SUMMER SCHOOL

Makerere–Kampala 2006

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Chapter 1

WEEK 1

1.1 Day 1

1.1.1 Claudio 1, Convex sets

We want to study certain geometric properties of convex polytopes. These are the generalization to any dimension of the usual elementary notion of convex polygones in 2-dimension or polyhedra in 3-dimensions.

Let us recall the basic definition. We work on a finite dimensional vector space over the real numbers, of some fixed dimension s which for simplicity we may think of as $\mathbb{R}^s$.

let us recall that, given two points $a, b \in V$ the closed segments which has a, b as the two extremes, is the set of *convex combinations*:

$$[a, b] := \{ta + (1 - t)b \,|\, 0 \leq t \leq 1.\}$$

In general given m points $a_1, \ldots, a_m \in V$ a *convex combination* of these points is any point

$$p = t_1 a_1 + t_2 a_2 + \cdots + t_m a_m, \; 0 \leq t_i \leq 1, \; \sum_{i=1}^{m} t_i = 1.$$

Definition 1.1.2. We say that a sunset $A \subset V$ is **convex**, if given any two points $a, b \in A$ the entire segment $[a, b]$ is contained in A.

We have an immediate

Proposition 1.1.3. *Given any set $A \subset V$ there is a unique minimal convex set $\tilde{A}$ containing A. This set is called the* **convex hull** *of A. $\tilde{A}$ is formed by all the convex combinations of points in A.*

Proof. Let us prove first that, given a convex set B and m points $a_1, \dots, a_m \in B$ any convex combination of them is in B. If $m = 2$ this is the definition of convexity. Otherwise let us work by induction.

Take a convex combination $p = t_1 a_1 + t_2 a_2 + \cdots + t_m a_m,\ 0 \le t_i \le 1,\ \sum_{i=1}^m t_i = 1$ we may assume all the $t_i > 0$. We write

$$p = t_1 a_1 + t_2 a_2 + \cdots + t_m a_m = t_1 a_1 + (1 - t_1)[t_2/(1 - t_1)a_2 + \cdots + t_m/(1 - t_1)a_m]$$

we have

$$t_2/(1 - t_1) + \cdots + t_m/(1 - t_1) = (t_2 + \dots t_m)/(1 - t_1) = 1$$

so that by induction $q := t_2/(1 - t_1)a_2 + \cdots + t_m/(1 - t_1)a_m \in B$ and then by convexity $p = t_1 a_1 + (1 - t_1)q \in B$.

As second point let us show that the set of convex combinations of elements of A is a convex set. Thus let us take two such convex combinations $p = \sum_i t_i a_i,\ q = \sum_i s_i a_i$, since we allow the coefficients to be possibly 0 we may assume that the combinations involve the same points a_i. A point in the segment $[p, q]$ is of the form $tp + (1 - t)q,\ 0 \le t \le 1$ we have:

$$tp+(1-t)q = \sum_i (tt_i+(1-t)s_i)a_i,\ 0 \le (tt_i+(1-t)s_i),\ \sum_i (tt_i+(1-t)s_i) = t\sum_i t_i+(1-t)\sum_i s_i = 1$$

thus any point of the segment $[p, q]$ is still a convex combination of the a_i. □

The prime example for us is the following, take a finite set of points $X = \{a_1, \dots, a_m\}$.

Definition 1.1.4. The convex hull of the set X is a convex polytope.

As we have seen such a polytope is the set of convex combinations of the points a_i. It is possible that, when we remove one of the a_i the convex hull of the remaining elements does not change. This happens when the element we have removed is a convex combination of the remaining ones.

If whenever we remove one of the a_i we get a strictly smaller convex hull we say that the elements a_i are the *extremal points of the polytope*. It can be seen that this notions is intrinsic.

The main point of our treatment comes from the fact that a convex polytope can be also defined *dually* as the set of points p where a finite set of linear functions f_i are positive, i.e. by a finite set of linear inequalities $f_i(p) \ge 0$. This is the form in which we shall study the polytopes.

1.1.5 Variable polytopes

Let us now take a finite set of vectors $X = \{a_1, \dots, a_m\}$ and assume now that V is a vector space over the real numbers and that 0 is not in the convex hull of the elements of X. This is equivalent to saying that there is a linear function $\phi \in V^*$ with $c_i := \langle \phi \mid a_i \rangle > 0,\ \forall a_i \in X$.

Let us think of the vectors a_i as the columns of a real matrix A.

As in linear programming from the system of linear equations

$$\sum_{i=1}^{m} a_i x_i = b, \quad \text{or} \quad Ax = b,$$

we deduce two families of variable convex bounded polytopes:

$$\Pi_X(b) := \{x \mid Ax = b, x_i \geq 0, \ \forall i\}$$

$$\Pi_X^1(b) := \{x \mid Ax = b, 1 \geq x_i \geq 0, \ \forall i\}.$$

The hypothesis made (that 0 is not in the convex hull of elements of X) is only necessary to insure that the polytopes $\Pi_X(b)$ are all bounded. In fact if $\sum_i x_i a_i = b$ we have $\sum_i x_i c_i = \langle \phi \mid b \rangle$ so that $\Pi_X(b)$ lies in the bounded region, $0 \leq x_i \leq \langle \phi \mid b \rangle c_i^{-1}$. No restriction is necessary if we restrict our study to the polytopes $\Pi_X^1(b)$.

Definition 1.1.6. We denote by $V_X(b)$, $V_X^1(b)$ the volumes of $\Pi_X(b)$, $\Pi_X^1(b)$ respectively.

Suppose furthermore, that the vectors a_i and b happen to be integral (and 0 is not in their convex hull). One has a third function, now on $\mathbb{Z}^s$, important for combinatorics, the *partition function* given by:

$$\mathcal{P}_X(v) := \#\{(n_1, \ldots, n_N) \mid \sum n_i a_i = v, \ n_i \in \mathbb{N}\}, \tag{1.1}$$

$$\mathcal{P}_X^1(v) := \#\{(n_1, \ldots, n_N) \mid \sum n_i a_i = v, \ n_i \in [0, 1]\}. \tag{1.2}$$

Thus $P_X(b)$, $P_X^1(b)$ count the number of integer points in the polytopes $\Pi_X(b)$, $\Pi_X^1(b)$.

Alternatively, $P_X(b)$ counts the number of ways in which we can decompose $b = t_1 a_1 + \cdots + t_m a_m$ with t_i not negative integers hence the name *partition function.*

A basic question that we shall treat is to compute explicitly all the previous functions and describe some of their qualitative features.

The method we shall follow can be summarized as follows:

- We interpret all the functions as *tempered distributions supported in the pointed cone* $C(X)$
- We apply Laplace transform and change the problem to one in algebra
- We solve the algebraic problems by module theory
- We interpret the results inverting Laplace transform

1.2 Splines

1.2.1 Ramadas 1. Tempered distributions; Fourier-(Laplace) transform.

Tempered distributions were introduced by Laurent Schwartz as the ideal setting for the Fourier transform.

We first recall some elementary linear algebra. Let $\mathcal{S}$ be a real (respectively, complex) topological vector space, $\mathcal{S}'$ the vector space of continuous "linear functionals" on $\mathcal{S}$. A "linear functiona" is a linear map from $\mathcal{S}$ to $\mathbb{R}$ (resp., $\mathbb{C}$); from now on we only consider continuous ones. Given a vector $f \in \mathcal{S}$ and a linear functional $T \in \mathcal{S}'$, we denote by $< T|f >$ the value of T on f. The space $\mathcal{S}'$ is called the *dual* space.

Let now $\mathcal{S}_1,\ \mathcal{S}_2$ be two vector spaces and $\mathcal{S}_1',\ \mathcal{S}_2'$ the respective duals. If $F : \mathcal{S}_1 \to \mathcal{S}_2$ is a continuous linear map, the *transpose* map $F^{tr} : \mathcal{S}_2' \to \mathcal{S}_1'$ is defined by

$$< F^{tr}(T_2)|f_1 >=< T_2|F(f_1) > \ \ T_2 \in \mathcal{S}_2', \quad f_1 \in \mathcal{S}_1$$

We say that F^{tr} is "defined by duality". (Note that in the case of *finite-dimensional* vector spaces, all linear maps are continuous.)

Consider now an $s-$dimensional vector space V with coordinates $x_1, \dots, x_s$, so we can identify V with $\mathbb{R}^s$. A "multiindex" α is a sequence $(h_1, \dots, h_s)$ of non-negative integers. One sets

$$x^\alpha := x_1^{h_1} \dots x_s^{h_s}, \quad \partial^\alpha := \frac{\partial^{h_1}}{\partial x_1} \dots \frac{\partial^{h_s}}{\partial x_s}$$

We let dx denote the Lebesgue measure $dx_1 dx_2 \dots dx_s$. We denote by U the vector space dual to V, with co-ordinates $y_1, \dots y_s$. We let $< y|x >$ denote the value of the linear functional y on x; the co-ordinates y_i are defined such that

$$< y|x >= \sum_i y_i x_i$$

Definition 1.2.2. The *Schwartz space* $\mathcal{S} = \mathcal{S}(V)$ *of "rapidly decreasing smooth functions" is the space of* C^∞ *functions* f *on* V*, such that, for every pair of multiindices* α, β *one has that* $x^\alpha \partial^\beta f$ *is a bounded function.*

It then easily follows that $|x^\alpha \partial^\beta f|$ takes its maximal value; this will be denoted by $|f|_{\alpha,\beta}$.

This gives a family of seminorms on $\mathcal{S}$ which induces thus a topology on $\mathcal{S}$, which then becomes a topological vector space, in fact a Fréchet space (and therefore metrisable) but *not* a Banach space.

Definition 1.2.3. A *tempered distribution is a linear functional on* $\mathcal{S}$ *continuous under all these seminorms.*

We let $\mathcal{S}' = \mathcal{S}'(V)$ the space of tempered distributions.

Remark 1.2.4. We have used coordinates to fix explicitly the seminorms, but one could have given a coordinate-free definition using any monomial in linear functions ϕ_i and derivatives D_{v_i}.

Note that there is an inclusion $\mathcal{S} \to \mathcal{S}'$, which we denote $g \mapsto T_g$, where

$$< T_g | f >= \int g(x) f(x) dx$$

In fact $\mathcal{S}$ is contained in the usual L^p spaces, and the above formula works for these spaces as well, so that we have dense inclusions

$$\mathcal{S} \hookrightarrow L^p \hookrightarrow \mathcal{S}'$$

By its very definition the Schwartz space is a module over the algebra $W(V)$ of differential operators with polynomial coefficients, and so is (by duality) the space of tempered distributions.:

$$< x^\alpha \partial^\beta T | f >:= (-1)^{|\beta|} < T | \partial^\beta x^\alpha f >$$

Note that we have for any $D \in \mathcal{S}$,

$$DT_g = T_{Dg}, g \in \mathcal{S}$$

That is, the inclusion $\mathcal{S} \to \mathcal{S}'$ given by $g \mapsto T_g$ is a map of Weyl modules.

The usual Fourier transform of a rapidly decreasing smooth function f is defined to be the function $\hat{f}$:

$$\hat{f}(y) := (2\pi)^{-s/2} \int_V e^{-i\langle y \,|\, x \rangle} f(x) dx. \tag{1.3}$$

A basic fact of Fourier analysis (cf. [62]), is that the Fourier transform is an continuous isomorphism of $\mathcal{S}$ to itself (as topological vector space) and thus it induces, by duality a Fourier transform on the space of tempered distributions: if T is a tempered distribution we define $\hat{T}$ by:

$$\langle \hat{T} \,|\, f \rangle := \langle T \,|\, \hat{f} \rangle \ .$$

In other words the Fourier transform on $\mathcal{S}'$ is the transpose of the Fourier transform on $\mathcal{S}$. The fact that $T_{\hat{g}} = \hat{T}_g$ follows from the identity, valid for $f, g \in \mathcal{S}$:

$$\int \hat{f} g dx = \int f \hat{g} dx$$

Thus $T \mapsto \hat{T}$ is an isomorphism of $\mathcal{S}'$ to itself which extends the Fourier transform on $\mathcal{S}$.

How does the Fourier transform interact with the Weyl algebra? Differentiating "under the integral sign" in the formula (1.3), which we can freely do since f is rapidly decreasing (prove this, using dominated convergence), we find

$$\frac{\partial}{\partial y_j} \hat{f}(y) = \widehat{-i x_j f}$$

And a simple integration by parts shows

$$\widehat{i \frac{\partial}{\partial x_j} f} = i y_j \hat{f}(y)$$

More generally

$$\begin{aligned}\partial_y^\alpha \hat{f}(y) &= \widehat{(-ix)^\alpha f} \\ \widehat{\partial_x^\alpha f}(y) &= (iy)^\alpha \hat{f}(y)\end{aligned} \tag{1.4}$$

Though the definition of Fourier transform of a tempered distribution is very elegant, it is rather indirect. There are special cases where a more direct formula exists. For example, as we noted above, if the distribution is of the form T_g, with $g \in \mathcal{S}$, the Fourier transform can be computed by an integral. More generally, this works if the distribution is given by a (finite) measure. *We will use this fact below without further comment.* Further,

- If g is in L^1, the formula (1.3) defines a continuous function $\hat{g}$ (vanishing at infinity) which in turn defines a tempered distribution $T_{\hat{g}}$ such that $\hat{T}_g = T_{\hat{g}}$. (Riemann-Lebesgue)
- If g is in L^2, there is a $\hat{g} \in L^2$ such that $\hat{T}_g = T_{\hat{g}}$, and $g \mapsto \hat{g}$ is an isometry of L^2. (Parseval.)

Recall the definition of *support* of a continuous function f:

$$Supp(f) = \overline{\{x | f(x) \neq 0\}}$$

where the bar denotes closure. An equivalent definition is

$$Supp(f) = (\cup W)^c$$

where the union is taken over all open sets W such that $f|_W = 0$, and the superscript ^{c} denotes the complement. This second definition extends to distributions: we define the support of a distribution T by

$$Supp(T) = (\cup W)^c$$

where the union is taken over all open sets W such that $T|_W = 0$. The last condition means that $< T|f >= 0$ for all Schwartz functions with support contained in W. If T has compact support, one can define the product with an *arbitrary* smooth function g by the formula.

$$< gT|f > := < T|gf >$$

Support properties of a distribution are reflected in analyticity properties of its Fourier transform. For example, if T has compact support, $\hat{T}$ is a smooth function which is in fact the restriction to $\mathbb{R}^s$ of an entire function on $\mathbb{C}^s$; the famous Payley-Wiener theorem makes this more precise and contains a converse statement. The entire function $\hat{T}$ is defined at $y = \zeta + i\eta$ by

$$\hat{T}(\zeta + i\eta) = \widehat{e^{<\eta|.>}T}(\zeta)$$

where the notation $e^{<\eta|.>}T$ refers to the distribution got by multiplying T by the smooth function $x \mapsto e^{<\eta|.x>}$.

To introduce the Laplace transform, which we shall use extensively in the course, we first consider the case when $s = 1$, and consider a tempered distribution T with support in the set of non-negative reals. In this case, $\hat{T}$ is not necessarily smooth, but there is a function defined and analytic on $y = \zeta + i\eta$ for $\eta < 0$, such that its "boundary value", as $\eta \to 0$, is equal to the tempered distribution $\hat{T}$. We continue to denote this function by $\hat{T}$, and call it the Fourier-Laplace transform. The Laplace transform itself we define by the formula

$$Lf(y) = (2\pi)^{s/2}\hat{f}(-iy) \ . \tag{1.5}$$

Clearly Lf is defined and analytic when its argument has real part negative. In particular the restriction of Lf to the positive real axis makes sense, and this is the classical Laplace transform.

What is the higher-dimensional analogue of this? We consider a tempered distribution supported in a pointed cone $C \subset V$. Recall that the dual cone $\hat{C}$ is defined by

$$\hat{C} := \{u \in U | < u|v > \geq 0\}$$

and that 'pointed' is equivalent to the condition that $\hat{C}$ has a non-empty interior. Then the basic result is that

Theorem 1.2.5. *If T is a tempered distribution supported in C, the Fourier transform extends to an analytic function of $z = \zeta - i\eta$ where $\eta \in \hat{C}^0$ (that is, for η belonging to the interior of $\hat{C}$).*

(Given a function $f(v) \in \mathcal{S}$ suppose that there is an open set $A \subset U$ such that, for all $z \in A$ we have that $e^{-\langle z\,|\,v\rangle} f(v) \in \mathcal{S}$ then the Fourier transform extends to the open set $U + iA$ of complex vectors.)

The analytically extended functions are sometimes called 'Fourier-Laplace transforms'. We shall simply refer to them as Laplace transform, except that as in the one-variable case, we make a small change of notation and normalisation factor to avoid cluttering the notation with i's.

1.2.6 The Laplace Transform

We make the formal definition in this separate section, for ease of reference. In particular, we make the definition co-ordinate free. Let V be a finite dimensional real vector space, U the dual. Fix a Euclidean structure on V which induces Lebesgue measures dv, du on V, U and all their linear subspaces. (The inner product also identifies V and U, but we often choose not to use this identification. If we choose co-ordinates x_i on V defined with respect to an orthonormal basis and y_i on U w.r.to the dual basis, we can identify both U and V with $\mathbb{R}^s$ as before.)

We set

$$Lf(u) := \int_V e^{-\langle u\,|\,v\rangle} f(v) dv. \tag{1.6}$$

on functions (and measures) whenever the formula makes sense. On distributions, we define the transform by duality. (When in doubt refer back to the Fourier transform.)

The transform L maps functions or tempered distributions with support in $C \subset V$ *injectively* to functions on $\hat{C}^0 + iU$.

For $p \in U$, $w \in V$, let p also denote the linear function (on V) defined by $v \mapsto \langle p \,|\, v\rangle$, and let D_w denote the directional derivative (of functions on V). Similarly, w denotes a linear function on U and p a direcytional derivative on U. With this notation, we have

$$L(D_w f)(u) = wLf(u),\ L(pf)(u) = -D_p Lf(u), \tag{1.7}$$

$$L(e^p f)(u) = Lf(u-p),\ L(f(v+w))(u) = e^w Lf(u). \tag{1.8}$$

We close this section with some very important observations.

1) The Laplace transform maps δ_0 to the function 1. Any distribution supported at the origin is a (finite) derivative of the delta function; so the above formulas show that

$$\{distributions\ supported\ at\ 0 \in V\} \stackrel{Laplace}{\leftrightarrow} \{Polynomial\ functtions\ on\ U\}$$

2) As the above example shows, the Laplace transform of a distribution supported on C can extend beyond $\hat{C}+iU$; in the examples which we encounter they will extend to rational functions (quotients of polynomials) with poles along affine hyperplanes.

1.2.7 Fanja 1. The cone and the box

Let us start with a definition.

Definition 1.2.8. We set:

$$C(X) := \{\sum_{a\in X} t_a a \,|\, 0 \le t_a,\ \forall a\}, \quad B(X) := \{\sum_{i=1}^N t_i a_i,\ 0 \le t_i \le 1\} \tag{1.9}$$

$C(X)$ is the cone of linear combinations of vectors in X with positive coefficients.

$B(X)$, the *shadow* of the cube $[0,1]^N$, is also called the *box* generated by X.

Recall that, in polyhedral geometry, one defines the Minkowsky sum of two polytopes A, B to be $A + B := \{a + b,\ a \in A,\ b \in B\}$.

Thus $B(X)$ is the Minkowsky sum of the intervals $[0, a_i]$. It is a special type of polytope called a *zonotope*, we shall go back to the geometry of $B(X)$ in section **??**.

If we assume that 0 is not in the convex hull of the vectors in X, we have that $C(X)$ does not contain lines and is contained entirely in some half space, it is thus a *pointed cone*. In fact there is a linear function ϕ with $\langle \phi \,|\, v\rangle > 0,\ \forall v \in C(X),\ v \neq 0$.

In general we define the set $\hat{C}(X) := \{\phi \in V^* \,|\, \langle \phi \,|\, a\rangle \ge 0,\ \forall a \in X\}$.

Proposition 1.2.9. *$\hat{C}(X)$ is also a convex cone generated by finitely many elements $\phi_1, \dots, \phi_M$. It is called the* **dual cone***.*

$\hat{C}(X)$ is a pointed cone provided that X spans V.

$\hat{\hat{C}}(X) = C(X)$.

insert proof

Remark 1.2.10. The previous duality tells us in particular that, a cone $C(X)$ defined as the positive linear combinations of finitely many vectors, can also be described as the set defined by finitely many linear inequalities.

Such a finitely generated cone is also called a polyhedral cone. These cones are connected with *piecewise linear* functions of the following type.

Take N linear functions $a_i(x), i = 1, \dots, N$ on V and set $s(x) := \max a_i(x)$. Then V decomposes in the (possibly overlapping) pieces $V_i := \{x \in V \mid s(x) = a_i(x)\}$. Of course V_i is defined by the inequalities $a_i(x) - a_j(x) \geq 0$, $\forall j \neq i$ and it is thus a polyhedral cone. We have therefore decomposed V into polyhedral cones on each of which $s(x)$ coincides with one of the linear functions $a_i(x)$.

Definition 1.2.11. A function f is piecewise linear with respect to a decomposition of cones C_i if it is linear on each C_i.

1.3 Day 2

1.3.1 Claudio 2. The box spline

Take a finite list $X := \{a_1, \dots, a_N\}$ of non zero vectors $a_i \in \mathbb{R}^s$, thought of as the columns of a matrix A. If X spans $\mathbb{R}^s$, one builds an important function for numerical analysis, the *box spline* $B_X(x)$ implicitly defined by the formula:

$$\int_{\mathbb{R}^s} f(x) B_X(x) dx := \int_0^1 \dots \int_0^1 f(\sum_{i=1}^N t_i a_i) dt_1 \dots dt_N, \tag{1.10}$$

where $f(x)$ is any continuous function.

If 0 is not in the convex hull of the vectors a_i then one has a simpler function $T_X(x)$, the *multivariate spline* (cf. [32]) characterized by the formula:

$$\int_{\mathbb{R}^s} f(x) T_X(x) dx = \int_0^\infty \dots \int_0^\infty f(\sum_{i=1}^N t_i a_i) dt_1 \dots dt_N, \tag{1.11}$$

where $f(x)$ has compact support or more generally it is exponentially decreasing on the cone $C(X)$.

If X does not span, we have to consider T_X and B_X as meausers on the subspace spanned by X. In fact it is best to consider them as tempered distributions as we shall see presently.

Using this more general point of view one proves that T_X and B_X are indeed functions as soon as X spans V, i.e. when the support of the distribution has maximal dimension.

Both B_X and T_X have a simple geometric interpretation as functions computing the volume of the variable polytopes introduced in the previous paragraph 1.1.5. We assume that X spans V.

Theorem 1.3.2. $V_X(x) = \sqrt{\det(AA^t)}\, T_X(x),\ V_X^1(x) = \sqrt{\det(AA^t)}\, B_X(x).$

Proof. Let us give the proof in the case of T_X, the other case being similar. Start by observing that, multiplying A by a suitable orthogonal matrix U we can obtain $A = (B, 0)U$ with B an $s \times s$ invertible matrix.

Think of the map $p_X : (t_1, \dots, t_N) \mapsto \sum_{i=1}^N t_i a_i$ as a composition

$$\mathbb{R}^N \xrightarrow{U} \mathbb{R}^N \xrightarrow{\pi} \mathbb{R}^s \xrightarrow{B} \mathbb{R}^s$$

π the projection of matrix $(1_s, 0)$.

Let us now apply Fubini's theorem to the function $\chi_+ f(p_X(t_1, \dots, t_N))$ where χ_+ denotes the characteristic function of the quadrant $\mathbb{R}_+^N$.

The right hand side of (1.13) equals $\int_{\mathbb{R}^N} \chi_+ f(p_X(t_1, \dots, t_N))dt$ which, by the orthogonality of U equals $\int_{\mathbb{R}^N} \chi_+ f(B\pi(t_1, \dots, t_N))dt = \int_{\mathbb{R}^s} V_X(Bx)f(Bx)dx$ by Fubini's theorem.

Since for any function g we have $\int_{\mathbb{R}^s} g(Bx)dx = |\det(B)|^{-1} \int_{\mathbb{R}^s} g(x)dx$:

$$\int_{\mathbb{R}^s} f(x)T_X(x)dx = \int_{\mathbb{R}_+^N} f(\sum_{i=1}^N t_i a_i)dt = |\det(B)|^{-1} \int_{\mathbb{R}^s} V_X(x)f(x)dx.$$

Finally $AA^t = (AU^{-1})(AU^{-1})^t = BB^t$, hence $\sqrt{\det(AA^t)} = |\det(B)|$ and the final formula follows. □

Remark 1.3.3. let M be an invertible $s \times s$ matrix, then:

$$\int_{\mathbb{R}^s} f(x)T_{MX}(x)dx = \int_{\mathbb{R}_+^N} f(\sum_{i=1}^N t_i M a_i)dt = \int_{\mathbb{R}^s} f(Mx)T_X(x)dx =$$

$$|\det(M)|^{-1} \int_{\mathbb{R}^s} f(x)T_X(M^{-1}x)dx \implies T_{MX}(x) = |\det(M)|^{-1}T_X(M^{-1}x).$$

1.3.4 Ramadas 2. Laplace transforms of splines.

Recall from Claudio's lecture the definition of Box spline and multivariate splines.

Take a finite list $X := \{a_1, \dots, a_N\}$ of non zero vectors $a_i \in \mathbb{R}^s$, which we sometimes think of as the columns of a $s \times N$ matrix A. For simplicity, we usually assume that

- X spans $\mathbb{R}^s$, and
- $C(X)$ is "pointed" (that is, the origin is not in the convex hull of the vectors a_i).

We let $C(X)$ denote the cone generated by X and $B(X)$ the "box" generated by X. Explicitly,

$$C(X) = \{t_1 a_1 + \dots t_s a_s | t_i \; real \; t_i \geq 0\}$$

and

$$B(X) = \{t_1 a_1 + \dots t_s a_s | 1 \geq t_i \; real \; t_i \geq 0\}$$

We then define two real-valued and non-negative functions

1. the box spline B_X, with support $B(X)$, and
2. the multivariate spline T_X, with support the cone $C(X)$.

To do this we first consider the map $p_X : \mathbb{R}^N \to \mathbb{R}^s$:

$$p_X(t_1, \dots, t_N) = t_1 a_1 + \dots + t_N a_N$$

(p_X is given by multiplying by the matrix A.) The function B_X is implicitly defined by

$$\int_{\mathbb{R}^s} f(x) B_X(x) dx := \int_{cube} f \circ p_X(t) dt \tag{1.12}$$

for any continuous function f. The function T_X is implicitly defined by

$$\int_{\mathbb{R}^s} f(x) T_X(x) dx = \int_{quadrant} f \circ p_X(t) dt \tag{1.13}$$

where $f(x)$ has compact support or more generally, is exponentially decreasing on the cone $C(X)$. By "cube" we mean the domain $\{t| = (t_1 \dots, t_N)|1 \geq t_i \geq 0\}$, and by "'quadrant" the domain $\{t| = (t_1 \dots, t_N)|t_i \geq 0\}$

<u>Basic Result:</u> There exist unique functions B_X and T_X satisfying (1.12) and (1.13) respectively provided we demand in addition that

1. B_X(respectively, T_X) is continuous on $B(X)$ (respectively, $C(X)$), and
2. $B_X = 0$ (respectively, $T_X = 0$) on $B(X)^c$ (respectively, $C(X)^c$).

Finally, we recall the basic example. When $N = s$ (so that the vectors a_i form a basis),

$$\begin{aligned} B_X(x) &= (det \; p_X)^{-1} \chi_{B(X)} \\ T_X(x) &= (det \; p_X)^{-1} \chi_{C(X)} \end{aligned} \tag{1.14}$$

where if Y is a subset of $\mathbb{R}^s$, we define χ_Y to be the "characteristic function" of Y:

$$\begin{aligned} \chi_T(x) &= 1 \; if \; x \in Y \\ &= 0 \; if \; x \notin Y \end{aligned} \tag{1.15}$$

The Laplace transform reduces the computation of the spline functions to algebra. Consider first the box spline. Since it has compact support, its Fourier transform should be an entire function, and indeed we find

$$\begin{aligned} \hat{B_X}(y) &= \frac{1}{(2\pi)^s} \int e^{-i<y|x>} B_X(x)dx \\ &= \frac{1}{(2\pi)^s} \int_{cube} e^{-i\sum_i t_i <y|a_i>} dt \\ &= \frac{1}{(2\pi)^s} \prod_i \frac{e^{-i<y|a_i>} - 1}{-i < y|a_i >} \end{aligned} \tag{1.16}$$

which is clearly an entire function of y. This yields, for the Laplace transform, the formula

$$L(B_X)(y) = \prod_{a\in X} \frac{1-e^{-a}}{a}$$

where a vector a is identified with the linear function $x \mapsto < a|x >$.

From now on, we work directly with the formula for the Laplace transform. Turning now to the multivariate spline, we find for its Laplace transform

$$\begin{aligned} L(T_X)(y) &= \int e^{-<u|x>} T_X(x)dx \\ &= \int_{quadrant} e^{-\sum_i t_i <y|a_i>} dt \\ &= \prod_i \frac{-1}{< y|a_i >} \\ &= \prod_{a\in X} \frac{1}{a} \equiv \frac{1}{d_A} \end{aligned} \tag{1.17}$$

The expressions for $L(B_X)$ and $L(T_X)$ are valid *a priori* for $Re(y) \in C(\hat{X})^0$, the interior of the dual cone, but in fact they clearly extend to define meromorphic functions with poles along the hyperplanes $a_i = 0$.

There is a simple combinatorial formula that relates the box spline and the multivariate spline:

Proposition 1.3.5. *For $a \in V$, set $\nabla_a f(x) := f(x) - f(x-a)$; and for every subset $S \subset X$ set $a_S := \sum_{a\in S} a$. Then*

$$B_X(x) = \prod_{a\in X} \nabla_a T_X(x) = \sum_{S\subset X} (-1)^{|S|} T_X(x - a_S) \tag{1.18}$$

where the equaliity holds possibly outside a set of measure zero, in particular in the interiors of the big cells in $B(X)$.

Proof. We give two proofs, to get used to the concepts.

First consider the difference $B_X(x) - \prod_{a\in X}\nabla_a T_X(x)$, and take its Laplace transform. Note that the relation $L(f(x-a)) = e^{-a}L(f)$ yields $L(\nabla_a f) = (1-e^{-a})L(f)$, and iteratively, $L(\prod_{a\in X}\nabla_a f) = \prod_{a\in X}(1-e^{-a})L(f)$. So

$$L(B_X(x) - \prod_{a\in X}\nabla_a T_X(x)) = 0$$

which means that $B_X(x) - \prod_{a\in X}\nabla_a T_X(x)$ must be zero as a tempered distribution, and therefore zero except possibly at the points of discontinuity.

As for the second proof, consider the characteristic functions (in $\mathbb{R}^N$) of the quadrant and the cube. One checks easily that

$$\chi_{cube} = \prod_{i=1,\dots,N}\nabla_i\chi_{quadrant}$$

and now the result follows from definitions. (exercise!) □

1.3.6 $E-$splines

It is useful to generalize these notions, introducing a *parametric version* called $E-$splines.

Fix parameters $\underline{\mu} := \{\mu_1,\dots,\mu_N\}$ and define the functions (or tempered distributions) on V by the implicit formulas:

$$\int_V f(x)B_{X,\underline{\mu}}(x)dx := \int_0^1\dots\int_0^1 e^{-\sum_{i=1}^N t_i\mu_i} f(\sum_{i=1}^N t_i a_i)dt_1\dots dt_N \tag{1.19}$$

$$\int_V f(x)T_{X,\underline{\mu}}(x)dx = \int_0^\infty\dots\int_0^\infty e^{-\sum_{i=1}^N t_i\mu_i} f(\sum_{i=1}^N t_i a_i)dt_1\dots dt_N. \tag{1.20}$$

The same proof as in Theorem 1.3.2, shows that these functions have a nice geometric interpretation as integrals on the polytopes $\Pi_X^0(x) = p_X^{-1}(w)\cap[0,1]^N$ and $\Pi_X(x) = p_X^{-1}(w)\cap[0,\infty]^N$

$$B_{X,\underline{\mu}}(x) = \int_{\Pi_X^0(x)} e^{-\sum_{i=1}^N t_i\mu_i}dz;\quad T_{X,\underline{\mu}}(x) = \int_{\Pi_X(x)} e^{-\sum_{i=1}^N t_i\mu_i}dz \tag{1.21}$$

Where the measure dz is induced from the standard Lebesgue measure on $\mathbb{R}^N$ multiplied by the normalization constant $\sqrt{\det(AA^t)}^{-1}$. Of course for $\underline{\mu} = 0$ we recover the previous definitions.

We come now to the main formula which will allow us to change the computation of these functions into a problem of algebra. An easy computation gives their Laplace transforms:

Theorem 1.3.7.

$$\int_V e^{-\langle x,y\rangle} B_{X,\underline{\mu}}(x)dy = \int_0^1 \dots \int_0^1 e^{-\sum_{i=1}^N t_i(\langle x,a_i\rangle+\mu_i)} dt_1 \dots dt_N \tag{1.22}$$
$$= \prod_{a\in X} \frac{1-e^{-a-\mu_a}}{a+\mu_a}.$$

and

$$\int_V e^{-\langle x,y\rangle} T_{X,\underline{\mu}}(x)dy = \int_0^\infty \dots \int_0^\infty e^{-\sum_{i=1}^N t_i(\langle x,a_i\rangle+\mu_i)} dt_1 \dots dt_N \tag{1.23}$$
$$= \prod_{a\in X} \frac{1}{a+\mu_a}.$$

We have written shortly $a := \langle x, a\rangle$, for the linear function on U. Most of our work will deal with effective methods to invert these Laplace transform.

In the course of this book we give an idea of the general algebraic calculus involving these functions. Under Laplace transform one can reinterpret the calculus in terms of the structure of certain algebras of rational functions (or exponentials) as $D-$modules.

Given $a \in V,\ \mu \in \mathbb{C}$ let us introduce the notation, which will be discussed more deeply in **Part 2**:

$$\nabla_a^\mu f(x) := f(x) - e^{-\mu} f(x-a).$$

From the expressions of the Laplace transforms one gets that, the box spline can be obtained from the multivariate spline by a simple combinatorial formula.

Proposition 1.3.8. *For every subset $S \subset X$ we set $a_S := \sum_{a\in S} a$, and $\mu_S := \sum_{a\in S} \mu_a$ then:*

$$B_{X,\underline{\mu}}(x) = \prod_{a\in X} \nabla_a^{\mu_a} T_{X,\underline{\mu}}(x) = \sum_{S\subset X} (-1)^{|S|} e^{-\mu_S} T_{X,\underline{\mu}}(x-a_S). \tag{1.24}$$

Proof. It follows from the basic rule (1.8) which gives the commutation relation between the Laplace transform and translations. □

1.3.9 Fanja 2. Faces of polytopes, examples

1.3.10 Demissu 1.

1.3.11 Algebraic Fourier transform

In this chapter by vector space we shall mean a finite dimensional vector space over the field of real numbers $\mathbb{R}$. It is convenient to take an intrinsic and base free approach to our problems.

Let us fix an $s-$dimensional vector space U, let us denote by $V := U^*$ its dual. We identify the symmetric algebra $S[V]$ with the ring of polynomial functions on U and sometimes denote it by A.

This algebra can also be viewed as the algebra of polynomial differential operators with constant coefficients on V.

Indeed, given a vector $v \in V$ we denote by D_v the corresponding directional derivative defined by

$$D_v f(x) := \frac{df(x+tv)}{dt}_{t=0}.$$

Remark that, if $\phi \in U$ is a linear function we have $\langle \phi \,|\, x + tv \rangle = \langle \phi \,|\, x \rangle + t \langle \phi \,|\, v \rangle$. Thus D_v is algebraically characterized, on $S[U]$, as the derivation which on each element $\phi \in U$, takes the value $\langle \phi \,|\, v \rangle$. It follows that $S[V]$ can be thus identified with the ring of differential operators which can be expressed as polynomials in the D_v.

Similarly $S[U]$ is the ring of polynomial functions on V, or polynomial differential operators with constant coefficients on U.

It is often very convenient to work over the complex numbers and use the complexified spaces:

$$V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}, \quad U_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C} = \hom(V, \mathbb{C}).$$

One can organize all these facts in the algebraic language of Fourier transform. Let $W(V), W(U)$ denote the two algebras of differential operators with complex polynomial coefficients on V and U respectively. Notice that, from a purely algebraic point of view they are both generated by $V \oplus U$.

In the first case V is thought of as the space of directional derivatives and then we write D_v instead of v, and U as the linear functions. In $W(U)$ the two roles are exchanged.

The relevant commutation relations are thus:

$$[D_v, \phi] = \langle \phi \,|\, v \rangle, \qquad [D_\phi, v] = \langle v | \phi \rangle.$$

Thus we see that we have a canonical isomorphism of algebras:

$$\mathcal{F} : W(V) \to W(U), \quad D_v \mapsto -v, \quad \phi \mapsto D_\phi.$$

One usually writes $\hat{a}$ instead of $\mathcal{F}(a)$.

This allows us, given a module M over $W(V)$, to consider its *Fourier transform* $\hat{M}$ as a module over $W(U)$ by $a.m := \hat{a}m$ and conversely.

In explicit coordinates $x_1, \ldots, x_s$ for V (the x_i are a basis of the dual space U), the dual basis in V is given by $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_s}$ and:

$$W(V) = W(s) := \mathbb{R}[x_1, \ldots, x_s; \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_s}].$$

The automorphism $\mathcal{F}$ is then:

$$\mathcal{F}: \; x_i \mapsto \frac{\partial}{\partial x_i}, \quad \frac{\partial}{\partial x_i} \mapsto -x_i.$$

The algebra $W(s)$ is also sometimes called the Weyl algebra. The algebra $W(s)$ is not commutative; its generators satisfy the canonical commutation relations (of quantum mechanics):

$$[x_i, \frac{\partial}{\partial x_i}] = x_i \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_i} x_i = -1, \quad [x_i, \frac{\partial}{\partial x_j}] = 0, \; \forall i \neq j, \tag{1.25}$$

$$[x_i, x_j] = [\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0, \; \forall i, j.$$

It is easy to see that each element of $W(n)$ can be written in a unique manner as a linear combination of elements:

$$x_1^{h_1} \ldots x_s^{h_s} \frac{\partial^{k_1}}{\partial x_1} \cdots \frac{\partial^{k_s}}{\partial x_s}, \quad h_i, k_j \in \mathbb{N}.$$

Remark: We use the notion of the *commutator* of two elements $[a, b] := ab - ba$ in an arbitrary algebra.

We will use some simple properties of the commutator:

Lemma 1.3.12.

$$[a, bc] = [a, b]c + b[a, c], \quad \textit{Leibnitz rule}$$

If $[a, b] = 1$ *then* $[a, b^n] = nb^{n-1}, \; \forall n \geq 1$.

Proof $[a, b]c + b[a, c] = (ab - ba)c + b(ac - ca) = abc - bca = [a, bc]$.

For the second identity, the case $n = 1$ is true by hypothesis, then we use induction and the Leibnitz rule:

$$[a, b^n] = [a, b]b^{n-1} + b[a, b^{n-1}] = b^{n-1} + b(n-1)b^{n-2} = nb^{n-1}.$$

Let us apply this lemma to $a = \frac{\partial}{\partial x_i}, \; b = x_i$, and then again to $a = -x_i, \; b = \frac{\partial}{\partial x_i}$, obtaining:

$$[\frac{\partial}{\partial x_i}, x_i^n] = nx_i^{n-1}, \; [x_i, \frac{\partial}{\partial x_i}^n] = -n \frac{\partial}{\partial x_i}^{n-1}.$$

Remark 1.3.13. It is not necessary to consider all variables and derivatives. Below, it shall be useful to consider other orderings of variables and derivatives, and even variables and derivatives alternatively. Using the commutation rules, these are bases obtained from each other by triangular changes of basis.

1.3.14 Claudio 3 Recursive expressions

It is costumary to compute both T_X and B_X recursively as follows. Consider $Y := \{a_1, \dots, a_{N-1}\}$ and set $z := a_N$ so that $X = \{Y, z\}$.

Proposition 1.3.15. *1. T_X and B_X are tempered distributions.*

2. *B_X is supported in the box $B(X)$ and T_X is supported in $C(X)$.*

3. *In both cases we can identify T_X and B_X to functions which are continuous on their support and given by*
$$T_X(x) = \int_0^\infty T_Y(x - tz)dt, \quad B_X(x) = \int_0^1 B_Y(x - tz)dt \tag{1.26}$$

4. *The cone $C(X)$ can be decomposed into the union of finitely many polyhedral cones C_i so that, T_X restricted to each C_i is a homogeneous polynomial of degree $N - s$, $s := \dim\langle X\rangle$.*

5. *The box $B(X)$ can be decomposed into the union of finitely many bounded convex polytopes B_i so that, B_X restricted to each B_i is a polynomial of degree $N - s$.*

Proof. Since the restriction of a function in Schwartz space to a subspace is a continuous map to the corresponding space of Schwartz functions we may assume that X spans V.

Then it is easy to see that a piecewise polynomial function determines a tempered distribution. Thus parts 1. and 2. follow clearly from 3, 4, 5.

Let us give the proof of 3 and 4 for T_X. In the case B_X 3 follows in an analogous way while 5 will be an immediate consequence of the result for T_X and Proposition 1.24.

Let us assume the statement by induction for Y. Let $W = \langle Y\rangle$ be the linear span of Y. By induction we have:

$$\int_{\mathbb{R}_+^N} f(\sum_{i=1}^N t_i a_i)dt_1 \dots dt_N = \int_0^\infty (\int_W T_Y(x)f(x + tz)dx)dt.$$

We distinguish two cases. If $z \notin W$ we have $W' = \langle X\rangle = W \oplus \mathbb{R}z$. Consider the map $i : W \oplus \mathbb{R} \to W'$ given by $(w, t) \mapsto w + tz$, we have then that the push forward of the measure $dxdt$ on $W \oplus \mathbb{R}$ is the Lebesgue measure of W' divided by the absolute value d of the normal component of z to W.

Thus
$$T_X(w + tz) = \begin{cases} 0 & \text{if } t < 0 \\ d^{-1}T_Y(w) & \text{if } t \geq 0 \end{cases}.$$

T_X corresponds under i to the product of T_Y times the characteristic function of $\mathbb{R}^+$ divided by d. In this case $C(X)$ corresponds under i to $C(Y) \times \mathbb{R}^+$ and T_X is clearly continuous on $C(X)$.

In this case 4 follows immediately by induction.

Let us now analyze the case $z \in W = W'$. In this case we have

$$\int_0^\infty (\int_W T_Y(x) f(x+tz)dx)dt = \int_0^\infty (\int_W T_Y(x-tz) f(x)dx)dt.$$

We claim that $\int_0^\infty T_Y(x-tz)dt$ is well defined and continuous on $C(X)$. This will allow us to identify it with $T_X(x)$.

We need to remark that, for each x the set $I_x := \{t \in \mathbb{R}^+ \,|\, x - tz \in C(Y)\}$ is empty unless $x \in C(X)$; if $x \in C(X)$, $I_x = [a(x), b(x)]$ is an interval, and both $a(x), b(x)$ are continuous functions on $C(X)$.

Let $C(Y) := \{v \,|\, \langle \phi_i \,|\, v\rangle \geq 0\}$ for some elements $\phi_1, \dots, \phi_M \in V^*$, we have that $I_x = \{t \geq 0 \,|\, \langle \phi_i \,|\, x\rangle \geq t\langle \phi_i \, z\rangle, \forall i\}$.

These inequalities can be written as:

$$t \geq \langle \phi_i \,|\, x\rangle \langle \phi_i \,|\, z\rangle^{-1}, \ \forall i \,|\, \langle \phi_i \,|\, z\rangle < 0$$
$$t \leq \langle \phi_i \,|\, x\rangle \langle \phi_i \,|\, z\rangle^{-1}, \ \forall i \,|\, \langle \phi_i \,|\, z\rangle > 0.$$

When $\langle \phi_i \,|\, z\rangle = 0$ we have either that every t or no t satisfies the inequality. Since $C(X)$ is a pointed cone we there is at least one i such that $\langle \phi_i \,|\, z\rangle > 0$ otherwise $-z \in C(Y)$.

This proves that I_x is a bounded closed interval.

Set

$$a(x) := \max\{0, \langle \phi_i \,|\, x\rangle \langle \phi_i \,|\, z\rangle^{-1}\}, \ \forall i \,|\, \langle \phi_i \,|\, z\rangle < 0$$
$$b(x) := \min\{\langle \phi_i \,|\, x\rangle \langle \phi_i \,|\, z\rangle^{-1}\}, \ \forall i \,|\, \langle \phi_i \,|\, z\rangle > 0.$$

The set I_x is empty as soon as $a(x) > b(x)$ and this happens exactly outside the cone $C(X)$. Clearly, the two functions $a(x), b(x)$ are continuous and piecewise linear with respect to a decomposition into cones of the space.

We claim that the function T_X is piecewise polynomial of homogeneous degree $N - s$ with $s := \dim\langle X\rangle$.

Again proceed by induction. We assume that $C(Y)$ is decomposed into polyhedral cones $C_i(Y)$ so that T_Y is a homogeneous polynomial $p_{i,Y}(x)$ homogeneous of degree $N-1-s$ on each $C_i(Y)$. Then as before we have piecewise linear functions $a_i(x), b_i(x)$ so that when $a_i(x) \leq b_i(x)$ the intersection $I_x^i := \{t \in \mathbb{R}^+ \,|\, x - tz \in C_i(Y)\}$ is the segment $[a_i(x), b_i(x)]$, otherwise it is empty.

Thus we can decompose the cone $C(X)$ into polyhedral cones on each of which all the functions a_i, b_i are linear.

When x varies on one of these cones C, the interval I_x is decomposed into intervals $[u_j(x), u_{j+1}(x)]$, $j = 1, \dots, k$ so that all the $u_j(x)$ are linear and when $t \in [u_j(x), u_{j+1}(x)]$ the point $x - tz$ lies in some given cone $C_j(Y)$. Therefore on C the function T_X is a sum:

$$T_X(x) = \sum_{j=1}^{k-1} \int_{u_j(x)}^{u_{j+1}(x)} T_Y(x - tz)dt$$

Clearly, since by induction each $T_Y(x) = p_{j,Y}(x)$ on $C_j(Y)$ is a homogeneous polynomial of degree $N-1-s$ we have that $p_{j,Y}(x-tz) = \sum_{k=0}^{N-1-s} t^k p_{j,Y}^k(x)$ with $p_{j,Y}^k(x)$ homogeneous of degree $N-1-s-k$ and

$$\int_{u_j(x)}^{u_{j+1}(x)} t^k p_{j,Y}^k(x)dt = (k+1)^{-1} p_{j,Y}^k(x)[u_{j+1}(x)^{k+1} - u_j(x)^{k+1}]$$

this is a homogeneous polynomial of degree $N-1-s+1 = N-s$.

□

Example Let $X = \{a_1, \dots, a_s\}$ be a basis, $d := |\det(a_1, \dots, a_s)|$:

$B(X)$ is the parallelepiped with edges the a_i, $C(X)$ is the positive quadrant generated by X.

$$B_X = d^{-1}\chi_{B(X)}, \qquad T_X = d^{-1}\chi_{C(X)} \tag{1.27}$$

where, for any given set A, we denote by χ_A its characteristic function.

Remark that $C(X) = \cup_{r\in\mathbb{R}^+} rB(X)$.

Remark 1.3.16. In the parametric case we have analogous recursive formulas:

$$T_{X,\underline{\mu}}(x) = \int_0^\infty e^{-\mu_z t} T_{Y,\underline{\mu}}(x-tz)dt, \quad B_{X,\underline{\mu}}(x) = \int_0^1 e^{-\mu_z t} B_{Y,\underline{\mu}}(x-tz)dt \tag{1.28}$$

1.3.17 Ramadas 3

Possibly part of the previous lecture should be done here. Plus some properties of the functions we are computing

1.3.18 Fanja 3

Complete the program

1.3.19 Demissu 2 Modules

We want to study some modules over the Weyl algebra.

In very simple terms, an algebra R can be described intrinsically or as an algebra of operators on a vector space. A module is essentially an *incarnation* of R as an algebra of operators.

in more formal terms, a module is a homomorphism $p : R \to End(V)$ from R to the algebra of endomorphisms of a vector space V.

Equivalently, a module is a *multiplication*:

$$p : R \times V \to V, \quad \text{written} \quad (r, v) \mapsto rv,$$

satisfying the following axioms:

p is bilinear. $1v = v$ for each $v \in V$,and $(rs)v = r(sv)$ for each $r, s \in R$ and $v \in V$.

The most natural module is the ring R itself, in which the multiplication p coincides with the usual multiplication in thye ring. This module is also called the *regular representation.*

Various operations are possible on modules. In particular, the direct sum of two modules M, N is the space of pairs $(m, n),\ m \in M,\ n \in N$ with $r(m, n) := (rm, rn)$.

A submodule $N \subset M$ of a module M is a (vector) subspace satisfying $rn \in N,\ \forall r \in R,\ \forall n \in N$, or, in other words, N *is stable with respect to the operators induced from* R.

An important costruction is *the quotient module* M/N. This refers to a new vector space obtained from M putting the vectors of N equal to 0; or in other words letting $m_1 = m_2$ if $m_1 - m_2 \in N$. If N is a submodule the action of R on M induces in anatural way an action on M/N. Sarting from a ring R we can construct iother modules by taking quotients and direct sums.

The *free module* is a direct sum $R^{\oplus I}$ (not necessarily finite) of copies of R. Formally, given a set I the module $R^{\oplus I}$ is the set of functions $f : I \to R$ with the property $f(i) = 0$ except for finitely many elements i (a condition which is vacuous if I is finite).

Given a module M and a set $m_i,\ i \in I$ of elements $m_i \in M$ indexed by I we have a map

$$\pi : R^{\oplus I} \to M, \quad \pi(f) = \sum_{i \in I} f(i) m_i.$$

We say that the elements m_i *generate* M if π is surjective.

In this case M is isomorphic to $R^{\oplus I}/K$ where $K = \{f \in R^{\oplus I} \,|\, \pi(f) = 0\}$ and the *kernel* di π is sometimes called the *module of relations among the lements* m_i.

A particularly important case arises when M is generated by a a single element. m. In this case the module is called *cyclic*. The module of relations of a cyclic module is a left ideal I of R, $I = \{r \in R \mid rm = 0\}$ alo known as the *annihillator* of m. We have $M = R/I$.

We have seen some important special types of modules First of all, the *irreducible* ones, modules M which do not contain a submodule N other than 0 or M. We see immediately that an irreducible module is necessarily cyclic and that the aanihilator of any nonzero element is a maximal left ideal.

Semisimple or *completely reducible* modules are direct sums of irreducible modules.

We have also stated some basic theorems on modules, in particular, the structure theorem for semisimple modules and the theorem of Jordan–Hölder.

1.3.20 Claudio 4 Partial fractions

Given some numbers $\underline{\mu} := \{\mu_a,\ a \in X\}$ consider the algebra $R_{X,\underline{\mu}} := S[V][\prod_{a\in X}(a+\mu_a)^{-1}]$ as that obtained from $S[V]$ by inverting all the elements $a + \mu_a, a \in X$.

$R_{X,\underline{\mu}}$ is the coordinate algebra of the complement of the arrangement of affine hyperplanes given by the equations $a + \mu_a = 0,\ a \in X$.

This algebra is important for our work since it contains the Laplace transforms of the functions $T_{Y,\underline{\mu}}$ when $Y \subset X$. It is thus useful to study a canonical expansion into partial fractions for the elements of this algebra. We do it first in an elementary way and then interpret thisin the language of module over the Weyl algebra.

In the linear case, where all $\mu_a = 0$ we drop the $\underline{\mu}$ and write simply R_X.

Let us first develop a basic identity. Take linearly dependent vectors $a_0 = \sum_{i=1}^k \alpha_i a_i$, in X. Set

$$\nu := \mu_{a_0} - \sum_{i=1}^{k} \alpha_i \mu_{a_i}. \tag{1.29}$$

If $\nu \neq 0$ we write:

$$\frac{1}{\prod_{i=0}^k (a_i + \mu_{a_i})} = \nu^{-1} \frac{a_0 + \mu_{a_0} - \sum_{i=1}^k \alpha_i (a_i + \mu_{a_i})}{\prod_{i=0}^k (a_i + \mu_{a_i})}. \tag{1.30}$$

When we develop the right hand side, we obtain a sum of $k+1$ terms in each of which one of the elements $a_i + \mu_{a_i}$ has disappeared.

Let us draw a first consequence on separation of variables:

Proposition 1.3.21. *Assume that X spans V. For suitable constants c_p we have:*

$$\prod_{a\in X} \frac{1}{a+\mu_a} = \sum_{p\in P(X,\underline{\mu})} c_p \prod_{a\in X_p} \frac{1}{a+\mu_a} = \sum_{p\in P(X,\underline{\mu})} c_p \prod_{a\in X_p} \frac{1}{a - \langle a \mid p\rangle} \tag{1.31}$$

Proof. This follows by induction applying the previous algorithm of separation of denominators.

Precisely, if X is a basis there is a unique point of the arrangement and nothing to prove. Otherwise we can write $X = \{Y, z\}$ where Y still spans V. By induction

$$\prod_{a\in X} \frac{1}{a+\mu_a} = \frac{1}{z+\mu_z} \prod_{a\in Y} \frac{1}{a+\mu_a} = \sum_{p\in P(Y,\underline{\mu})} c'_p \frac{1}{z+\mu_z} \prod_{a\in Y_p} \frac{1}{a+\mu_a}.$$

We need to analyze one product

$$\frac{1}{z+\mu_z} \prod_{a\in Y_p} \frac{1}{a+\mu_a}.$$

If $\langle z \,|\, p\rangle + \mu_z = 0$ then $p \in P(X, \underline{\mu})$, $X_p = \{Y_p, z\}$ and we are done. Otherwise choose a basis $y_1, \dots, y_s$ from Y_p and apply the previous proposition to the incompatible list $z, y_1, \dots, y_s$. Using this we write the previous product as a linear combination of products involving one less factor. We then proceed by induction. □

Now let us go back to 1.29. If $\nu = 0$ we can write

$$\frac{1}{\prod_{i=0}^k (a_i + \mu_{a_i})} = \frac{\sum_{j=1}^k \alpha_j (a_j + \mu_{a_j})}{(a_0 + \mu_{a_0}) \prod_{i=0}^k (a_i + \mu_{a_i})} = \sum_{j=1}^k \frac{\alpha_j}{(a_0+\mu_{a_0})^2 \prod_{i=1,\ i\neq j}^k (a_i + \mu_{a_i})}.$$

So, repeating this algorithm, we see that:

Proposition 1.3.22. *For every $p \in P(X, \underline{\mu})$ we can write any fraction $1/(\prod_{a\in X_p} (a+\mu_a)^{h_a})$, $h_a > 0$ as a linear combination of fractions where the elements $a_i + \mu_i$ appearing in the denominator form a basis.*

The degree of the denominators is always $\sum_a h_a$.

In particular R_X is spanned by the functions of the form $f/(\prod_i b_i^{k_i+1})$, with f a polynomial, $B = \{b_1, \dots b_t\} \subset X$ a linearly independent subset and $k_i \geq 0$ for each $i = 1, \dots, t$.

At this point we are still far from having a normal form for the partial fraction expansion. In fact a dependency relation $\sum_{i=0}^k \alpha_i (a_i + \mu_i) = 0$ produces a basic relation:

$$\sum_{j=0}^k \frac{\alpha_j}{\prod_{i=0,\ i\neq j}^k (a_i + \mu_{a_i})} = 0. \tag{1.32}$$

We shall use these relations and the idea of unbroken bases, in order to normalize the fractions.

Let $\underline{c} := a_{i_1}, \dots, a_{i_k} \in X$, $i_1 < i_2 \cdots < i_k$, be a sublist of linearly independent elements.

Definition 1.3.23. We say that a_i *breaks* $\underline{c}$ if there is an index $1 \leq e \leq k$ such that:

- $i \leq i_e$.
- a_i is linearly dependent on $a_{i_e}, \dots, a_{i_k}$.

In particular, given any basis $\underline{b} := a_{i_1}, \dots, a_{i_s}$ extracted from X, we set $B(\underline{b}) := \{a \in X \mid a \quad \text{breaks} \quad \underline{b}\}$ and $n(\underline{b}) = |B(\underline{b})|$ the cardinality of $B(\underline{b})$.

Definition 1.3.24. We say that $\underline{b}$ is *unbroken* if $B(\underline{b}) = \underline{b}$ or $n(\underline{b}) = s$.

Let us denote by $\mathcal{B}(X)$ the set of all bases extracted from X. We shall consider the map $\underline{b} \mapsto n(\underline{b})$ as a *statistic* on $\mathcal{B}(X)$.

Remark 1.3.25. The combinatorics of unbroken bases is usually very complicated. We will see later how to unravel it for the list of positive roots of type A_n.

So let us take k linearly independent elements $a_{i_j}, i_1 < i_2 < \cdots < i_k, a_i \in X$, such that the subspace W of the arrangement given by $a_{i_j} + \mu_{i_j} = 0$ is non empty.

If there is an index $i \neq i_j, \ \forall j$ so that $a_i + \mu_i$ vanishes on W and a_i breaks the list a_{i_j} (see definition 1.3.23) we shall say that these elements are *broken* on W or simply broken. Otherwise we shall refer to the list $a_{i_j} + \mu_{i_j}$ as an *unbroken basis on* W and to the a_{i_j} as an *unbroken basis in* X_W.

When a_i breaks the list a_{i_j} we can apply the previous identity 1.32. Thus the fraction $\prod_{j=1}^k (a_{i_j} + \mu_{i_j})^{-1}$ equals a sum of fractions in which one of the terms $a_{i_j} + \mu_{i_j}$ has been replaced by $a_i + \mu_i$ and $i < i_j$. In an obvious sense, the fraction has been replaced by a sum of fractions which are *lexigographycally strictly inferior*.

One deduces then:

Theorem 1.3.26. *Consider a fraction $F := 1/(\prod_i (a_i + \mu_i)^{h_i})$ where the equations $a_i + \mu_i$ define a subspace W of the arrangement.*

F is a linear combination of fractions where the elements $a_i + \mu_i$ appearing in the denominator form an unbroken basis on W (see **??***).*

1.3.27 Ramadas 4 Cells and singular points

Recall that a *partition* of a set V is a family of disjoint subsets V_μ such that V is the union of the V_μ. We often express this by writing

$$V = \sqcup_\mu V_\mu$$

Equivalently, a partition corresponds to a relation of equivalence, with the subsets being the equivalence classes. A *stratification* of a topological space is a partition satisfying some additional conditions, which can vary from context to context. For the moment we will require only that each V_μ must be *locally closed*, that is, the intersection of a closed set and an open set. A standard example of a stratification is given by the following one of $\mathbb{R}^s$. The strata are

1. the open stratum $\{(x_1, \dots x_s) | x_i < 0 \; \forall \; i\}$, and

2. the 2^s strata parametrised by $\mu \subset \{1, \dots, s\}$

$$\{(x_1, \dots, x_s) | x_i = 0 \; i \in \mu, and \; x_i > 0 \; i \notin \mu\}$$

of which there is one closed startum, the origin (corresponding to $\mu = \{1, \dots, s\}$) and there is one open stratum (corresponding to $\mu = \phi$).

Note that the startification consists of $2^s + 1$ subsets. The latter 2^s sets give a stratification of the set $\{(x_1, \dots, x_s) | x_i \geq 0\}$, which we call the positive quadrant. By a *face* of this quadrant we mean a closure of a stratum; faces are parametrised by $\mu \subset \{1, \dots, s\}$:

$$f^\mu := \{(x_1, \dots, x_s) | x_i = 0 \; i \in \mu, and \; x_i \geq 0 \; i \notin \mu\}$$

Exercise: Let $V = \sqcup_\mu V_\mu$ and $V = \sqcup_\nu V'_\nu$ be two partitions of V. Show that $V = \sqcup_{(\mu,\nu)} V_\mu \cap V_\nu$ is a partition. If V is a topological space and the V_μ and V'_ν locally closed, so is $V_\mu \cap V'_\nu$.

We now turn to the geometry of the cone $C(X)$.

Given any basis $\underline{b} = \{b_1, \dots, b_s\}$ of V denote by $C(\underline{b})$ the cone (which we will also sometimes call the "quadrant") generated by $\underline{b}$. Using the basis we can identify V with $\mathbb{R}^s$ and this gives a stratification of V with $2^s + 1$ strata. Clearly, the strata consist of

1. the complement of the cone, that is, $C(\underline{b})^c := \{\sum_i x_i b_i | x_i < 0 \; \forall \; i\}$,

2. the 2^s strata parametrised by $\mu \subset \{1, \dots, s\}$

$$C(\underline{b})^\mu := \{\sum_i x_i b_i | x_i = 0 \; i \in \mu, and \; x_i > 0 \; i \notin \mu\}$$

of which there is one closed stratum, the origin, and there is one open stratum, the interior of the cone.

We will let $f^\mu(\underline{b})$ denote the 2^s faces of the cone $C(\underline{b})$, and $\mathcal{F}(\underline{b})$ denote the set of faces.

We will work with the stratification induced by the cones $C(\underline{b})$ as $\underline{b}$ varies among all the bases that one can extract from X.

It is useful to make this explicit, at the cost of introducing some notation. Let $\mathcal{I} = \{\underline{b}_1, \dots, \underline{b}_N\}$ denote a family of bases $\underline{b}$ extracted from X. For each such $\underline{b}$, let $C(\underline{b})^\mu$ ($\mu = c$ or $\mu \subset \{1, \dots, s\}$) denote the strata defined as above. Then the stratification generated by the set $\mathcal{I}$ of bases consists of the strata

$$C(\underline{b}_1)^{\mu_1} \cap \dots \cap C(\underline{b}_N)^{\mu_N}$$

We denote the corresponding stratification by $\mathcal{S}_\mathcal{I}$.

In our case, when we take $\mathcal{I}$ to be the set of *all* bases extracted from X, we claim that the resulting partition, which we denote simply by $\mathcal{S}$, is formed by the complement of $C(X)$ and other sets which are relative interiors of pointed convex polyhedral cones. (This stratification has the property that the closure of a stratum is the union of strata, and from now on we shall mean by stratification such a partition by locally closed sets.)

In order to justify our claim let us start with a

Lemma 1.3.28. *The cone $C(X)$ is the union of the quadrants $C(\underline{b})$ as $\underline{b}$ varies among the bases extracted from X.*

Proof. We proceed by induction on the cardinality of X. If X is a basis there is nothing to prove. Otherwise we can write $X = \{Y, z\}$ where Y still generates V. Take now an element $u = \sum_{y \in Y} t_y y + tz$, by induction we can rewrite $u = \sum_{a \in \underline{b}} t_a a + tz$ where $\underline{b}$ is a basis. We are thus reduced to the case $X = \{v_0, \ldots, v_s\}$, $u = \sum_{i=0}^s t_i v_i$. Let $\sum_{i=0}^s \alpha_i v_i = 0$ be the linear relation. We may assume that $a_i > 0, \forall i \leq k$ for some $k \geq 0$ and $a_i < 0, \forall i > k$. Let us consider the minimum element among the t_i/α_i, $i \leq k$ for instance assume this is taken for $i = 0$, then substituting v_0 with $-\sum_{i=1}^s \alpha_i/\alpha_0^{-1} v_i$ we see that u is expressed as combination of $v_1, \ldots, v_s$ with non negative coefficients. □

Let a set $\mathcal{I}$ of bases be given. Define a relation of equivalence on the set V as follows. Let $\mathcal{F}_{\mathcal{I}} = \cup_{\underline{b} \in \mathcal{I}} \mathcal{F}(\underline{b})$; this is the set of faces of all the quadrants defined by the bases belonging to $\mathcal{I}$. For $x \in V$, define

$$Z(x) := \{f \in \mathcal{F}(\underline{b}) | x \in f\} \ .$$

Then the relation of equivalence: is $x_1 \sim x_2$ iff $Z(x_1) = Z(x_2)$.

Theorem 1.3.29. *(1) The partition corresponding to the above relation of equivalence coincides with that given by the stratification $\mathcal{S}_{\mathcal{I}}$*

(2) Each stratum (of $\mathcal{S}$) inside $C(X)$ is the relative interior of a convex polyhedral cone.

(3) The closure of each stratum is a union of strata.

Proof. The fact that the complement of $C(X)$ is a stratum is immediate since by the previous Lemma it is the intersection of the complements of the various $C(\underline{b})$. The rest of (1) and (2) follows from the fact that $C(X)$ is the union of the quadrants $C(\underline{b})$.

As for (3) let us consider a stratum $S = F_1 \cap \cdots \cap F_k$ where the F_i are relatively open faces of some of the quadrants. We claim that the closure $\overline{S}$ of S is the intersection of the closures of the F_i and this suffices. Clearly $\overline{S} \subset \cap \overline{F}_i$. Thus let a be a point in $\cap \overline{F}_i$ and let $b \in S$. By convexity the segment $[a, b]$ is contained in all the sets $\overline{F}_i$, now $(a, b]$ is contained in all the sets F_i hence $(a, b] \subset S$ and so $a \in \overline{S}$ □

The open strata in $C(X)$ are called the *big cells*, the points of $C(X)$ in the big cells are called *regular*. The remaining points of $C(X)$ are called *singular points*.

Proposition 1.3.30. *(1) Given a regular point p the big cell containing p is the intersection of the interiors of all the quadrants $C(\underline{b})$ containing p.*

(2) The set of singular points is the union of the cones $C(A)$ generated by subsets $A \subset X$ which do not span V.

Proof. If x is a regular point, clearly $Z(x)$ is the set of (closed) quadrants containing x (necessarily in their interior, since x is regular). Conversely, if $Z(x)$ consists of s-dimensional faces only, x is a regular point. The claims (1) and (2) now follow. □

Recall the notion of an *unbroken basis* extracted from $X = (a_1, \dots a_N)$. For $(i_1 < i_2 < \dots i_s)$ a subset of $(1, \dots, N)$, consider the decreasing family of subspaces of V:

$$Span(a_{i_1}, \dots, a_{i_s}) \supseteqq Span(a_{i_2}, \dots, a_{i_s}) \supseteqq \dots Span(a_{i_s})$$

(By decreasing we do not necessarily mean strictly decreasing.) When we say that $\underline{a} = (a_{i_1}, \dots, a_{i_s}) \subset X$ is an unbroken basis, we mean that

- $\underline{a}$ is a basis (of course in this case the above sequence of spaces *strictly* decreasing).
- For each i_l, the vectors $(a_1, \dots, a_{i_l - 1})$ lie *outside* the subspace $Span(a_{i_l}, \dots a_{i_s})$

(In particular in this case $i_1 = 1$.)

Let $\mathcal{N}$ denote the set of unbroken bases extracted from X. An important theorem of which we omit the proof says that the stratifications $\mathcal{S}$ and $\mathcal{S}_{\mathcal{N}}$ coincide. In particuilar,

Theorem 1.3.31. *Two points are in the same big cell if and only if they belong to the interior of the quadrants $C(\underline{b})$ for all the* **unbroken** *bases $\underline{b}$.*

1.3.32 Demissu 3

All the modules over Weyl algebras which will appear are built out of some basic irreducible modules, in the sense that they will have finite composition series in which these basic modules appear. It is thus useful to quickly discuss these basic modules. We can work directly over the complex numbers for simplicity.

1.3.33 The polynomials

Let us take $W(V) = W(s) = \mathbb{C}[x_1, \dots, x_s; \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_s}]$, the differential operators on V. The most basic module on $W(V)$ is the polynomial ring $S[U] = \mathbb{C}[x_1, \dots, x_s]$.

The following facts are immediate but we want to stress them since they will be generalized soon.

Proposition 1.3.34. *$\mathbb{C}[x_1, \dots, x_s]$ is a cyclic module generated by 1.*

$\mathbb{C}[x_1, \dots, x_s]$ is an irreducible module.

The annihilator ideal of 1 is generated by all the derivatives D_v, $v \in V$ that is by all the derivatives $\frac{\partial}{\partial x_i}$.

An element $a \in \mathbb{C}[x_1, \dots, x_s]$ is annihilated by all the derivatives, if and only if it is a constant multiple of 1.

We shall use a simple corollary of this fact.

Corollary 1.3.35. *Let M be any $W(V)$ module.*

If $m \in M$ is a non 0 element satisfying $D_v m = 0$, $\forall v \in V$ then the $W(V)$ module generated by m is isomorphic to $S[U]$ under a unique isomorphism mapping 1 to m.

Let $m_1, \dots, m_k$ be linearly independent elements in M satisfying the equations $D_v m = 0$, $\forall v \in V$ then the modules $W(V)m_i$ form a direct sum.

Proof. The first part is clear, if I denotes the left ideal generated by the elements D_v we have $W(V)/I = S[U]$ and a morphism $W(V)/I \to W(V)m$ mapping 1 to m. Since $S[U]$ is irreducible, this map is an isomorphism.

Let us show the second part. Any element of $\sum_i W(V)m_i$ can clearly be written (by part 1) as a sum $\sum_i f_i m_i$, $f_i \in S[U]$ polynomials. We have to show that, if $\sum_i f_i m_i = 0$ then all the $f_i = 0$.

Assume this is not the case and so let us choose a monomial x^α of degree $r \geq 0$, maximal for all f_i and appearing in at least one of the polynomials. Apply ∂^α to $\sum_i f_i m_i$, since $D_v m_i = 0$ for all v we have that $\partial^\alpha(f_i m_i) = \partial^\alpha(f_i)m_i$. Thus $0 = \partial^\alpha(\sum_i f_i)m_i$ is a linear combination with non zero constant coefficients of the m_i, a contradiction.

□

1.3.36 Automorphisms

From the commutation relations it is clear that the subspace $V \oplus U \oplus \mathbb{C} \subset W(V)$ is closed under commutators, thus it is a Lie subalgebra (called the Heisenberg algebra).

The Lie product takes values in $\mathbb{C}$ and induces the natural symplectic form on $V \oplus U$ given by $[(v, u), (v', u')] = \langle v, u' \rangle - \langle v', u \rangle$ for $v, v' \in V$, $u, u' \in U$.

The group $G = (V \oplus U) \ltimes Sp(V \oplus U)$ (where $Sp(V \oplus U)$ denotes the symplectic group of linear transformations preserving the given form) acts as a group of automorphisms of the algebra $W(V)$

preserving the space $V \oplus U \oplus \mathbb{C}$. Let us identify some automorphisms in G.

- Translations. Given $u \in U$ we get the automorphism γ_u defined by $\gamma_u(v) = v + \langle v, u \rangle$ for $v \in V$ while $\gamma_u(u') = u'$ if $u' \in U$ and similarly for the automorphism γ_v associated to an element $v \in V$.
- Linear changes. Given an element $g \in Gl(V)$ then we can consider the symplectic transformation
$$\begin{pmatrix} g & 0 \\ 0 & g^{*-1} \end{pmatrix}$$
where $g^* : U \to U$ is the adjoint to g.
- Partial Fourier transform. If we choose a basis x_j, $1 \leq j \leq s$ of V and denote by $\partial/\partial x_i$ the dual basis of U, we have for a given $0 \leq k \leq s$ the symplectic linear transformation ψ_k defined by
$$x_i \mapsto \frac{\partial}{\partial x_i}, \ \frac{\partial}{\partial x_i} \mapsto -x_i, \ \forall i \leq k.$$
$$x_i \mapsto x_i, \frac{\partial}{\partial x_i} \mapsto \frac{\partial}{\partial x_i}, \forall i > k.$$

We can use automorphisms in the following way.

Definition 1.3.37. Given a module M over a ring R and an automorphism ϕ of R one defines ${}^{\phi}M$ the *module twisted by* ϕ as the same abelian group (or vector space for algebras), M with the new operation $r \circ_{\phi} m := \phi^{-1}(r)m$.

The following facts are trivial to verify:

- ${}^{\phi}(M \oplus N) =^{\phi} M \oplus^{\phi} N$
- If M is cyclic generated by m and with annihilator I then ${}^{\phi}M$ is cyclic generated by m and with annihilator $\phi(I)$.
- M is irreducible if and only if ${}^{\phi}M$ is irreducible.
- If ϕ, ψ are two automorphisms ${}^{\phi\psi}M =^{\phi} ({}^{\psi}M)$.

What will really appear in our work are some twists, under automorphisms of the previous type, of the basic module of polynomials.

Precisely, for each affine subspace S of U we have an irreducible module N_S which is defined as follows.

Write $S = W + p$ where W is a linear subspace of U and $p \in U$.

We can take coordinates $x_1, \dots, x_s$ so that

$$W = \{x_1 = x_2 = \dots = x_k = 0\}$$

and $\langle x_i, p\rangle = 0$ for $i > k$.

N_S is generated by an element δ_S satisfying:

$$x_i\delta_S = \langle x_i, p\rangle\delta_S,\ i \leq k, \quad \frac{\partial}{\partial x_i}\delta_S = 0,\ i > k.$$

To our basis we associate the linear transormation ψ_k and to p the translation γ_p. It is clear that the the two transformations commute and their composition give a well defined element $\phi \in G$ and hence a well defined automorphism of $W(V)$ which we denote by the same letter.

It is clear that:

Proposition 1.3.38. *1. N_S is the twist of the polynomial ring under the automorphism ϕ defined above.*

2. N_S is freely generated by the element δ_S, transform of 1, over the ring

$$\phi(\mathbb{C}[x_1, \dots, x_s]) = \mathbb{C}[\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}, x_{k+1}, x_{k+2}, \dots, x_s].$$

3. N_S depends only on S and not on the coordinates chosen.

Proof. The first two statements follow from the definitions. As for the last statement, the elements $x_i - \langle x_i, p\rangle$ vanishing on S span the intrinsically defined space $S^\perp$ of all linear equations vanishing on S. While the derivatives $\frac{\partial}{\partial x_i},\ i > k$ span the space intrinsically defined $D_u,\ u \in W$. These two spaces generate the left ideal annihilator of the element δ_S. □

Remark 1.3.39. We can also interpret N_S in the language of distributions, leaving to the reader to verify that in this language δ_S should be thought of as the δ function of S, given by $\langle \delta_S \mid f\rangle = \int_S f dx$
.

1.3.40 Rik Enrico

computer lab, also possible example:

Example 1.3.41. A_3 ordered as: $\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3$.

We have 6 n.b.b all contain necessarily α_1:

$$\alpha_1,\ \alpha_2,\ \alpha_3,$$

$$\alpha_1,\ \alpha_2,\ \alpha_2+\alpha_3.$$

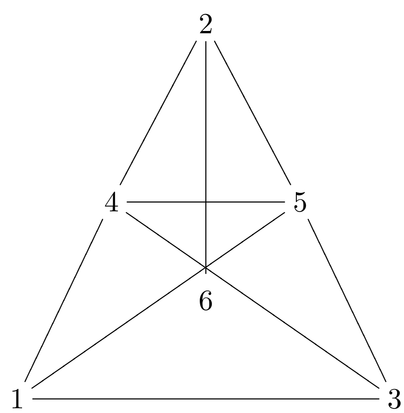
$$\alpha_1,\ \alpha_2,\ \alpha_1+\alpha_2+\alpha_3.$$

$$\alpha_1,\ \alpha_3,\ \alpha_1+\alpha_2.$$

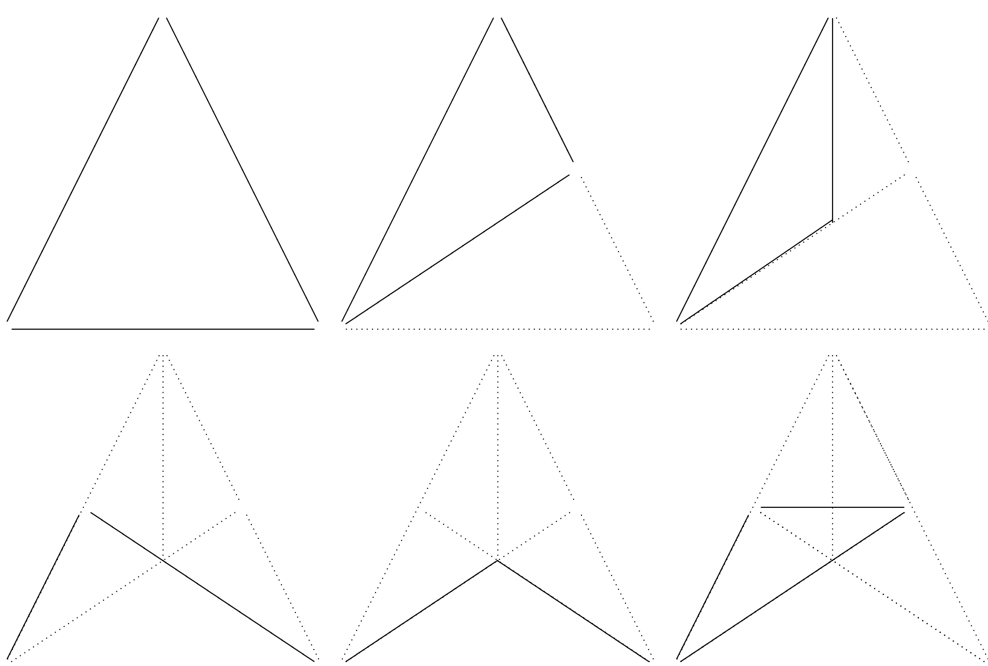
$$\alpha_1,\ \alpha_3,\ \alpha_1+\alpha_2+\alpha_3.$$

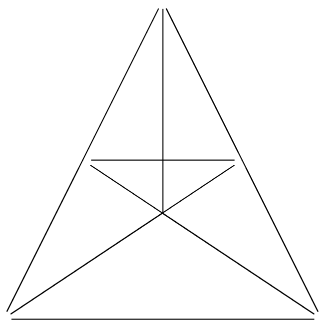
$$\alpha_1,\ \alpha_1+\alpha_2,\ \alpha_2+\alpha_3.$$

The big cells:



The decomposition obtained superposing the cones associated to unbroken bases:





1.4 Day 5

To be decided

Chapter 2

Week 2

2.1 Day 1

2.1.1 Claudio 1 The generic case

Let $\mathcal{B}_X$ be the family of subsets $\sigma := \{i_1, \dots, i_s\} \subset \{1, \dots, N\}$ so that the set $b_\sigma := \{a_{i_1}, \dots, a_{i_s}\}$ is a basis extracted from the list X. Consider the corresponding point $p_\sigma : a_{i_k} + \mu_{i_k} = 0,\ k = 1, \dots, s$. To say that the parameters are generic is equivalent to saying that these points are all distinct. In particular we have that $\langle a_i \,|\, p_\sigma \rangle + \mu_i = 0, \iff i \in \sigma$.

Set

$$d_\sigma := \prod_{k \in \sigma} (a_k + \mu_k).$$

Then according to the previous theory, we can apply recursively the formula (1.30) and by (1.31) there are coefficients c_σ such that:

$$\prod_{i=1}^{N} \frac{1}{(a_i + \mu_i)} = \sum_{\sigma} \frac{c_\sigma}{d_\sigma}. \tag{2.1}$$

One can easily derive an explicit formula for the constants c_σ as follows. Take one set $\sigma_0 \in \mathcal{B}_X$, we have

$$\prod_{i=1,\ i \notin \sigma_0}^{N} \frac{1}{(a_i + \mu_i)} = \sum_{\sigma \neq \sigma_0} \frac{c_\sigma d_{\sigma_0}}{d_\sigma} + c_{\sigma_0}. \tag{2.2}$$

Evaluating both terms in p_{σ_0} we finally have that:

$$\prod_{i=1,\ i \notin \sigma_0}^{N} \frac{1}{\langle a_i \,|\, p_{\sigma_0} \rangle + \mu_i} = c_{\sigma_0}. \tag{2.3}$$

Finally

$$\prod_{i=1}^{N} \frac{1}{(a_i + \mu_i)} = \sum_{\sigma} \frac{1}{d_\sigma \prod_{i=1,\ i\notin\sigma}^{N} (\langle a_i \,|\, p_\sigma\rangle + \mu_i)}. \tag{2.4}$$

This formula has a geometric meaning, in the sense that the function $\prod_{i=1}^{N} 1/(a_i + \mu_i)$ has simple poles at the points p_σ and the residues have a fairly simple formula. When the parameters μ_i specialize or even become all 0 the poles coalesce and become higher order poles. It is still possible to interpret the formula of the volume as a residue but, in order to justify it geometrically it is necessary to perform certain geometric constructions called *blow up*.

Since we are in the generic case, the set $P(X, \underline{\mu})$ coincides with the set of points p_σ as $\sigma \in \mathcal{B}_\Delta$ runs on the subsets indexing bases. For such a p_σ we have that X_{p_σ} coincides with the basis $a_i,\ i \in \sigma$. $T_{X_{p_\sigma}} = L^{-1}(d_\sigma^{-1})$ is equal to $v_\sigma^{-1}\chi_\sigma$ where χ_σ is the characteristic function of the positive cone C_σ generated by the vectors $a_i,\ i \in \sigma$ and v_σ the volume of the parallelepiped they generate. We deduce the following explicit formula.

Proposition 2.1.2. *On a point h in the cone generated by the a_i the function $T_{X,\mu}(h)$ equals the sum*

$$\sum_{\sigma \mid\ h\in C_\sigma} \frac{e^{\langle h \,|\, p_\sigma\rangle}}{v_\sigma \prod_{i=1,\ i\notin\sigma}^{N} (\langle a_i \,|\, p_\sigma\rangle + \mu_i)}$$

Remark that the points p_σ depend on the parameters μ_i and can of course be explicited by Cramer's rule.

Now let us vary the generic parameters as $t\mu_i$ with t a parameter that we will let go to 0. We then have that the corresponding points p_σ vary as tp_σ.

The factors $v_\sigma \prod_{i=1,\ i\notin\sigma}^{N} (\langle a_i \,|\, p_\sigma\rangle + \mu_i)$ vary as

$$v_\sigma \prod_{i=1,\ i\notin\sigma}^{N} (\langle a_i \,|\, tp_\sigma\rangle + t\mu_i) = t^{N-s} v_\sigma \prod_{i=1,\ i\notin\sigma}^{N} (\langle a_i \,|\, tp_\sigma\rangle + \mu_i)$$

while the numerator

$$e^{\langle h \,|\, tp_\sigma\rangle} = \sum_{k=0}^{\infty} t^k \langle h \,|\, p_\sigma\rangle^k / k!$$

From formula 1.21 it follows that $T_{X,t\mu}(h)$ is a holomorphic function in t around 0 and that its value for $t = 0$ is the (normalized) volume $V_X(h)$ of the polytope $\Pi_X(h)$. We deduce that:

$$\sum_{\sigma,\ h\in C_\sigma} \frac{\langle h \,|\, p_\sigma\rangle^k}{v_\sigma \prod_{i=1,\ i\notin\sigma}^{N} (\langle a_i \,|\, p_\sigma\rangle + \mu_i)} = 0,\ \forall k < N - s$$

and a *Formula for the volumes:*

$$V_X(h) = \sum_{\sigma,\ h\in C_\sigma} \frac{\langle h \,|\, p_\sigma\rangle^{N-s}}{(N-s)!\, v_\sigma \prod_{i=1,\ i\notin\sigma}^{N} (\langle a_i \,|\, p_\sigma\rangle + \mu_i)} \tag{2.5}$$

This formula, (from [18]), contains parameters μ_i and the corresponding p_σ (determined by a generic choice of the μ_i), although it is independent of these parameters. This formula, although explicit, it is probably not the best formula since it contains many terms (indexed by all bases σ, with $h \in C_\sigma$) and the not so easy to compute auxiliary p_σ. We shall discuss in the next section a direct approach to the volume.

2.1.3 The general case

We can now pass to the general case and are indeed ready to state and prove the main formula one can effectively use for computing the function T_X.

Let us denote by $\mathcal{NB}$ the set of all unbroken bases extracted from X.

Theorem 2.1.4. *There exist uniquely defined polynomials $p_{\underline{b},X}(x)$ homogeneous of degree $|X| - s$ and indexed by the unbroken bases in X, with*

$$\frac{1}{d_X} = \sum_{\underline{b}\in\mathcal{NB}} p_{\underline{b},X}(\partial_x)\frac{1}{d_{\underline{b}}}, \qquad d_{\underline{b}} := \prod_{a\in\underline{b}} a. \tag{2.6}$$

Proof. We have to apply to the fraction d_X^{-1} the algorithm used to prove Proposition 1.3.26. We first remark that, when we apply this algorithm to a fraction $\prod_{a\in X} a^{-h_a}$ with the property that the set of a's for which $h_a > 0$ spans V, the same is true for all the terms resulting by applying a step of the algorithm.

As a result we shall get in the end an expansion

$$d_X^{-1} = \sum_{\underline{b}=\{b_1,\dots,b_s\}\in\mathcal{NB}} \sum_{h_1,\dots,h_s,\ h_i\in\mathbb{N}} c_{h_1,\dots,h_s} b_1^{-1-h_1}\dots b_s^{-1-h_s}$$

Now we have $b_1^{-1-h_1}\dots b_s^{-1-h_s} = \prod_{i=1}^s(-1)^{h_i}h_i!\dfrac{\partial}{\partial b_i}(\prod_{i=1}^s b_i)^{-1}$ hence

$$p_{\underline{b},X}(\partial_x) = \sum_{h_1,\dots,h_s,\ h_i\in\mathbb{N}} c_{h_1,\dots,h_s}\prod_{i=1}^s(-1)^{h_i}h_i!\frac{\partial}{\partial b_i} \tag{2.7}$$

□

Remark 2.1.5. A way of understanding this formula comes from the fact that the module of polar parts is a free $S[U]$ module with basis the classes of the elements $d_{\underline{b}}^{-1}$ as $\underline{b}$ run over the unbroken bases. Thus the polynomials $p_{\underline{b},X}(\partial_x)$ are the coordinates of the class of d_X^{-1} with respect to this basis.

Example 2.1.6. We write the elements of X as linear functions.

$$X = [x+y, x, -x+y, y] = [x, y]\begin{vmatrix} 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$$

$$\frac{1}{(x+y)\,x\,y\,(-x+y)} = -\frac{1}{y\,(x+y)^3} + \frac{4}{(x+y)^3(-x+y)} + \frac{1}{x\,(x+y)^3} =$$

$$1/2[-\frac{\partial^2}{\partial^2 x}(\frac{1}{y\,(x+y)}) + (\frac{\partial}{\partial x}+\frac{\partial}{\partial y})^2(\frac{1}{(x+y)(-x+y)}) + \frac{\partial^2}{\partial^2 y}(\frac{1}{x\,(x+y)})]$$

2.1.7 Ramadas 1. formula for T_X

We can now interpret Theorem 2.1.4 as:

Theorem 2.1.8. *Given a point x in the closure of a big cell $\mathfrak{c}$ we have*

$$T_X(x) = \sum_{\underline{b}\ \mid\ \mathfrak{c}\subset C(\underline{b})} |\det(\underline{b})|^{-1} p_{\underline{b},X}(-x), \tag{2.8}$$

where for each unbroken basis $\underline{b}$, $p_{\underline{b},X}(x)$ is the homogeneous polynomial of degree $|X|-s$ a uniquely defined in Theorem 2.1.4.

Proof. We apply the inversion of the Laplace transform (cf. 1.27, 1.23) to formula 2.6 obtaining:

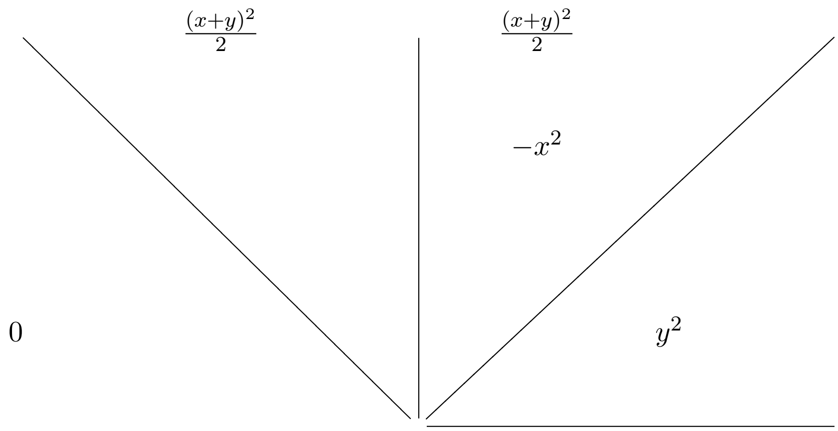
$$L^{-1}(d_A^{-1}) = \sum_{\underline{b}\in\mathcal{NB}} p_{\underline{b},A}(-x_1,\dots,-x_n)|\det(\underline{b})|^{-1}\chi_{C(\underline{b})}.$$

This function is clearly continuous only on the set of regular points and, a priori, coincides with T_X only outside a subset of measure zero. By the continuity of T_X in $C(X)$ (part 3 of Proposition 1.3.15), it is sufficient to show our claim in the interior on each big cell. This last fact is clear. □

Thus we see that the decomposition into big cells, which is essentially a geometric and combinatorial picture, corresponds to a decomposition of the cone $C(X)$ into regions in which the multivariate spline T_X has a polynomial nature. Moreover, the contributions to the polynomials on a given cell $\mathfrak{c}$ come only from the unbroken bases $\underline{b}$ such that $\mathfrak{c}\subset C(\underline{b})$. This is in accordance with theorem 1.3.31 which states that one can use only the quadrants associated to unbroken bases in order to decompose the cone into big cells.

Example 2.1.9. In example 2.1.6 we have thus:

$$T_X(x,y) = 1/2[-x^2\chi_{C((0,1),(1,1))} + \frac{(x+y)^2}{2}\chi_{C((1,1),(-1,1)} + y^2\chi_{C((1,0),(1,1))}]$$



From now on, unless there is a possibility of confusion, we shall write $p_{\underline{b}}$ instead of $p_{\underline{b},X}$.

Remark 2.1.10. (1) The polynomials $p_{\underline{b}}(x)$, with $\underline{b}$ a unbroken basis, will be characterized by differential equations in section (**??**).

(2) We will also show (Corollary **??**) that the polynomials $p_{\underline{b}}(x)$ are a basis of the space spanned by the polynomials coinciding with T_X on the big cells.

(3) It is easily seen that the number of big cells is usually much larger than the dimension of this last space. This reflects into the fact that the polynomials coinciding with T_X on the big cells satisfy complicated linear dependency relations which can be determined by the incidence matrix between big cells and cones generated by unbroken bases.

2.1.11 The case of numbers: volume

Let us discuss the simple case in which $s = 1$. In other words, X is a row vector $A := (h_1, \dots, h_m)$ of positive numbers. We want to compute the volume function whose Laplace transform is $|A| \prod h_i^{-1} y^{-m}$. The computation is rather easy:

- The normalization constant in formula 1.21 is $\sqrt{\det(AA^t)} = \sqrt{\sum_i h_i^2}$.
- The polyhedron $P_X(b)$ associated to the numbers h_i, b is a *simplex*, given by:

$$\{(x_1, \dots, x_m) \,|\, x_i \geq 0,\ \sum_i x_i h_i = b.\}$$

- $P_{\underline{h}}(b)$ is the convex envelope of the vertices $P_i = (0, 0, \dots, 0, b/h_i, 0, \dots, 0)$.

We could compute the volume of $P_{\underline{h}}(b)$ directly but let us find the *function whose Laplace trasform is y^{-m}*. From the basic formulas if χ denotes the characteristic function of the half line $x \geq 0$ we have:

$$L((-x)^k\chi) = (-1)^k k! y^{-k-1}.$$

IN CONCLUSION:

Theorem 2.1.12. *The volume of $P_{\underline{h}}(b)$ is given by*

$$\boxed{\frac{|A|b^{m-1}}{(m-1)!\prod h_i}}.$$

afternoon

2.1.13 Rik or Enrico

Basic examples of partition function, for instance the example $2h + 3k = n$ in detail (I have some little notes).

2.2 Day 2

2.2.1 Claudio 2. The Partition functions

In our work the word $s-$dimensional lattice will mean a free abelian group of finite dimension s, in other word an abelian group Λ isomorphic to $\mathbb{Z}^s$. As in the theory of vector spaces, such an explicit isomorphism corresponds to a choice of an integral basis a_i for Λ. A new integral basis b_j of Λ is related to the basis a_i by a matrix $b_j = \sum_i \alpha_{ji} a_i$ (α_{ij} is an integral matrix with integral inverse, i.e. with determinant ± 1.

Associated to Λ we have the $\mathbb{Q}$ vector space $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ isomorphic to $\mathbb{Q}^s$. A basis of $\mathbb{Q}^s$ contained in Λ will be referred to as a *rational basis*, or sometimes just a basis.

Definition 2.2.2. We identify a function f on Λ with the distribution

$$\tilde{f} := \sum_{\lambda \in \Lambda} f(\lambda)\delta_\lambda \tag{2.9}$$

where δ_v is the Dirac distribution supported at v.

Recall that we have defined the partition function on Λ by

$$\mathcal{P}_X(v) := \#\{(n_1, \dots, n_N) \mid \sum n_i a_i = v,\ n_i \in \mathbb{N}\}.$$

This is then identified with the tempered distribution

$$\mathcal{T}_X := \sum_{v \in \Lambda} \mathcal{P}_X(v)\delta_v \tag{2.10}$$

We can then compute its Laplace transform obtaining

$$L\mathcal{T}_X = \sum_{v\in\Lambda} \mathcal{P}_X(v)e^{-v} = \prod_{a\in X} \frac{1}{1-e^{-a}} \tag{2.11}$$

We shall think of $\mathcal{T}_X$ as a discrete analogue of the multivariate spline T_X. We also have an analogue of B_X. Namely, setting

$$\mathcal{Q}_X(v) := \#\{(n_1,\dots,n_N) \mid \sum n_i a_i = v,\ n_i \in \{0,1\}\}.$$

$$\mathcal{B}_X := \sum_{v\in\Lambda} \mathcal{Q}_X(v)\delta_v \tag{2.12}$$

with Laplace transform

$$L\mathcal{B}_X = \sum_{v\in\Lambda} \mathcal{Q}_X(v)e^{-v} = \prod_{a\in X}(1+e^{-a}) = \prod_{a\in X} \frac{1-e^{-2a}}{1-e^{-a}} \tag{2.13}$$

which implies that

$$\mathcal{B}_X(x) = \sum_{S\subset X} (-1)^{|S|}\mathcal{T}_X(x-2a_S) \tag{2.14}$$

with $a_S = \sum_{a\in S} a$.

2.2.3 Ramadas 2. Euler recursion

Given positive numbers $\underline{h} := (h_1,\dots,h_m)$, the problem of counting the number of ways in which a positive integer n can be written as a linear combination $\sum_{i=1}^n k_i h_i$ goes back at least to Euler who showed that this function, denoted for the moment u_n satisfies a simple recursive relation which at least in principle allows us to compute it. The relation is classically expressed as follows, consider the polynomial $\prod_i (1-u^{h_i})$ expand it and the substitute formally to each u^r the expression u_{n-r}, set the resulting expression to 0 getting the required relation. We shall presently explain the meaning of these types of recursions also in the general multidimensional case. let us now formalize in a different way.

First of all remark:

Lemma 2.2.4. *u_n is the coefficient of t^n in the power series expansion of:*

$$P_{\underline{h}}(t) := \prod_{i=1}^m \frac{1}{1-t^{h_i}}.$$

Proof.

$$\prod_{i=1}^m \frac{1}{1-t^{h_i}} = \prod_{i=1}^m \sum_{k=0}^\infty t^{kh_i} = \sum_{k_1,\dots k_m} t^{\sum_{i=1}^m k_i h_i} = \sum_{k=0}^\infty u_k t^k.$$

□

In general for a sequence $\underline{u} := \{u_n\}$ we shall set $G_{\underline{u}}(t) := \sum_{n=-\infty}^{\infty} u_n t^n$ to be its *generating function*.

Let us now introduce the *shift operator* on sequences $\underline{u} := \{u_n\}$ defined by $(\tau \underline{u})_n := u_{n-1}$ we clearly have:

$$G_{\tau\underline{u}}(t) = tG_{\underline{u}}(t).$$

Given a polynomial $p(\tau) = a_0\tau^n + a_1\tau^{n-1} + \cdots + a_{n-1}\tau + a_n$ we say that $\underline{u}$ satisfies the recursion $p(\tau)$ if $p(\tau)(\underline{u}) = 0$.

In terms of generating functions thus $p(t)G_{\underline{u}}(t) = 0$.

In our case the function $P_{\underline{h}}(t)$ satisfies a non homogeneous recursion equation:

$$p(\tau)P_{\underline{h}}(t) = \delta_0 \tag{2.15}$$

where $\delta_0 = 1$ is the sequence $u_i = 0,\ \forall i \neq 0,\ u_0 = 1$.

A priori as for any system of non homogeneous linear equations, to a given solution one may add any solution of the homogeneous equation. In our case though we know that $u_n = 0,\ \forall n < 0$, this is easily seen to imply that the solution to 2.15 is unique with this extra condition.

The number of ways in which n is written as combination of numbers a_i is called by Sylvester a *denumerant* and the first results on denumerants are due to Cayley and Sylvester, who proved that such a denumerant is a polynomial in n of degree $m-1$ plus a periodic function called an *undulant*, see Chapter 3 of [37] or the original papers of Cayley Sylvester in Kap. 3 [4]. The theory has been formalized by Bell in (1943) [16] and we present a variant of this in the next two sections. A more precise description of the leading polynomial part and of the periodic corrections is presented in **??**.

2.2.5 Two strategies

Fix positive numbers $\underline{h} := (h_1, \ldots, h_m)$, and given an integer n, we want to compute the coefficient of x^n in series expansion of the function $\prod_i \frac{1}{1-x^{h_i}}$; equivalently, of x^{-1} in the expansion of $\frac{x^{-n-1}}{\prod_i 1-x^{h_i}}$.

We have two essentially equivalent strategies, to be analyzed separately from the algorithmic point of view.

1. Develop $F_{\underline{h}}(x)$ in partial fractions.

2. Compute the residue $\frac{1}{2\pi i} \oint \frac{x^{-n-1}}{\prod_i 1-x^{h_i}} dx$ around 0.

In both cases first we must expand in a suitable way the function $F_{\underline{h}}(x)$. Given k let us denote by

$\zeta_k := e^{\frac{2\pi i}{k}}$, a primitive $k-$th root of 1; we have then the identity:

$$1 - x^k = \prod_{i=0}^{k-1}(1 - \zeta_k^i x) = \prod_{i=0}^{k-1}(\zeta_k^i - x).$$

Using this, we can write

$$F_{\underline{h}}(x) := \prod_{i=1}^{m}\prod_{j=0}^{h_i-1}\frac{1}{\zeta_{h_i}^j - x} = \prod_{i=1}^{m}\prod_{j=0}^{h_i-1}\frac{1}{1 - \zeta_{h_i}^j x}. \tag{2.16}$$

Let μ be the least common multiple of the numbers h_i, and write $\mu = h_i k_i$. If $\zeta = e^{2\pi i/\mu}$, we have $\zeta_{h_i} = \zeta^{k_i}$ therefore we have

Lemma 2.2.6.

$$\prod_i \frac{1}{1 - x^{h_i}} = \prod_{i=1}^{m}\prod_{j=0}^{h_i-1}\frac{1}{1 - \zeta^{k_i j}x} = \prod_{i=0}^{\mu-1}\frac{1}{(1 - \zeta^i x)^{b_i}}$$

where the integers b_i can be computed easily from the numbers h_j.

- Thus given i we must count how many numbers k_i are divisors of i. This number is b_i.
- In particular the function $\prod_i \frac{x^{-n-1}}{1-x^{h_i}}$, $n \geq 0$ has poles at 0 and at the $m-$th roots of 1 (but not at ∞)

2.2.7 First method: development in partial fractions.

The classical method starts from the fact that there exist numbers c_i for which:

$$\prod_{i=0}^{\mu-1}\frac{1}{(1 - \zeta^i x)^{b_i}} = \sum_{i=0}^{\mu-1}\frac{c_i}{(1 - \zeta^i x)^{b_i}}. \tag{2.17}$$

In order to compute them we use (for instance) recursively the simple identities, (valid if $a \neq b$ are two numbers):

$$\frac{1}{(1 - ax)(1 - bx)}\frac{1}{a - b}\left[\frac{a}{(1 - ax)} - \frac{b}{(1 - bx)}\right]$$

and then:

$$\frac{1}{(1 - t)^k} = \sum_{h=0}^{\infty}\binom{k - 1 + h}{h}t^h \tag{2.18}$$

to get:

$$\prod_{i=0}^{m-1} \frac{1}{(1-\zeta^i x)^{b_i}} = \sum_{i=0}^{m-1} c_i(\sum_{h=0}^{\infty} \binom{b_i - 1 + h}{h} (\zeta^i x)^h). \tag{2.19}$$

We have thus obtained a formula for the coefficient

$$S_{\underline{h}}(b) = \sum_{i=0}^{m-1} c_i(\zeta^{ib}) \binom{b_i - 1 + b}{b}.$$

Let us remark now that

$$\binom{b_i - 1 + b}{b} = \frac{(b+1)(b+2)\dots(b+b_i-1)}{(b_i-1)!}$$

is a polynomial of degree $b_i - 1$ in b, while the numbers ζ^{ib} depend only from the coset of b modulo m. Given an $0 \le a < m$ and restricting us to the numbers $b = mk + a$, we have:

Theorem 2.2.8. *The function $S_{\underline{h}}(mk+a)$ is a polynomial in the variable k of degree $\le \max(b_i)$, that can be computed.*

Sometimes we express the fact that $S_{\underline{h}}(b)$ is a polynomial on every coset saying that $S_{\underline{h}}(b)$ is a *quasi–polynomial* or a periodic polynomial.

afternoon

2.2.9 Enrico Rik some computer work

2.3 Day 3

2.3.1 Claudio 3

Explain without proof a formula passing from the multivariate spline to the partition function . Possibly this is too much and it is better to expand on what I already have put into these notes.

2.3.2 Ramadas 3 Second method: computation of residues

Here the strategy is the following: shift the computation of the residue to the remaining poles, exploiting the fact that the sum of residues at all the poles of a rational function is 0.

From the theory of residues we have:

$$\frac{1}{2\pi i}\oint\prod_i \frac{x^{-n-1}}{1-x^{h_i}}dx = -\sum_{j=1}^m \frac{1}{2\pi i}\oint_{C_j}\prod_{i=1}^m \frac{x^{-n-1}}{(1-\zeta^i x)^{b_i}}dx$$

where C_j is a small circle around ζ^{-j}. Here we use the fact that the residue at ∞ is 0, given that $n \geq 0$. In order to compute the term $\frac{1}{2\pi i}\oint_{C_j}\prod_{i=1}^m \frac{x^{-n-1}}{(1-\zeta^i x)^{b_i}}dx$ we perform a change of coordinates $x = w + \zeta^{-j}$ obtaining:

$$\frac{1}{2\pi i}\oint_{C_j}\prod_{i=1}^m \frac{(w+\zeta^{-j})^{-n-1}}{(1-\zeta^{i-j}-\zeta^i w)^{b_i}}dw.$$

Now $\prod_{i=1,\ i\neq j}^m \frac{1}{(1-\zeta^{i-j}-\zeta^i w)^{b_i}}$ is holomorphic around 0 and we can explicitly expand in power series $\sum_{h=0}^\infty a_{j,h}w^h$ while:

$$(w+\zeta^{-j})^{-n-1} = \zeta^{j(n+1)}\sum_{k=0}^\infty (-1)^k\binom{n+k}{k}(\zeta^j w)^k.$$

Finally we have, for the $j-$th term:

$$-\frac{(-1)^{b_j}}{2\pi i}\oint_{C_j}\zeta^{j(n+1-b_j)}\left(\sum_{k=0}^\infty(-1)^k\binom{n+k}{k}\zeta^{jk}w^k\right)\left(\sum_{h=0}^\infty a_{j,h}w^h\right)w^{-b_j}dw$$

$$= -(-1)^{b_j}\zeta^{j(n+1-b_j)}\sum_{k+h=b_j-1}(-1)^k\zeta^{jk}\binom{n+k}{k}a_{j,h}.$$

Summing over j we obtain an explicit formula for $S_{\underline{h}}(b)$, again as a *quasi–polynomial*.

Remark that, in order to develop these formulae it suffices to compute a finite number of coefficients $a_{j,h}$..

In these formulae appear roots of unity while the final partition functions are clearly integer valued. An algorithmic problem remains. When we write out our expressions we get, for each coset, a polynomial that takes integer values but that a priori is expressed with coefficients that are expressions in the root ζ. We need to know how to manipulate such an expression. This is an elementary problem but it has a certain level of complexity and requires some manipulations on cyclotomic polynomials. We shall explain in **??** how to sum over roots of unity by computing so called Dedekind sums.

afternoon

OFF

2.4 Day 4

2.4.1 Claudio 4

In this lecture, the word $D-$module is used to denote a module over one of the two Weyl algebras $W(V), W(U)$ of differential operators with polynomial coefficients on V, U respectively. The purpose of this chapter is to determine an expansion in partial fractions of the rational function on the complement of a hyperplane arrangement. This is done using the $D-$module structure of the algebra of regular functions. Finally by inverse Laplace transform all this is interpreted as a calculus on the corresponding distributions.

2.4.2 A prototype $D-$module

We start by considering, for some $t \leq s$, the algebra of partial Laurent polynomials

$$L := \mathbb{C}[x_1^{\pm 1}, \ldots, x_t^{\pm 1}, x_{t+1}, \ldots, x_s]$$

as a module over the Weyl algebra $W(s)$.

A linear basis of L is given by the monomials $M := x_1^{h_1} \ldots x_s^{h_s},\ h_i \in \mathbb{Z}, \forall i \leq t, h_i \in \mathbb{N}, \forall i > t$. For such a monomial M define $p(M) := \{i \,|\, h_i < 0\}$, the *polar set of* M and, for any subset $A \subset \{1, \ldots, t\}$ let L_A be the subspace with basis the monomials M with $p(M) \subset A$.

Define furthermore $L_k := \sum_{|A| \leq k} L_A$.

Proposition 2.4.3. *L_A is a $W(s)$ submodule for each A.*

L_k/L_{k-1} has as basis the classes of the monomials M with $|p(M)| = k$.

For each A let W_A be the space defined by the linear equations $x_i = 0,\ \forall i \in A$, we shall denote by N_A the module N_{W_A}.

Define M_A to be the subspace of L_k/L_{k-1} with basis the monomials M with $p(M) = A$ then:

Theorem 2.4.4. $L_k/L_{k-1} = \oplus_{|A|=k} M_A$

M_A is canonically isomorphic to N_A where the basic generator δ_A is the class of $1/\prod_{i \in A} x_i$.

Proof. All the statements are essentially obvious. Since L_k has as basis the monomials M with $|p(M)| \leq k$ and L_{k-1} has as basis the monomials M with $|p(M)| \leq k-1$, clearly L_k/L_{k-1} has as basis the classes of the monomials M with $|p(M)| = k$. These in turn distribute among the various subsets A of cardinality k.

The class δ_A of $\frac{1}{\prod_{i\in A} x_i}$ clearly satisfies the equations $x_i\delta_A = 0, \forall i \in A$ and $\frac{\partial}{\partial x_j}\delta_A = 0,\ \forall j \notin A$, so it generates a submodule isomorphic to N_A. Given $\prod_{j\notin A} x_j^{h_j} \prod_{i\in A} \frac{\partial^{k_i}}{\partial x_i}, h_i, k_i \geq 0$ we have explicitly

$$\prod_{j\notin A} x_j^{h_j} \prod_{i\in A} \frac{\partial^{k_i}}{\partial x_i} \frac{1}{\prod_{i\in A} x_i} = \prod_{j\notin A} x_j^{h_j} \prod_{i\in A} (-1)^{h_i} h_i! x_i^{-1-h_i}$$

giving the corresponding basis for N_A.

□

We can apply a linear change of coordinates and a translation to obtain a similar theorem with any basis a_i and elements μ_i for the corresponding algebra $S[V][\prod_{i=1}^s (a_i + \mu_i)^{-1}]$. We get a filtration by polar order with successive quotients direct sum of the modules corresponding again to subsets A but now associated to an affine linear subspace of equations $a_i + \mu_i = 0,\ \forall i \in A$.

2.4.5 Ramadas 4

2.4.6 The filtration by polar order

Our goal is to describe the algebra $R_{X,\underline{\mu}}$ as a module over the Weyl algebra.

In order not to have a very complicated notation, unless there is a risk of confusion we shall simply write $R = R_{X,\underline{\mu}}$.

As preliminary take a subset $Y \subset X$ and set $\underline{\mu}(Y)$ to be the family $\underline{\mu}$ restricted to Y. We have the inclusion of algebras $R_{Y,\underline{\mu}(Y)} \subset R_{X,\underline{\mu}}$, and clearly this is an inclusion of $W(U)$ modules.

Let us now introduce a filtration in $R = R_{X,\underline{\mu}}$ by $D-$submodules which we will call the *filtration by polar order.*

This is defined algebraically as follows. One puts in filtration degree $\leq k$ all the fractions $f \prod_{a\in X}(a+\mu_a)^{-h_a},\ h_a \geq 0$ for which the set of vectors a, with $h_a > 0$, spans a space of dimension $\leq k$.

In other words:

$$R_k = \sum_{Y\subset X,\ \dim\langle Y\rangle\leq k} R_{Y,\underline{\mu}(Y)}. \tag{2.20}$$

Notice that $R_s = R$.

Theorem 2.4.7. *R_k/R_{k-1} is a direct sum of copies of Fourier transforms of the modules N_W, as W runs over the subspaces of the arrangement of codimension k.*

For given W the isotypic component of M_W is canonically isomorphic to $N_W \otimes \Theta_W$ where Θ_W is the vector space spanned by the classes of the elements $\prod_{a\in \underline{c}}(a+\mu_a)^{-1}$ as $\underline{c}$ runs on all the bases of X_W.

The space Θ_W has as basis the classes of the elements $\underline{c} \subset X_W$ as $\underline{c}$ runs on all the unbroken bases in X_W.

Proof. We prove the statement by induction on k. For $k = 0$ we have as only space of codimension 0 the entire space V. Then $R_0 = S[V] = N_V$ as desired.

Next, using the expansion 2.20 and Theorem 1.3.26 it follows that:

$$R_k = \sum_{W,\underline{c}} R_{\underline{c},\underline{\mu}(\underline{c})}$$

as W runs over the subspaces of the arrangement of codimension $\leq k$ and $\underline{c}$ over the unbroken bases in the subset X_W.

Consider, for each $W, \underline{c}$, with codimension of $W = k$, the map $R_{\underline{c},\underline{\mu}(\underline{c})} \subset R_k \to R_k/R_{k-1}$.

Clearly this factors to a map:

$$i_{\underline{c}} : (R_{\underline{c},\underline{\mu}(\underline{c})})_k/(R_{\underline{c},\underline{\mu}(\underline{c})})_{k-1} \to R_k/R_{k-1}$$

By the discussion in section 2.4.2 we know that $(R_{\underline{c},\underline{\mu}(\underline{c})})_k/(R_{\underline{c},\underline{\mu}(\underline{c})})_{k-1}$ is identified to the irreducible module N_W, thus either the map $i_{\underline{c}}$ is an injection or it is 0. In this last case, the module N_W appears in a composition series of the module R_{k-1} By induction, R_{k-1} has a composition series for which each irreducible factor has as characteristic variety the conormal space to some subspace of codimension $\leq k-1$.

If $i_{\underline{c}} = 0$ then N_W is isomorphic to one of these factors. This is a contradiction, since the characteristic variety of N_W is the conormal space to W, a space of codimension k. It follows that $i_{\underline{c}}$ is an injection.

As a consequence we can deduce at least that

$$R_k/R_{k-1} = \sum_{W,\underline{c}} Im(i_{\underline{c}})$$

as W runs over the subspaces of the arrangement of codimension k and $\underline{c}$ over the unbroken bases in the subset X_W.

Since, for two different W_1, W_2 the two modules N_{W_1}, N_{W_2} are not isomorphic it follows that, for given W the sum $\sum_{\underline{c}} Im(i_{\underline{c}})$ as $\underline{c}$ runs over the unbroken bases in the subset X_W gives the isotypic component of type N_W of the module R_k/R_{k-1}, and this module is the direct sum of these isotypic components. Thus the first two parts of the theorem are proved and it only remains to check that the sum $\sum_{\underline{c}} Im(i_{\underline{c}})$ is direct.

For this, using the third part of corollary 1.3.35 (which as noted extends to N_W), it is enough to verify the condition that the classes in R_k/R_{k-1} of the elements $\prod_{a\in\underline{c}}(a+\mu_a)^{-1}$ are linearly independent, as $\underline{c}$ runs over the unbroken bases of X_W. This last point is non trivial and requires a special argument, which we prove separately in the next Proposition.

□

Proposition 2.4.8. *For given W of codimension k, the classes of the elements $\prod_{a\in\underline{c}}(a+\mu_a)^{-1}$ as $\underline{c}\subset X_W$ runs over the unbroken bases are linearly independent modulo R_{k-1}*

The proof of this proposition is not simple and we omit it.

afternoon

2.4.9 Rik Enrico computer lab

2.5 Day 5

2.5.1 I suggest that we use this day to review the entire program

ILLUSTRATIVE EXAMPLE $a=6,\ b=3,\ c=2,\ A=1,\ B=2,\ C=3$:

$$(6abc)^{-1}\{m^3+3\frac{a+b+c+1}{2}m^2+6[\frac{a^2+b^2+c^2+1}{12}+\frac{ab+bc+ac+a+b+c}{4}]m\}+$$
$$[\frac{\sigma(A;a)}{bc}+\frac{\sigma(B;b)}{ac}+\frac{\sigma(C;c)}{ab}]m+1$$

$$\sigma(A;a):=\sum_{k\mid A}\sum_{\zeta\in\Gamma_k}\frac{1}{1-\zeta^{-1}}\frac{1}{1-\zeta^{-a}}=\sum_{k\mid A}\sum_{\zeta\in\Gamma_k}\frac{1}{1-\zeta}\frac{1}{1-\zeta^{a}}.$$

I conti espliciti:

$$\{1+3\frac{6+3+2+1}{2}6^{-1}+6^{-1}[\frac{6^2+3^2+2^2+1}{12}+\frac{18+6+12+6+3+2}{4}]\}+$$
$$[\frac{\sigma(A;6)}{6}+\frac{\sigma(2;3)}{12}+\frac{\sigma(3;2)}{18}]6+1$$

$$\{1+3\frac{12}{2}6^{-1}+6^{-1}[\frac{36+9+4+1}{12}+\frac{47}{4}]\}+$$

$$[\frac{\sigma(A;6)}{6}+\frac{\sigma(2;3)}{12}+\frac{\sigma(3;2)}{18}]6+1$$

$$\{1+3\frac{12}{2}6^{-1}+6^{-1}[\frac{50}{12}+\frac{47}{4}]\}+$$

$$[\frac{\sigma(1;6)}{6}+\frac{\sigma(2;3)}{12}+\frac{\sigma(3;2)}{18}]6+1$$

$$\{5+6^{-1}[\frac{50}{12}+\frac{47}{4}]\}+$$

$$[\frac{\sigma(1;6)}{6}+\frac{\sigma(2;3)}{12}+\frac{\sigma(3;2)}{18}]6$$

$$\{5+6^{-1}[\frac{50}{12}+\frac{47}{4}]\}+$$

$$[\frac{\sigma(1;6)}{6}+\frac{\sigma(2;3)}{12}+\frac{\sigma(3;2)}{18}]6$$

$$\sigma(1;6)=0,\ \sigma(2;3)=1/4,\ \sigma(3;2)=2/3$$

$$5+6^{-1}[\frac{50}{12}+\frac{141}{12}]+$$

$$[\frac{1/4}{12}+\frac{2/3}{18}]6$$

$$5+\frac{191}{72}+\frac{1}{8}+\frac{2}{9}$$

$$5+\frac{191}{72}+\frac{9}{72}+\frac{16}{72}$$

$$5+\frac{191+9+16}{72}$$

$$5+\frac{216}{72}=8$$

$$m_{a\nmid b}$$

$$d_{\underline{b},\phi}(x) := \frac{e(\phi)(x)}{\prod_{a\in\underline{b}}(1-e^{-\langle a\,|,\phi+x\rangle})} = \frac{1}{\prod_{a\in\underline{b}}\langle a\,|,\phi+x\rangle}F(x)$$

Example $n = 2h + 3k$.

$2, 3$

the quasi polynomials are

0	$\frac{n}{6} + 1$
1	$\frac{n}{6} - \frac{1}{6}$
2	$\frac{n}{6} + \frac{2}{3}$
3	$\frac{n}{6} + \frac{1}{2}$
4	$\frac{n}{6} + \frac{1}{3}$
5	$\frac{n}{6} + \frac{1}{6}$

The roots of the arrangement are

$$1,\ -1,\ \zeta,\ \zeta^2, \quad \zeta = e^{2\pi i/3}$$

The multivarite splines are:

$$x/6,\ 1/2,\ 1/3$$

the operator at 1 is (set $d := \frac{d}{dx}$)

$$\frac{2d}{1 - e^{-2d}} \frac{3d}{1 - e^{-3d}} = (1 + d + \dots)(1 + 3/2d + \dots) = 1 + 5/2d + \dots$$

which applied to $x/6$ gives $x/6 + 5/12$.

The local operator at 2 is

$$(-1)^x \frac{1}{1 - (-1)^3 e^{-3d}} = (-1)^x/2 + \dots$$

which applied to $1/2$ gives $(-1)^x/4$.

The local operator at 3 is

$$\zeta^x \frac{1}{1 - \zeta^{-2} e^{-2d}} + \zeta^{2x} \frac{1}{1 - \zeta^{-4} e^{-2d}} = \zeta^x \frac{1}{1 - \zeta^{-2}} + \zeta^{2x} \frac{1}{1 - \zeta^{-4}} + \dots$$

we have

$$\zeta^x \frac{1}{1 - \zeta^{-2}} + \zeta^{2x} \frac{1}{1 - \zeta^{-4}} = \zeta^x \frac{1}{1 - \zeta} + \zeta^{2x} \frac{1}{1 - \zeta^2} = 1/3[\zeta^x(1 - \zeta^2) + \zeta^{2x}(1 - \zeta)]$$

Applying this to 1/3 we get

0	$\frac{2+1}{9} = \frac{1}{3}$
1	$\frac{-2-1}{9} = -\frac{1}{3}$
2	$\frac{-1+1}{9} = 0$
3	$\frac{2-(-1)}{9} = \frac{1}{3}$
4	$\frac{-2-1}{9} = -\frac{1}{3}$
5	$\frac{-1+1}{9} = 0$

thus we have to add to $x/6$ the constant

0	$\frac{5}{12} + \frac{1}{4} + \frac{1}{3}$	$= 1$
1	$\frac{5}{12} - \frac{1}{4} - \frac{1}{3}$	$= -\frac{1}{6}$
2	$\frac{5}{12} + \frac{1}{4}$	$= \frac{2}{3}$
3	$\frac{5}{12} - \frac{1}{4} + \frac{1}{3}$	$= \frac{1}{2}$
4	$\frac{5}{12} + \frac{1}{4} - \frac{1}{3}$	$= \frac{1}{3}$
5	$\frac{5}{12} - \frac{1}{4}$	$= \frac{1}{6}$

in accord with the previous computation.

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