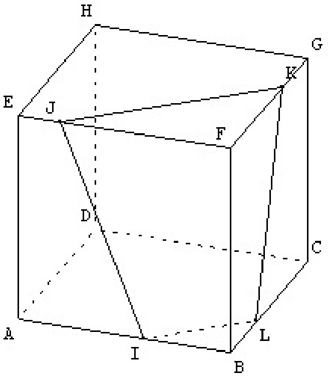
Geometric picture

The polytopes $\Pi_A^1(b)$ are sections of a hypercube,
for instance the simple case of a cube *A* a list of 3 numbers:



Geometric pictures





The object of study

Basic functions

- Set $T_A(x)$, $B_A(x)$ to be the volume of $\Pi_A(x), \Pi^1_A(x)$.
- If A, b have integer coordinates

Arithmetic case, A, x with integer coefficients

In this case set $P_A(x)$ to be the number of solutions of the system in which the coordinates x_i are non negative integers.

Up to a multiplicative normalization constant:
$T_A(x)$ is the Multivariate–spline
$B_A(x)$ the Box–spline
$P_A(x)$ is called the *partition function*

We are interested in

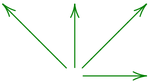
Computing the three functions $T_A(x)$, $B_A(x)$, $P_A(x)$ and describe their qualitative properties.

Applications of these functions to arithmetic, numerical analysis, Lie theory.

EXAMPLE, the C^1 function T_A, case ZP

In the literature of numerical analysis the Box spline associated to the root system B_2 is called the Zwart–Powell or ZP element:

$$A = \begin{vmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 1 \end{vmatrix}$$



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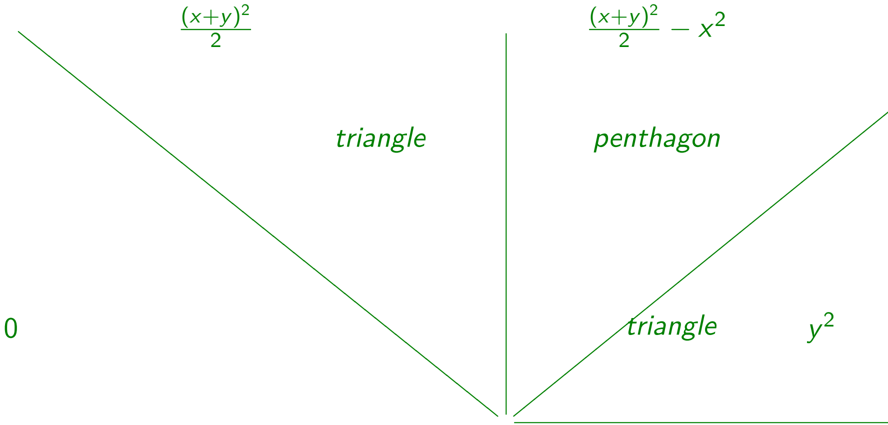
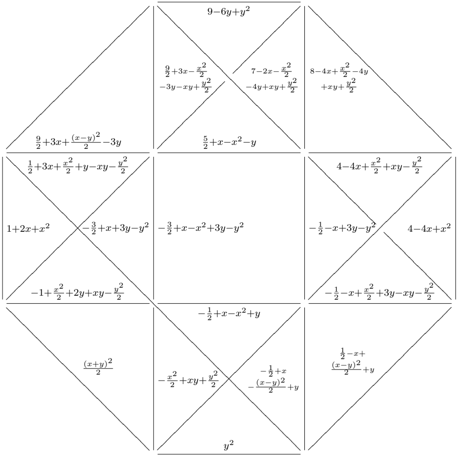


The box–spline for type B_2, ZP element



WHAT DO WE SEE

The computation of the box–spline has some geometric, combinatorial and algebraic flavor. It appears as a piecewise polynomial function on a compact polyhedron.

From simple data we get soon a complicated picture!

THE PARTITION FUNCTION

When A, b have integer elements it is natural to think of an expression like:
$b = t_1 a_1 + \cdots + t_m a_m$ with t_i not negative integers as a:

partition of b with the vectors a_i,

in $t_1 + t_2 + \cdots + t_m$ parts, hence the name partition function for the number $P_A(b)$, thought of as a function of the vector b.

SIMPLE EXAMPLE $m = 2,\ n = 1,\ A = \{2, 3\}$

Parts are 2 and 3

In how many ways can you write a number b as:

$$b = 2x + 3y, \quad x, y \in \mathbb{N} \ ?$$

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Parts are 2 and 3

In how many ways can you write a number b as:

$$b = 2x + 3y, \quad x, y \in \mathbb{N}\ ?$$

ANSWER (Quasi polynomial!)

It depends on the class of n modulo 6.

$n \cong 0$	$\frac{n}{6} + 1$
$n \cong 1$	$\frac{n}{6} - \frac{1}{6}$
$n \cong 2$	$\frac{n}{6} + \frac{2}{3}$
$n \cong 3$	$\frac{n}{6} + \frac{1}{2}$
$n \cong 4$	$\frac{n}{6} + \frac{1}{3}$
$n \cong 5$	$\frac{n}{6} + \frac{1}{6}$

What is a quasi–polynomial?

Two equivalent definitions

FIRST

A function on a lattice Λ which is a polynomial on each coset of some sublattice M of finite index

SECOND

The restriction to a lattice Λ of a function which is a sum of products of a polynomial with an exponential function, periodic on some sublattice M of finite index

PARTIAL FRACTIONS

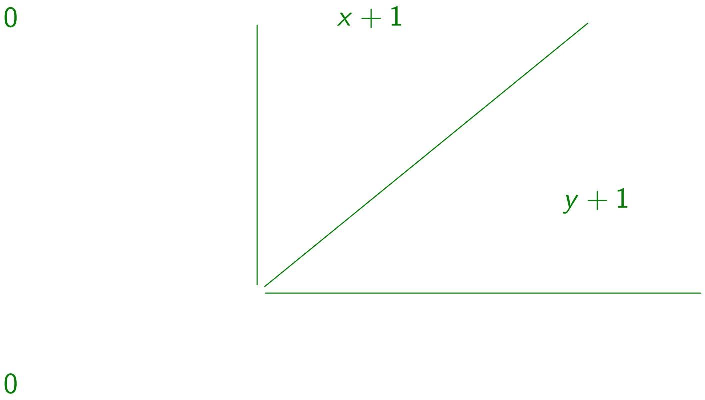
The classical method of study of the partition function associated to a list of numbers $a_1, \dots, a_i, \dots, a_m$ is to expand in *partial fractions* the generating function:

$$\prod_{i=1}^{m} \frac{1}{1 - x^{a_i}}$$

This is done by using the decomposition $1 - x^n = \prod_{k=0}^{n-1}(1 - \zeta_n^k x)$ where $\zeta_n = e^{\frac{2\pi i}{n}}$.

OTHER EXAMPLE, THE HAT FUNCTION

$A = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}$ the corresponding partition function P_A is piecewise polynomial with top degree coinciding with T_A



IMPORTANT EXAMPLE Lie theory

In Lie theory the *Kostant partition function* counts in how many ways can you decompose a weight as a sum of positive roots.

This is used in many computations.

THE THEOREMS

Main

Theorems

There are several general formulas to compute the previous functions which are obtained by a mixture of techniques.

Assume A spans $\mathbb{R}^n$.

Let h or $m(X)$ be the minimum number of columns that one can remove from A so that the remaining columns do not span $\mathbb{R}^n$.

The basic function $T_A(x)$ is a spline

1. $T_A(x)$ has support on the cone $C(A)$
2. $T_A(x)$ is of class $m(X) - 2$
3. $T_A(x)$ coincides with a homogeneous polynomial of degree $m - n$ on each big cell.

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SUMMARIZING

In order to compute $T_A(x)$ we need to

1. Determine the decomposition of $C(A)$ into cells
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GENERAL FORMULA FOR T_X

One can find explicit polynomials $p_{\underline{b},A}(x)$, indexed by a combinatorial object called ***unbroken bases*** and characterized by certain explicit differential equations so that: Given a point x in the closure of a big cell $\mathfrak{c}$ we have

Jeffry-Kirwan residue formula

$$T_A(x) = \sum_{\underline{b} \,\mid\, \mathfrak{c} \subset C(\underline{b})} |\det(\underline{b})|^{-1} p_{\underline{b},A}(-x).$$

From T_A one computes B_A

For a given subset S of A define $a_S := \sum_{a\in S} a$

the basic formula is:

$$B_A(x) = \sum_{S\subset A} (-1)^{|S|} T_A(x - a_S).$$

So T_A is the fundamental object.

Notice that the local pieces of B_A are no more homogeneous polynomials.

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THE FORMULA FOR THE PARTITION FUNCTION

There is a parallel theory, as a result we can compute a set of polynomials $q_{\underline{b},\phi}(-x)$ indexed by pairs, a character ϕ of finite order and a unbroken basis in $A_\phi = \{a \in A \,|\, \phi(e^a) = 1\}$.

The analogue of the Jeffrey–Kirwan formula is:

Theorem

Given a point x in the closure of a big cell $\mathfrak{c}$ we have

Residue formula for partition function

$$P_A(x) = \sum_{\phi \in P(A)} e^{\phi} \sum_{\underline{b} \in \mathcal{NB}_{A_\phi} \,|\, \mathfrak{c} \subset C(\underline{b})} q_{\underline{b},\phi}(-x)$$

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From volumes to partition functions

One can deduce the partition function from some combinatorics and multivariate splines (with parameters):

Theorem

Given a point x in the closure of a big cell $\mathfrak{c}$ we have

$$\mathcal{P}_A = \sum_{\phi \in P(A)} \widehat{Q}_\phi T_{A_\phi, \underline{\phi}}$$

$$Q_\phi = \prod_{a \notin A_\phi} \frac{1}{1 - e^{-a}} \prod_{a \in A_\phi} \frac{a - \langle \phi \,|\, a \rangle}{1 - e^{-a + \langle \phi \,|\, a \rangle}}$$

FIRST

FIRST STEP

SOME COMBINATORICS

IN THE COMBINATORIAL PART

one can explain the ideas relating unbroken bases with cells and all bases with the structure of the polyhedron where the box–spline has its support.

Let us illustrate these ideas through examples.

Unbroken bases

From the theory of matroids to cells.

Let $\underline{c} := a_{i_1}, \ldots, a_{i_k} \in A, \; i_1 < i_2 \cdots < i_k$, be a sublist of linearly independent elements.

Definition

We say that a_i *breaks* $\underline{c}$ if there is an index $1 \leq e \leq k$ such that:

- $i \leq i_e$.
- a_i is linearly dependent on $a_{i_e}, \ldots, a_{i_k}$.

In matrois theory one says that a_i is *externally active*.

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EXAMPLE of the elements breaking a basis

Example

Take as A the list of positive roots for type A_3.

$$A = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}.$$

Let us draw in Red one particular basis and in Green the elements which break it.

We have 16 bases

10 broken and 6 unbroken

FIRST THE 10 broken

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We have 6 unbroken, all contain necessarily α_1:

$$\color{red}{\alpha_1}, \ \color{red}{\alpha_2}, \ \color{red}{\alpha_3}, \ \alpha_1+\alpha_2, \ \alpha_2+\alpha_3, \ \alpha_1+\alpha_2+\alpha_3.$$

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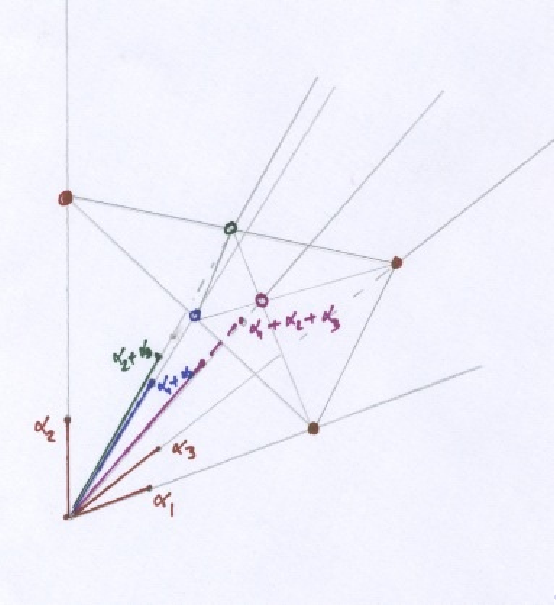
A 3-dimensional example

Let us apply the theory to cells in the same example:

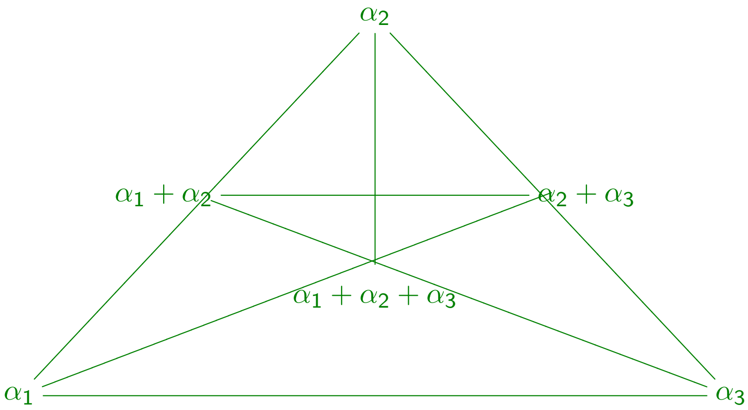
We do everything on a transversal section, where the cone looks like a bounded convex polytope and then project.

We want to decompose the cone $C(A)$ into *big cells* and define its singular and regular points.

PROJECTING A POLYHEDRON TO FORM A CONE

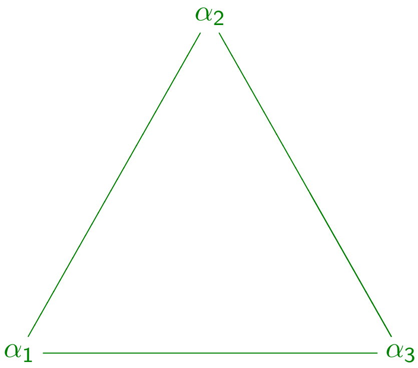


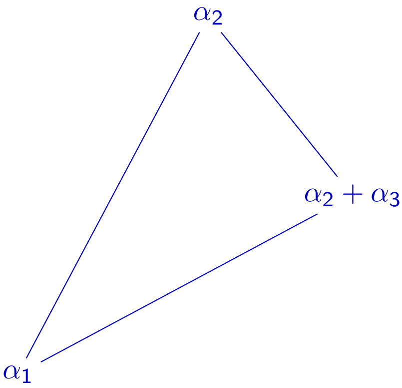
EXAMPLE Type A_3 in section (big cells):

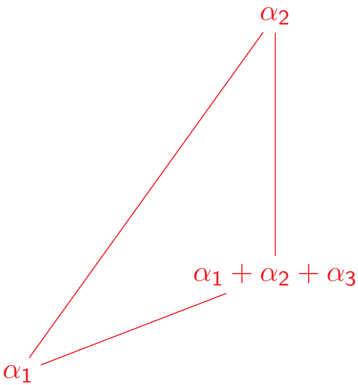


We have 7 big cells.

Let us visualize the simplices generated by the 6 unbroken:




α_2
$\alpha_2 + \alpha_3$
α_1


α_2
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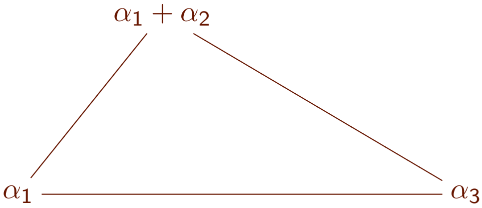
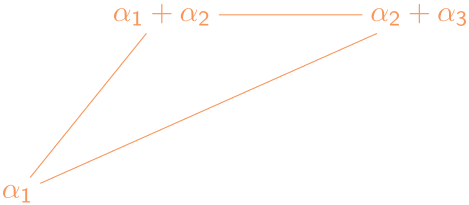

$\alpha_1 + \alpha_2$
α_1
α_3

$$\begin{tikzcd} & \alpha_1 + \alpha_2 + \alpha_3 \arrow[dl, dash] \arrow[dr, dash] & \\ \alpha_1 \arrow[rr, dash] & & \alpha_3 \end{tikzcd}$$


$\alpha_1 + \alpha_2$
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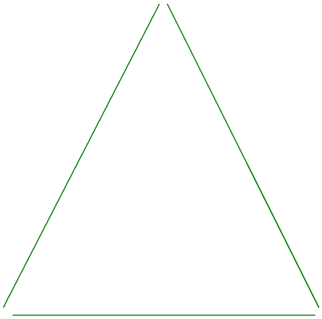
Decomposition into big cells and unbroken bases

Let us visualize the decomposition into big cells, obtained overlapping the cones generated by unbroken bases.

A remarkable fact



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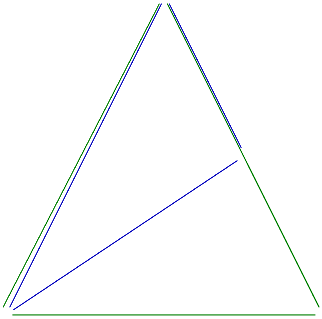

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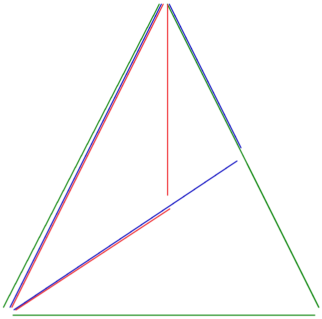

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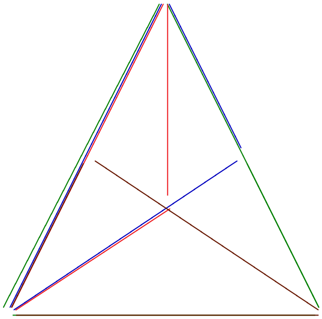

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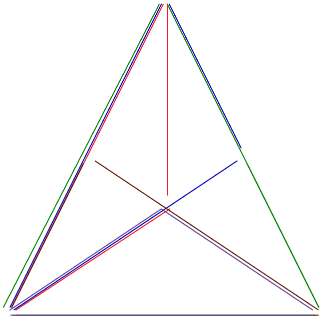

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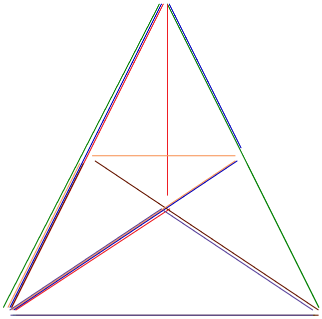

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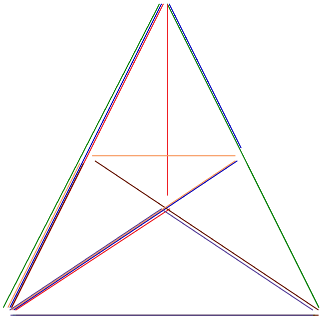


The overlapping theorem

You have visually seen a theorem we proved in general:

by overlapping the cones generated by the unbroken bases one obtains the entire decomposition into big cells!!

HYPERPLANE ARRANGEMENTS

A list of vectors a_i can be thought of as a

list of linear equations

defining a

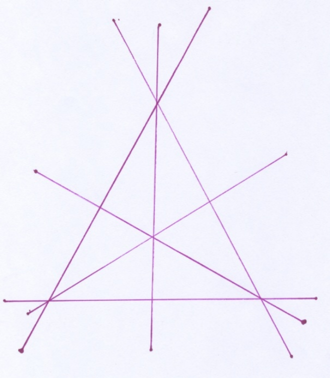
set of linear hyperplanes.

These intersect in a complicated pattern giving rise to a second interesting combinatorial geometric object, the set of all their intersections called:

HYPERPLANE ARRANGEMENT

PROJECTIVE PICTURE OF THE ARRANGEMENT A_3

Same example but as linear functions, or hyperplane arrangement:



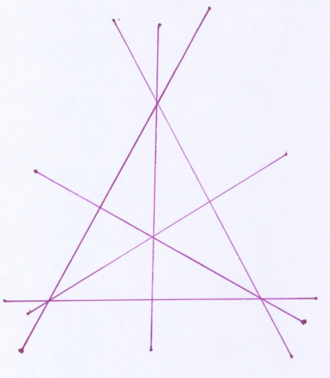
We have drawn in the projective plane of 4−tuples of real numbers with sum 0, the 6 lines

$$x_i - x_j = 0, \ 1 \leq i < j \leq 4$$

the 7 intersection points are also subspaces of the arrangement!

PROJECTIVE PICTURE OF THE ARRANGEMENT A_3

Same example but as linear functions, or hyperplane arrangement:



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$$x_i - x_j = 0, \ 1 \leq i < j \leq 4$$

the 7 intersection points are also subspaces of the arrangement!

PROBLEM

Describe the previous pictures for root systems.

For type A_n the unbroken bases are known and can be indexed by certain binary graphs or by permutations of n elements.
The decomposition into cells is unknown.

THE BOX

The box spline $B_A(x)$ is supported in the compact polytope:

$\boxed{\textit{THE BOX} \quad B(A)}$

The Box $B(A)$

that is the compact convex polytope

$$B(A) := \{\sum_{i=1}^{m} t_i a_i\}, 0 \leq t_i \leq 1, \ \forall i.$$

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The box $B(A)$ has a nice combinatorial structure, and can be paved by a set of parallelepipeds indexed by:
all the bases which one can extract from A!

Example

In the next example

$$A = \begin{vmatrix} 1 & 0 & 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 1 & 2 \end{vmatrix}$$

we have 15 bases and 15 parallelograms.

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EXAMPLE *paving the box*

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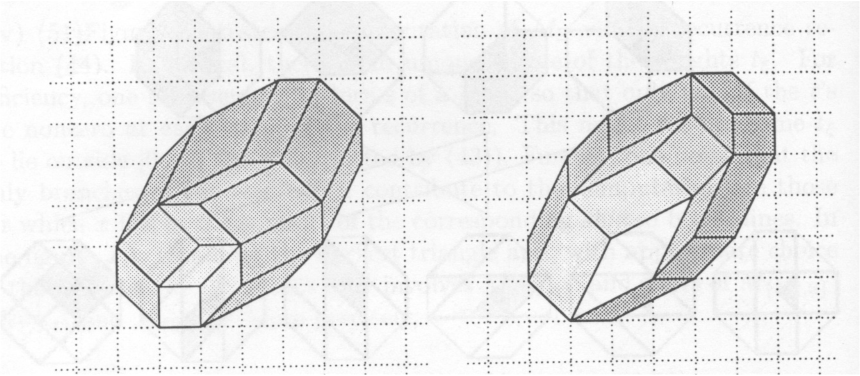
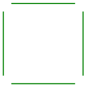


Step–wise paving of the box

$$= \begin{vmatrix} 1 & 0 & 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 1 & 2 \end{vmatrix}$$

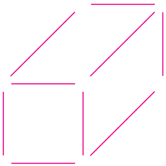
START WITH

$$A = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$



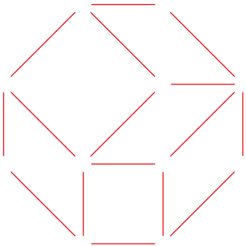
Step–wise paving of the box

$$A = \left|\begin{matrix} 1 & 0 & 1 & \\ 0 & 1 & 1 & \end{matrix}\right|$$



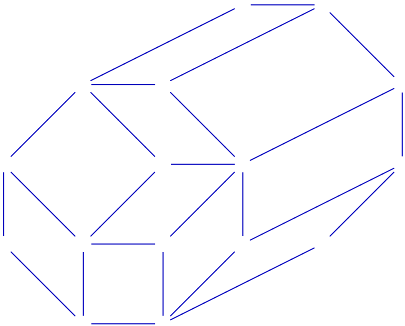
Step–wise paving of the box

$$A = \begin{vmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{vmatrix}$$



Step–wise paving of the box

$$A = \begin{vmatrix} 1 & 0 & 1 & -1 & 2 \\ 0 & 1 & 1 & 1 & 1 \end{vmatrix}$$



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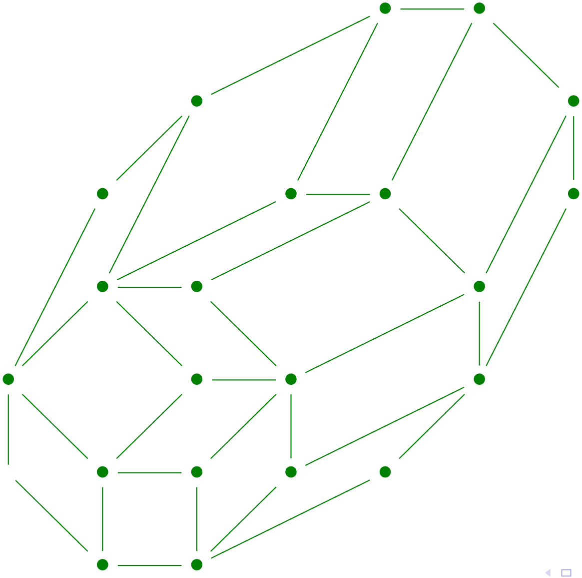

SECOND

SECOND STEP

THE SPLINES

SPLINES

The volume of the variable polytope $\Pi_A(x)$ equals, up to the constant $\sqrt{\det AA^t}$ to the

multivariate spline

that is the function $T_A(x)$ characterized by the formula:

$$\int_{\mathbb{R}^n} f(x)T_A(x)dx = \int_{\mathbb{R}^m_+} f(\sum_{i=1}^m t_i a_i)dt,$$

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WARNING

We have given a weak definition for $T_A(x)$

In general

$T_A(x)$ is a tempered distribution, supported on the cone $C(A)$!

Only when A has maximal rank $T_A(x)$ is a function.

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BOX SPLINES

While the function $T_A(x)$ is the basic object, the more interesting object for numerical analysis is the

box spline

that is the function $B_A(x)$ characterized by the formula:

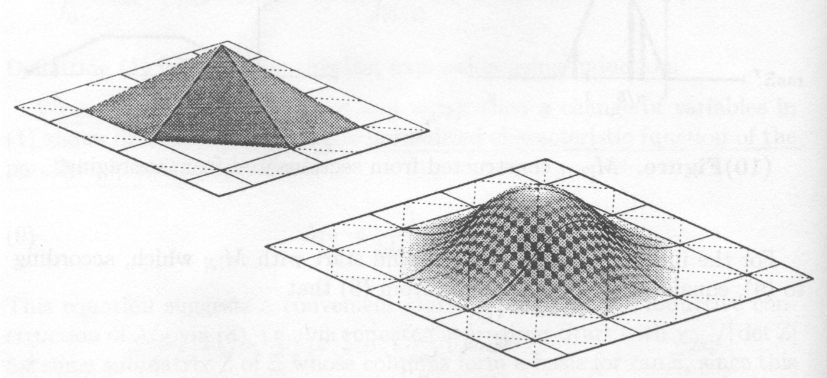
$$\int_{\mathbb{R}^n} f(x)B_A(x)dx = \int_{[0,1]^m} f(\sum_{i=1}^m t_i a_i)dt,$$

where $f(x)$ is any continuous function.

two box splines of class C^0, $h = 2$ and C^1, $h = 3$

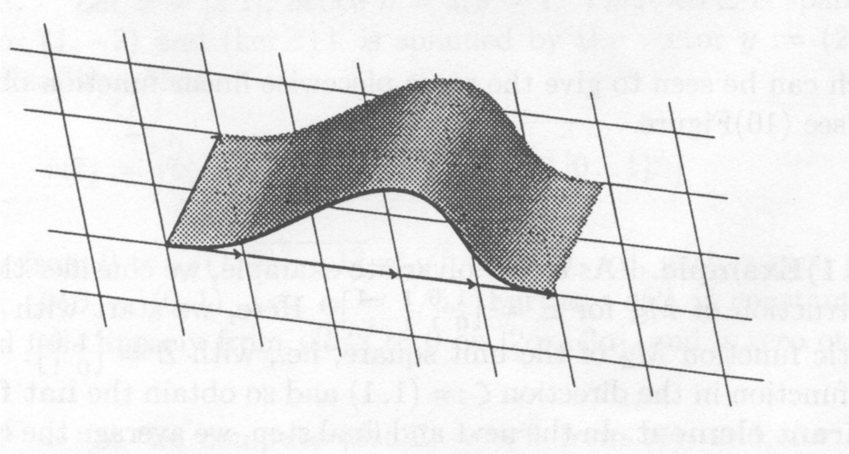
$A = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}$ Hat function or Courant element

$A = \begin{vmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{vmatrix}$ Zwart-Powell element



EXAMPLE OF A two dimensional box spline

Non continuous, $A = \begin{vmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix}$ $(h = 1)$



3 reasons WHY the BOX SPLINE ?

$$\int_{\mathbb{R}^n} B_A(x)dx = 1$$

recursive definition

$$B_{[A,v]}(x) = \int_0^1 B_A(x - tv)dt$$

in the case of integral vectors, we have

PARTITION OF UNITY

The translates $B_A(x - \lambda)$, λ runs over the integral vectors form a *partition of 1*.

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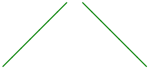
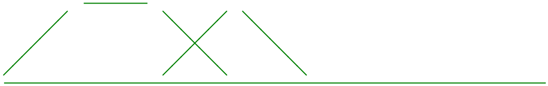
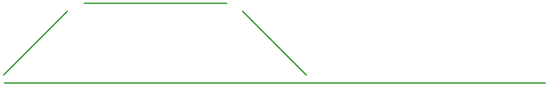
TRIVIAL EXAMPLE

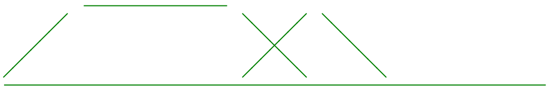
$A = \{1, 1\}$ we have for the box spline



Now let us add to it its translates!





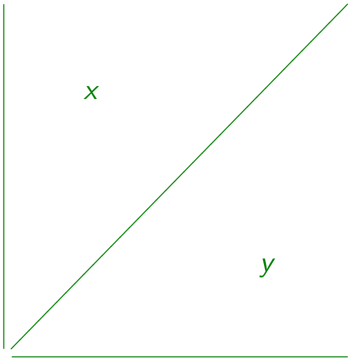


EXAMPLE OF THE HAT FUNCTION

$$A = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

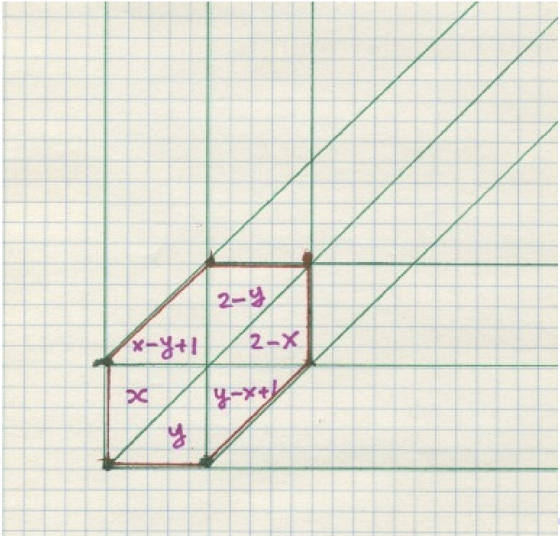
From the function T_A

0

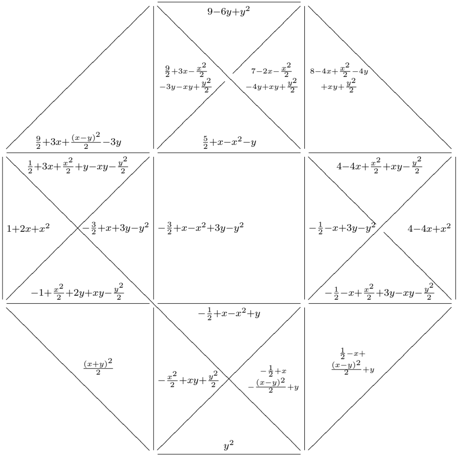


EXAMPLE OF THE HAT FUNCTION

we get the box spline *hat function* summing over the 6 translates of T_A



Recall the box–spline for type B_2



APPROXIMATION THEORY

The box spline, when the a_i are integral vectors, can be effectively used in the *finite element method* to approximate functions.
This follows from the fact that it satisfies the STRANG-FIX condition:

The transform

$$f \mapsto \sum_{i \in \mathbb{Z}^s} B_X(x-i) f(i)$$

transforms polynomials of degree $\leq h-1$ into polynomials (of the same degree).

THIRD

THIRD STEP

FROM ANALYSIS TO ALGEBRA

Here comes the algebra

How to compute T_A? or the partition function P_A?

We use the Laplace transform which will change the analytic problem to one in

algebra

LAPLACE TRANSFORM from $\mathbb{R}^s = V$ to $U = V^*$.

$$Lf(u) := \int_V e^{-\langle u \,|\, v\rangle} f(v)dv.$$

basic properties

$p \in U, w \in V$, write p, D_w for the linear function $\langle p \,|\, v\rangle$ and the directional derivative on V

$$L(D_w f)(u) = wLf(u), \qquad L(pf)(u) = -D_p Lf(u),$$

$$L(e^p f)(u) = Lf(u-p), \qquad L(f(v+w))(u) = e^w Lf(u).$$

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Basic transformation rules

Consider any $a \in V$ as a LINEAR FUNCTION ON U we have:

An easy computation gives the Laplace transforms:

$$LB_A = \prod_{a \in A} \frac{1 - e^{-a}}{a}$$

and

$$LT_A = \prod_{a \in A} \frac{1}{a}$$

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MORE PARTIAL FRACTIONS

We need to rewrite $LT_A = \prod_{a \in A} \frac{1}{a}$ for this we need to develop a theory of partial fractions in several variables, in this case for the algebra

$$S[V][\prod_{a \in X} a^{-1}]$$

WE DO THIS BY NON COMMUTATIVE ALGEBRA!!!!

BASIC NON COMMUTATIVE ALGEBRAS

Algebraic Fourier transform

Weyl algebras

Set $W(V), W(U)$ be the two Weyl algebras of differential operators with polynomial coefficients on V and U.

Fourier transform

There is an algebraic Fourier isomorphism between them, so any $W(V)$ module M becomes a $W(U)$ module $\hat{M}$

$D-$modules in Fourier duality:

Two modules Fourier isomorphic

1. The $D-$module $\mathcal{D}_A := W(V)T_A$ generated, in the space of tempered distributions, by T_A under the action of the algebra $W(V)$ of differential operators on V with polynomial coefficients.

2. The algebra $R_A := S[V][\prod_{a\in A} a^{-1}]$ obtained from the polynomials on U by inverting the element $d_A := \prod_{a\in A} a$

$$R_A = W(U)d_A^{-1}$$

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$$R_A = W(U)d_A^{-1}$$

R_A and it is the coordinate ring of the open set $\mathcal{A}_A$
complement of the union of the hyperplanes of U
of equations $a = 0, \ a \in A$.

It is a cyclic module under $W(U)$ generated by d_A^{-1}.

The module R_A

Localization

It is well known that once we invert an element in a polynomial algebra we get a holonomic module over the algebra of differential operators. $\boxed{R_A \text{ is holonomic!}}$

In particular R_A has a finite composition series and it is cyclic

We want to describe a composition series of R_A.

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The building blocks

The irreducible module N_W

For each subspace W of U we have an irreducible module N_W generated by the δ function of W.

As W runs over the subspaces of the hyperplane

arrangement given by the equations $a_i = 0, \ a_i \in A$

the N_W run over all the composition factors of R_A (with multiplicities).

Take coordinates $x_1, \dots, x_n$

$$W = \{x_1 = x_2 = \ldots = x_k = 0\}$$

N_W is generated by an element δ_W satisfying:

$$x_i \delta_W = 0,\ i \leq k, \quad \frac{\partial}{\partial x_i}\delta,\ i > k.$$

N_W is free of rank 1 generated by δ_W over:

$$\mathbb{C}[x_1, x_2, \ldots, x_k, \frac{\partial}{\partial x_{k+1}}, \ldots, \frac{\partial}{\partial x_n}].$$

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The filtration of R_A by polar order

Definition

R_A is filtered by the $W(U)-$submodules:

filtration degree $\leq k$

$R_{A,k}$ the span of all the fractions $f\prod_{a\in A} a^{-h_a}, \;\; h_a \geq 0$ for which the set of vectors a, with $h_a > 0$, spans a space of dimension $\leq k$.

We have $R_{A,s} = R_A$.

The filtration of R_A by polar order

For all k we have that $R_{A,k}/R_{A,k-1}$ is semisimple.

- The isotypic components of $R_{A,k}/R_{A,k-1}$ are of type N_W as W runs over the subspaces of the arrangement of codimension k.

$R_{A,s}/R_{A,s-1}$ is a free module over

$$S[U] = \mathbb{C}[\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_s}]$$

-

It is important to choose a basis.

The basis theorem

Theorem

A basis for $R_{A,s}/R_{A,s-1}$ over $S[U]$ is given by

the classes of the elements $\prod_{a \in \underline{b}} a^{-1}$ as $\underline{b}$ runs over the set of unbroken bases.

The expansion of $d_A^{-1} = \prod_{a \in A} a^{-1}$

Denote by $\mathcal{NB}$ the unbroken bases extracted from A.

We have a more precise THEOREM.

$$\frac{1}{\prod_{a \in A} a} = \sum_{\underline{b} \in \mathcal{NB}} p_{\underline{b}} \prod_{a \in \underline{b}} a^{-1}, \; p_{\underline{b}} \in S[U] = \mathbb{C}[\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_s}]$$

EXAMPLE Courant element

Example

USE COORDINATES x, y SET

$$A = [x+y, x, y] = [x, y] \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$\frac{1}{(x+y)\,x\,y} = \frac{1}{x\,(x+y)^2} + \frac{1}{y\,(x+y)^2} =$$

$$-\frac{\partial}{\partial y}\Big(\frac{1}{x\,(x+y)}\Big) - \frac{\partial}{\partial x}\Big(\frac{1}{y\,(x+y)}\Big)$$

EXAMPLE ZP element

Example

$$A = [x+y, x, y, -x+y] = [x, y] \begin{vmatrix} 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & 1 \end{vmatrix}$$

$$\frac{1}{(x+y)\,x\,y\,(-x+y)} = \frac{1}{(x+y)^3 x} + \frac{4}{(x+y)^3(-x+y)} - \frac{1}{(x+y)^3 y} =$$

$$1/2[\frac{\partial^2}{\partial^2 y}(\frac{1}{(x+y)x}) + (\frac{\partial}{\partial x} + \frac{\partial}{\partial y})^2(\frac{1}{(x+y)(-x+y)}) - \frac{\partial^2}{\partial^2 x}(\frac{1}{(x+y)\,y})]$$

We now need the basic inversion

Let $X = \{a_1, \dots, a_n\}$ be a basis, $d := |\det(a_1, \dots, a_s)|$

$\chi_{C(X)}$ the characteristic function of the positive quadrant $C(X)$ generated by X.

Basic example of inversion

$$L(d^{-1}\chi_{C(X)}) = \prod_{i=1}^{n} a_i^{-1}$$

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We are ready to invert!

We want to invert

$$d_A^{-1} = \sum_{\underline{b} \in \mathcal{NB}} p_{\underline{b},A}\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_s}\right) \prod_{a \in \underline{b}} a^{-1}$$

From the basic example and the properties!

We get

$$\sum_{\underline{b} \in \mathcal{NB}} p_{\underline{b},A}(-x_1, \ldots, -x_s) d_{\underline{b}}^{-1} \chi_{C(\underline{b})}$$

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EXAMPLE ZP element

Inverting

$$1/2[\frac{\partial^2}{\partial^2 y}(\frac{1}{x\,(x+y)})+(\frac{\partial}{\partial x}+\frac{\partial}{\partial y})^2(\frac{1}{(x+y)(-x+y)})-\frac{\partial^2}{\partial^2 x}(\frac{1}{y\,(x+y)})]$$

we get

$$1/2[y^2\chi_{C((1,0),(1,1))}+\frac{(x+y)^2}{2}\chi_{C((1,1),(-1,1)}-x^2\chi_{C((0,1),(1,1))}]$$

The theory of Dahmen–Micchelli

We need a basic definition of combinatorial nature

Definition

We say that a sublist $Y \subset A$ is a *cocircuit*, if the elements in $A - Y$ do not span V.

The basic differential operators

For such Y set $D_Y := \prod_{a \in Y} D_a$, a differential operator with constant coefficients.

(D_a is directional derivative, first order operator)

The theory of Dahmen–Micchelli

We need a basic definition of combinatorial nature

Definition

We say that a sublist $Y \subset A$ is a *cocircuit*, if the elements in $A - Y$ do not span V.

The basic differential operators

For such Y set $D_Y := \prod_{a \in Y} D_a$, a differential operator with constant coefficients.

(D_a is directional derivative, first order operator)

For a given unbroken circuit basis $\underline{b}$, consider the element

$$D_{\underline{b}} := \prod_{a \notin \underline{b}} D_a.$$

Characterization by differential equations

The polynomials $p_{\underline{b},A}$ are characterized by the differential equations

$$D_Y p = 0, \ \forall Y, \quad \text{a cocircuit in } A$$

$$D_{\underline{b}} p_{\underline{c},A}(x_1, \ldots, x_s) = \begin{cases} 1 & \text{if} \quad \underline{b} = \underline{c} \\ 0 & \text{if} \quad \underline{b} \neq \underline{c} \end{cases}$$

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The space $D(A)$

A remarkable space of polynomials

$$D(A) := \{p \mid D_Y p = 0, \ \forall Y, \quad \text{a cocircuit in } A\}$$

The theorem of Dhamen Micchelli

$\dim D(A)$ equals the total number of bases extracted from A

We have a more precise theorem

The polynomials $p_{\underline{b},A}$ form a basis for the top degree part $(m-n)$ of $D(A)$.

The graded dimension of $D(A)$ is given by

$$H_A(q) = \sum_{\underline{b} \in \mathcal{B}(A)} q^{m-n(\underline{b})}.$$

$n(\underline{b})$ is the number of elements of A breaking $\underline{b}$.

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For A_3 we get

The graded dimension is:

$$6q^3 + 6q^2 + 3q + 1$$

Remark that for all polynomials in three variables it is:

$$\ldots + 10q^3 + 6q^2 + 3q + 1$$

Interpretation

The theorem of Dahmen–Micchelli can be formulated both in terms of commutative as well as non–commutative algebra.

APPROXIMATION THEORY

APPROXIMATION POWER

THE STRANG-FIX CONDITIONS

THE STRANG-FIX CONDITIONS

The interest of the space of polynomials $D(A)$ comes in approximation theory from the problem of studying the approximation of a function $f(x)$ by the *finite element method* :

discrete convolution

$$f(x) \mapsto \sum_{\underline{i} \in \mathbb{Z}^n} B_A(x - \underline{i}) f(\underline{i})$$

Or at order $n \in \mathbb{N}$:

$$f(x) \mapsto \sum_{\underline{i} \in \mathbb{Z}^n} B_A(nx - \underline{i}/n) f(\underline{i}/n)$$

More generally we want to find *weights* $c_{\underline{i}}$ and approximate f by $f(x) \mapsto \sum_{\underline{i} \in \mathbb{Z}^n} B_A(nx - \underline{i}/n)c_{\underline{i}}$ and determine a constant $k \in \mathbb{N}$ so that (on some bounded region):

$$|f(x) - \sum_{\underline{i} \in \mathbb{Z}^s} B_A(nx - \underline{i}/n)c_{\underline{i}}| \leq Cn^{-k}$$

The maximum k is the

approximation power of B_A.

THE CARDINAL SPLINE SPACE

Given a spline $M(x)$ on $\mathbb{R}^s$, with compact support one may define *THE CARDINAL SPLINE SPACE* to be the space $\mathcal{S}_M$ of all (infinite) linear combinations:

$$\mathcal{S}_M := \{\sum_{\underline{i}\in\mathbb{Z}^s} M(x-\underline{i})c_{\underline{i}}\}.$$

The approximation power of $M(x)$ is related to two questions:
For which polynomials $f(x)$ we have that $\sum_{\underline{i}\in\mathbb{Z}^s} M(x-\underline{i})f(\underline{i})$ is a polynomial?
2. Which polynomials lie in the cardinal spline space?

Strang–Fix conditions

The Strang–Fix conditions is a general statement:

The approximation power of a function M is the maximum r such that the space of all polynomials of degree $\leq r$ is contained in the cardinal space $\mathcal{S}_M$.

The case of B_A

For the cardinal spline space in the case $B_A(x)$ with A integral we have:

1. $D(A)$ is characterized as the space of polynomials $f(x)$ which *reproduce, i.e.map to polynomials* under the *discrete convolution*.

2. $D(A)$ is also characterized as the space of polynomials lying in the cardinal spline space.

Strang-Fix conditions

The power of approximation by discrete convolution is measured by the maximum degree of the space of polynomials which reproduce under discrete convolution.

SUPERFUNCTIONS

Consider the following algorithm applied to a function g:

$$g_h := \sum_{\underline{i} \in \Lambda} F(x/h - \underline{i}) g(h\underline{i})$$

There are functions F in the cardinal spline space such that this transformation is the identity on polynomials of degree $< m(X)$, these are the super–functions. For such functions the previous algorithm satisfies the requirements of the Strang–Fix approximation

Theorem

We have, under the explicit algorithm previously constructed that, for any domain G:

$$||f_h - f||_{L^\infty(G)} = O(h^{m(X)}).$$

For every multi–index $\alpha \in \mathbb{N}^s$ with $|\alpha| \leq m(X) - 1$, we have:

$$||\partial^\alpha f_h - \partial^\alpha f||_{L^\infty(G)} = ||\partial^\alpha (f_h - f)||_{L^\infty(G)} = O(h^{m(X)-|\alpha|}).$$

THE DISCRETE CASE

PARTITION FUNCTIONS

THE DISCRETE CASE

THE DISCRETE CASE

We think of the partition function

$$P_A(b) = \#\{t_1, \dots, t_m \in \mathbb{N} \mid \sum_{i=1}^{m} t_i a_i = b\}$$

as a distribution

$$\sum_{\lambda \in \Lambda} P_A(\lambda)\delta_\lambda.$$

with Laplace transform

$$\sum_{\lambda \in \Lambda} P_A(\lambda)e^{-\lambda} = \prod_{a \in A} \frac{1}{(1 - e^{-a})}.$$

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One can see that the leading part of the formula for the partition function is given by the multivariate spline T_A and also that there are formulas using differential operators to pass from the functions T_A to the partition functions.

OTHER FOURIER ISOMORPHIC ALGEBRAS

The periodic Weyl algebras $\tilde{W}(U)$ and $W^{\#}(\Lambda)$

$$\tilde{W}(U) = \mathbb{C}[e^{\pm t_1}, \dots, e^{\pm t_n}, \frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_n}]$$

algebra of difference operators

$$W^{\#}(\Lambda) = \mathbb{C}[x_1, \dots, x_n, \nabla_{\pm x_1}, \dots, \nabla_{\pm x_n}]$$

$$\nabla_{x_i}(x_j) = \begin{cases} x_j \ \ if \ \ i \neq j \\ x_i - 1 \end{cases} .$$

$D-$modules in Fourier duality:

Two modules Fourier isomorphic

1. The $W^\#-$module $\mathcal{D}_A^\# := W^\#(\Lambda)\mathcal{P}_A$ generated, in the space of tempered distributions, by the partition distribution $\mathcal{P}_A$.

2. The $W(U)$ module $S_A = \mathbb{C}[\Lambda][\prod_{a\in A}(1-e^{-a})^{-1}]$ is the algebra obtained from the character ring $\mathbb{C}[\Lambda]$ by inverting $u_A := \prod_{a\in A}(1-e^{-a})$.

The toric arrangement

T the torus of character group Λ

$S_A = \mathbb{C}[\Lambda][u_A^{-1}]$ is the coordinate ring of the open set $\mathcal{T}_A \subset T$ complement of the union of the subgroups of T of equations $e^a = 1, \; a \in A$.

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The toric arrangement

The toric arrangement

The toric arrangement is the finite set consisting of all the connected components of the subvarieties obtained by intersecting the subgroups of T of equations $e^a = 1, \ a \in A$.

EXAMPLE $s = 1, \ T = \mathbb{C}^*, \ A = \{5, 3\}$
The arrangement consists of the connected components of the variety $x^5 = 1$ or $x^3 = 1$, i.e. of the five, fifth roots of 1 and the three third roots of 1.

Points of the arrangement

The elements of the toric arrangement are ordered by reverse inclusion, particular importance is given to the

points of the arrangement, $P(A)$

which are the zero-dimensional, i.e. points, elements of the arrangement.

A very special case is when $P(A)$ reduces to the point 1, this is the *unimodular case*.

Each point $p \in P(A)$ determines a sublist:

$$A_p := \{a \in A \mid e^a(p) = 1\}.$$

EXAMPLE ZP

$$A = \begin{vmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 1 \end{vmatrix}$$

The subgroups are

$$xy = 1,\ x = 1,\ y = 1,\ x^{-1}y = 1.$$

We have *two points in* $P(A)$

$$(1, 1),\ (-1, -1).$$

$$A_{(1,1)} = A, \quad A_{(-1,-1)} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}.$$

THE FILTRATION

We have as for hyperplanes a filtration by polar orders on S_A.

Each graded piece is semisimple.

The isopypic components appearing in grade k correspond to the connected components of the toric arrangement of codimension k

For the top part $S_{A,n}/S_{A,n-1}$ we have a sum over the *points of the arrangement* $P(A)$.

The isotypic component associated to a point e^ϕ decomposes as direct sum of irreducibles indexed by the unbroken bases in

$$A_\phi := \{a \in A \mid e^{\langle a \mid \phi \rangle} = 1\}.$$

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$$A_{\phi} := \{a \in A \mid e^{\langle a \mid \phi \rangle} = 1\}.$$

The previous formula shows in particular, that the partition function is on each cell a

Local structure of $\mathcal{P}_A$

LINEAR COMBINATION OF POLYNOMIALS TIME PERIODIC EXPONENTIALS.

Such a function is called a QUASI POLINOMIAL

Basic equations

As for the case of the multivariate spline:

The quasi polynomials appearing in the formula for $\mathcal{P}_A$ satisfy

special difference equations

DIFFERENCE OPERATORS

For $a \in \Lambda$ and f a function on Λ we define the the

difference operator:

$$\nabla_a f(x) = f(x) - f(x-a), \; \nabla_a = 1 - \tau_a.$$

Example

As special functions we have the characters, eigenvectors of difference operators.

DIFFERENCE EQUATIONS

Parallel to the study of $D(A)$, one can study the

system of difference equations

$$\nabla_Y f = 0, \text{ where } \nabla_Y := \prod_{v \in Y} \nabla_v$$

as $Y \in \mathcal{E}(A)$ runs over the cocircuits.

Let us denote the space of solutions by:

$$\nabla(A) := \{f : \Lambda \to \mathbb{C}, \mid \nabla_Y(f) = 0, \ \forall Y \in \mathcal{E}(A)\}.$$

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SECOND THEOREM OF DAHMEN–MICCHELLI

Weighted dimension

The dimension of $\nabla(A)$ is the *Volume of the box* $B(A)$*!* $\delta(A)$ we have:

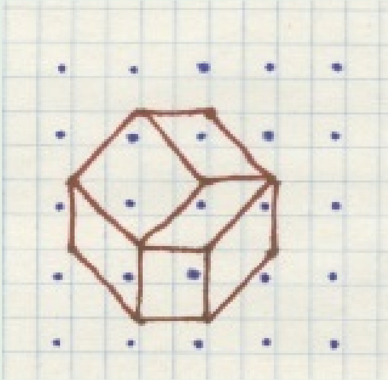
$$\delta(A) := \sum_{\underline{b} \in \mathcal{B}(A)} |\det(\underline{b})|.$$

This formula has a strict connection with the paving of the box.

Example

Let us take

$$A = \begin{vmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \end{vmatrix}$$



See that $\delta(A) = 1+1+1+1+1+2 = 7$ is the number of points in which the box $B(A)$, shifted generically a little, intersects the lattice!

FROM DIFFERENCE TO DIFFERENTIAL EQUATIONS

logarithm isomorphism

There is a formal machinery which allows us to interpret, locally around a point, difference equations as restriction to the lattice of differential equations, we call it the

logarithm isomorphism

We have this for any module over the periodic Weyl algebra $\mathbb{C}[\frac{\partial}{\partial x_i}, e^{x_i}]$ as soon as for algebraic reasons (nilpotency) we can deduce from the action of e^{x_i} also an action of x_i.

Recursion

Finally Dahmen–Micchelli prove that the partition function on a given big cell coincides with the quasi polynomial in $\nabla(A)$ which takes values 1 at 0 and 0 at the points of the lattice lying in the $x_0 - B(X)$ where x_0 is in the big cell very close to 0.

ALGEBRAIC GEOMETRY

WONDERFUL MODELS

RESIDUES

There is an approach to compute the partition function based on residues

Start from the case of numbers. Fix positive numbers $\underline{h} := (h_1, \dots, h_m)$.

Given an integer n,

the number of ways $n = \sum_i k_i h_i$ is the coefficient of x^{-1} in

$$P_{\underline{h}}(x) = \prod_i \frac{x^{-n-1}}{1 - x^{h_i}};$$

Thus it is a residue.

We can use the residue theorem by passing to the other poles which are roots on one.

multidimensional residues

In order to define multidimensional residues we need divisors with normal crossings or a function which in some coordinates x_i has a pole on the hyperplanes $x_i = 0,\ i = 1, \dots, s$.
The residue is the coefficient of $\prod_i x_i^{-1}$.

Definition

Given a subset $A \subset X$ the list $\overline{A} := X \cap \langle A \rangle$ will be called the completion of A. In particular A is called complete if $A = \overline{A}$.

The space of vectors $\phi \in U$ such that $\langle a | \phi \rangle = 0$ for every $a \in A$ will be denoted by $A^{\perp}$. Notice that clearly $\overline{A}$ equals to the list of vectors $a \in X$ which vanish on $A^{\perp}$.
From this we see that we get a bijection between the complete subsets of X and subspaces of the arrangement defined by X.

A central notion in what follows is given by

Definition

Given a complete set $A \subset X$, a **decomposition** is a decomposition $A = A_1 \cup A_2$ in non empty sets, such that:

$$\langle A \rangle = \langle A_1 \rangle \oplus \langle A_2 \rangle.$$

Clearly the two sets A_1, A_2 are necessarily complete.

We shall say that :

a complete set A is irreducible if it does not have a non trivial decomposition.

Theorem

Every set A can be decomposed as $A = A_1 \cup A_2 \cup \cdots \cup A_k$ with the A_i irreducible and:

$$\boxed{\langle A\rangle = \langle A_1\rangle \oplus \langle A_2\rangle \oplus \cdots \oplus \langle A_k\rangle.}$$

This decomposition is unique up to order.

$A = A_1 \cup A_2 \cup \cdots \cup A_k$ is called the ***decomposition into irreducibles*** of A.

Example

An interesting example is that of the configuration space of s-ples of point in a line (or the root system A_{s-1}). In this case $X = \{z_i - z_j | 1 \leq i < j \leq s\}$.
In this case, irreducible sets are in bijection with subsets of $\{1, \dots, s\}$ with least 2 elements. If S is such a subset the corresponding irreducible is $I_S = \{z_j - z_j | \{i, j\} \subset S\}$.
Given a complete set C, the irreducible decomposition of C corresponds to a family of disjoint subsets $S_1, \dots, S_k$ of $\{1, \dots, s\}$ each with at least 2 elements.

Definition

A family $\mathcal{S}$ of irreducibles A_i is called *nested* if, given elements $A_{i_1}, \ldots, A_{i_h} \in \mathcal{S}$ mutually incomparable we have that $C := A_1 \cup A_2 \cup \cdots \cup A_i$ is complete and $C := A_1 \cup A_2 \cup \cdots \cup A_i$ is its decomposition into irreducibles.

Consider the hyperplane arrangement $\mathcal{H}_X$ and the open set

$$\mathcal{A}_X = U/(\cup_{H \in \mathcal{H}_X} H)$$

complement of the union of the given hyperplanes.
Let us denote by $\mathcal{I}$ the family of irreducible subsets in X.
We construct a minimal smooth variety Z_X containing $\mathcal{A}_X$ as an open set with complement a normal crossings divisor, plus a proper map $\pi : Z_X \to U$ extending the identity of $\mathcal{A}_X$.

For any irreducible subset $A \in \mathcal{I}$ take the vector space $V/A^{\perp}$ and the projective space $\mathbb{P}(V/A^{\perp})$.
Notice that, since $A^{\perp} \cap \mathcal{A}_X = \emptyset$ we have a natural projection $\pi_A : \mathcal{A}_X \to \mathbb{P}(V/A^{\perp})$. If we denote by $j : \mathcal{A}_X \to U$ the inclusion we get a map

The model

$$i := j \times (\times_{a \in \mathcal{I}} \pi_A) : \mathcal{A}_X \to U \times (\times_{a \in \mathcal{I}} \mathbb{P}(U/A^{\perp}))$$

Definition

The model Z_X is the closure of $i(\mathcal{A}_X)$ in $U \times (\times_{a \in \mathcal{I}} \mathbb{P}(U/A^{\perp}))$.

There is a very efficient approach to computations by

residue at points at infinity in the wonderful compactification of the associated hyperplane arrangement.

Divisors at infinity correspond to irreducible subsets.

Points at infinity correspond to maximal nested sets.

A basis of the corresponding residues corresponds to special points indexed by unbroken bases.

The non linear coordinates

Given a MNS $\mathcal{S}$ choose a basis $\underline{b} := b_1, \dots, b_s$ from X so that if A_i is the minimal element of $\mathcal{S}$ containing b_i we have all the A_i distinct.
Construct new coordinates $z_A, \ A \in \mathcal{S}$ using the monomial expressions:

$$b_A := \prod_{B \in \mathcal{S},\ A \subseteq B} z_B. \tag{2}$$

The residue at the point 0 for these coordinates is denoted by $res_{\underline{b}}$.
There is a similar theory for toric arrangements. Now we need to build such data for each point of the arrangement getting residues $res_{\underline{b},\phi}$.

The residue

The polynomials building the multivariate spline and the partition functions are

residues:

Spline

$$\boxed{p_{\underline{b},X}(-y) = det(\underline{b}) res_{\underline{b}}\left(\frac{e^{\langle y|x\rangle}}{\prod_{a\in X}\langle x \mid a\rangle}\right).} \tag{3}$$

Partition function

$$\boxed{q_{\underline{b},X_\phi}(-y) = det(\underline{b}) res_{\underline{b},\phi}\left(\frac{e^{\langle y|z\rangle}}{\prod_{a\in X}(1 - e^{-a(z)-\langle \phi \mid a\rangle})}\right).} \tag{4}$$

SUMMARIZING

Given a list of vectors A.

- We have the two functions T_A, B_A.
- T_A is supported on the cone $C(A)$ and coincides on each big cell with a homogeneous polynomial in the space $D(A)$ defined by the differential equations $D_Y f = 0, \; Y \in \mathcal{E}(A)$.
- The space $D(A)$ has as dimension the number $d(A)$ of bases extracted from A.
- There are explicit formulas to compute T_A on each cell.
- B_A is deduced from T_A.

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When A is in a lattice we have the partition function P_A.

- P_A is supported on the intersection of the lattice with the cone $C(A)$ and coincides on each big cell with a quasi polynomial in the space $\nabla(A)$ defined by the difference equations $\nabla_Y f = 0, \ \ Y \in \mathcal{E}(A)$.
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- P_A is supported on the intersection of the lattice with the cone $C(A)$ and coincides on each big cell with a quasi polynomial in the space $\nabla(A)$ defined by the difference equations $\nabla_Y f = 0, \;\; Y \in \mathcal{E}(A)$.
- The space $\nabla(A)$ has as dimension $\delta(A)$ the weighted number of bases extracted from A.
- There are explicit formulas to compute P_A on each cell.

We have shown that there are interesting constructions in commutative and non commutative algebra associated to the study of these functions.

A particularly interesting case is when we take for A the list of positive roots of a root system, or multiples of this list. In this case one has applications to Clebsh–Gordan coefficients.

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