POLYTOPES
VOLUME AND INTEGER POINTS
REF: MATH AG/0505351, MATH.AG/0406290, CO/0412130

NOTES BY CLAUDIO PROCESI.

INTRODUCTION

These notes are an expansion of a short *corso di eccellenza* I held in September 2005 in the "Dipartimento di Produzione ed Economia dell'Azienda" of Politecnico di Torino.

My interest in the subject arose from conversations with Velleda Baldoni and Michelle Vergne (and reading [4]),

Together with Corrado De Concini we found a tie between this theory and the theory of wonderful models of hyperplane arrangements. From this we wrote some papers partly exposed in these notes.

The notes were written for a computer presentation thus retain the informal structure of such presentation.

Date: 2005.

INDEX

Contents

Part 1. Partition Function

1. Convex polytopes

We are interested in dealing with two fundamental problems under many point of view. The problem can be formulated in various different ways.

We begin with a system of linear equations:

$$\sum_{i=1}^{m} \alpha_i x_i = b, \quad \text{o} \quad Ax = b, \quad A := (\alpha_1, \dots, \alpha_m) \tag{1}$$

The *columns* α_i, b are vectors with n coordinates

$$(\alpha_{j,i}, b_j, \ j = 1, \dots, n).$$

Let us assume first that the matrix A is $n \times m$ and with real coefficients.

The object of study As in Linear Programming Theory we want to study the set $P_A(b)$ of solutions of (1) where the coordinates $x_i \geq 0$ are not negative real numbers.

We shall also study the more difficult case in which we assume that: The elements of the matrix A and of the given vector b are integers, i.e.:

$$\text{(column vectors) } \alpha_1, \dots \alpha_m, b \in \mathbb{Z}^n.$$

Arithmetic case In this case we want compute the number $S_A(b)$ of solutions of the system $(x_1, \dots, x_m)$ in which the coordinates x_i are non negative integers.

The problems we intend to study are interesting only if we assume that the set:

- $$P_A(b) := \{x \mid Ax = b, x_i \geq 0\}$$
  is *bounded* (or compact) for every b.
- In any case $P_A(b)$ is obviously convex. If it is also bounded it is to *convex bounded polyhedron*.
- In this case its volume is finite and, if A, b have integer coordinates, the set $S_A(b)$ consists of points in $P_A(b)$ with integer coordinates. Hence it is obviously a finite set.

Usually one assumes that the columns α_i generate the space $\mathbb{R}^n$ (otherwise one can easily reduce oneself to this case).

dimension: If there exists at least a solution with all coordinates strictly positive, the polyhedron $P_A(b)$ has *dimension* $m-n$, in the $m-n$ dimensional space V_b, of all solutions of $Ax = b$.

There exists a simple necessary and sufficient criterion which guarantees that the polyhedron $P_A(b)$ is always compact. We must have that:

Criterion 1.1. *There exist numbers* $\ell = (\ell_1, \dots, \ell_n)$ *such that*

$$\sum_{j=1}^{n} \ell_j \alpha_{j,i} > 0, \quad \forall i = 1, \dots m. \tag{2}$$

for every $i = 1, \dots m$*.*

In this case a necessary (but not sufficient) restriction on b for $P_A(b)$ to be not empty and not reduced to $\{0\}$ is that

$$\sum_{j=1}^{n} \ell_j b_j > 0$$

Remark 1.2. *One and only one of the following possibilities can occur for a matrix* A *with columns* α_i*:*

(1) *There exists a row vector* ℓ *such that* ℓA *has positive coordinates.*
(2) 0 *is a convex linear combination of the columns* α_i*.*

Moreover 1. is equivalent to the existence of a hyperplane containing the origin such that all the columns α_i *are in the same side with respect to that hyperplane, i.e. the convex hull of the points* α_i *does not intersect the hyperplane.*

In other words the vectors $\alpha_1, \dots \alpha_m$ define a map:

$$f : \mathbb{R}^m \to \mathbb{R}^n.$$

$$f(a_1, \dots a_m) := a_1\alpha_1 + \cdots + a_m\alpha_m.$$

Therefore, in this language, the polytope:

$$P_A(b) = f^{-1}(b) \cap \mathbb{R}^{m+},$$

where $\mathbb{R}^{m+}$ denotes the positive quadrant.

Our goals are therefore:

(1) 1. To compute the volume of $P_A(b)$.
(2) 2. If A, b have integer elements, to compute the number $S_A(b)$ of points with integer coordinates in $P_A(b)$.

Obviously it is possible to compute such a number $S_A(b)$ even if A, b do not have integer coordinates.

However the problem becomes extremely complicated and it is out of our scope.

When A, b have integer elements it is natural to think of an expression like:

$b = a_1\alpha_1 + \cdots + a_m\alpha_m$ with a_i not negative integers as a:

partition of b with the vectors α_i, in $a_1 + a_2 + \cdots + a_m$ parts, hence the name *partition function* for the number $S_A(b)$, thought of as a function of the vector b.

Partition functions appear in many different fields of mathematics, like representation theory and statistical mechanics.

2. Variable polihedron

From a geometric point of view it is better to think of the variable polyhedron $P_A(b) = f^{-1}(b) \cap \mathbb{R}^{m+}$ as a *variable polyhedron in a single space, the space $f^{-1}(0)$.*

rather than a polyhedron in the variable subspace $f^{-1}(b)$.

This can be done if we give a canonical way to identify the spaces $f^{-1}(b)$ ad $f^{-1}(0)$.

A geometrical useful way to do it is to choose in $\mathbb{R}^m$ a subspace W complementary to $f^{-1}(0)$, for example the orthogonal complement.

In this way we have to linear function $i : \mathbb{R}^n \to \mathbb{R}^m$ whose image is W such that $f \circ i$ is the identityof $\mathbb{R}^n$. By using this map we identify:

$$f^{-1}(b) = f^{-1}(0) + i(b),$$

$$P_A(b) - i(b) \subset f^{-1}(0)$$

the variable polytope in $f^{-1}(0)$.

In the arithmetic case we must choose i so that $i(b)$ has integer coordinates if the same is true for b.

Digression:

- In general a convex polytope is defined as the convex hull of a finite number of points.
- Such to polytope can also be defined as an envelope i.e. with to finite number of linear inequalities$\sum_i a_{j,i} x_i \geq b_i,\ j = 1, \dots, n.$

If we assume that the polytope is contained in the positive quadrant we obtain immediately our chosen representation.

In fact, if we add a variable y_j for each inequality, we can reformulate the inequality as:

$$\sum_i a_{j,i} x_i - y_i = b_i,\ y_i \geq 0,\ j = 1, \dots, n.$$

This is the way we have chosen, to represent polytopes.

Part 2. Polytopes and networks

3. Oriented graphs

We discuss now a rich class of examples.

Let $\Gamma := (V, L)$ be an oriented graph.

- V the set of its vertices
- L the set of edges.
- The orientation consists in assuming that each edge $a \in L$ has an initial vertex $i(a)$ and a final vertex $f(a)$.

Let us assume moreover that there are no edges with $i(a) = f(a)$ and that at most one edge connect two vertices.

In order to associate, to such graphs, polytopes we define:

The associated space and vectors

(1) Let us build a vector space with basis the elements e_v as v varies in the vertices, $v \in V$.
(2) Let us consider the set of vectors $\Delta_\Gamma := \{x_a := e_{f(a)} - e_{i(a)}\}$ for $a \in L$.
(3) Let us denote by V_Γ the space generated by the vectors x_a.

Lemma 3.1. *The vectors x_a generate to space V_Γ of dimension $|V| - b_0$ where:*

- *$|V|$ is the number of elements of V*
- *b_0 is the number of connected components of Γ.*

Proof Since the spaces spanned by the vectors in the different connected components form to direct sum, one reduces immediately to the case Γ connected, i.e. $b_0 = 1$. In this case the vectors generate a subspace U spanned by vectors for which the sum of the coordinates in the basis e_v is 0.

We claim that U coincides with this subspace. In fact, let us choose to vector e_v and add it to U. By connection every $e_w \in U + Re_v$ hence the result.

Remark 3.2. *If Γ is connected:*

$|V| - 1$ edges a_i are such that the vectors x_{a_i} are to basis of V_Γ if and only if these edges generate to maximal subtree.

If, with the given orientation, the vectors x_a generate a pointed cone, we shall speak of a network and we can define the polytopes associated to the network.

Next proposition gives us an idea of when an orientation produces a network and when one can make such a construction on a given graph.

Proposition 3.3. *We have:*

(1) A way to get to network consists in arbitrarily fixing to total ordering on all the vertexes and orienting all the edges according to this ordering.
(2) An oriented graph is to network if and only if it does not contain any closed oriented circuit.
(3) In each network we can order the vertices in a way compatible with the ordering.

Proof.

(1) Let us suppose to have a total ordering on the vertices. Take a vector α with strictly increasing coordinates with respect to this ordering. We have therefore that the scalar product of α with every x_a is strictly positive.

(2) An oriented circuit $a_1, \dots, a_k$ gives vectors $x_{a_1}, \dots, x_{a_k}$ such that $x_{a_1} + \cdots + x_{a_k} = 0$ i.e. 0 is a convex combination of these vectors, hence we do not have a network.
(3) Conversely if there are no circuits, take a maximal oriented chain $v_1, \dots, v_k$; v_1 is a source, otherwise we can get a longer chain. Take this source as minimal vertex, remove such source (and all the edges originating from it) hence proceed recursively.

4. Two examples:

(1) Let us take the complete graph over the vertices $1, 2, \dots, n$ oriented according to the given orientation.

Hence the vectors x_a are all vectors $e_i - e_j,\ i < j$.

This object appears also as the set of positive roots for type A_{n-1}.

(2) Let us take now two disjoint sets A, B with h and k elements respectively. We take the oriented graph with vertices $A \cup B$ and edges given by all edges that join a point A (initial) and one of B (final).

Thus the vectors x_a are all the elements$e_b - e_a,\ b \in B,\ a \in A$.

A vector $\sum_{a\in A,\ b\in B} r_{b,a}(e_b - e_a)$ of the space having as basis these vectors, can also be thought as a

matrix R with row indices B and column indexes A.

Let us consider the vector $r(\sum_{a\in A} e_a) + s(\sum_{b\in B} e_b)$ with r, s positive numbers.

The condition $\sum_{a\in A,\ b\in B} r_{b,a}(e_b - e_a) = r(\sum_{a\in A} e_a) + s(\sum_{b\in B} e_b)$ asserts that:

the matrix R is a kind of magic rectangle

- That is the sums over the rows have value s and those over the columns have value r.
- In particular when A, B have the same number of elements we have a magic square.

Part 3. Generating Functions

5. The positive cone

Obviously, the set of vectors b such that $P_A(b)$ is not empty is equal, by definition, to the set:

$$C_A := \{\sum_{i=1}^{m} x_i \alpha_i \mid x_i \geq 0\} \tag{3}$$

Obviously C_A is to convex cone. The condition 2 means that the cone C_A minus the point 0, is all contained in the half space $\sum_{i=1}^{n} \ell_i y_i > 0$.

C_A is usually said to be to pointed cone and 0 is its vertex.

It is useful to consider also the dual cone $\hat{C}_A$ of C_A.

In the dual space it consists of the (row) vectors that have non negative scalar product with all vectors α_i. I.e. with all vectors of the cone C_A. Under the hypothesis (2), $\hat{C}_A$ is obviously a cone with non empty interior.

6. Integer points

When A, b have integers elements there is to simple formula for the numbers $S_A(b)$:

Theorem 6.1. *The number $S_A(b)$ is the coefficient of $\prod_i x_i^{b_i}$ in the expansion in power series of the rational function:*

$$\prod_j (1 - \prod_{k=1}^n x_k^{a_{k,j}})^{-1} = \prod_j (\sum_{h=0}^{\infty} \prod_{k=1}^n x_k^{h a_{k,j}}). \tag{4}$$

Remark 6.2. *Let us note that expression 4 does not only make sense formally but also converges in the region $\mathcal{C}$ of spacewhere $\prod_{k=1}^r x_k^{a_{k,j}} < 1, \ \forall j$.*

In logarithmic coordinates $x_k = e^{\theta_k}$.

Given a vector $\alpha := (a_1, \ldots, a_n)$ with integer coordinates set e^{α}:

$$e^{\alpha}(\theta_1, \ldots, \theta_n) := e^{\sum_{j=1}^n a_j \theta_j}$$

Of course we have, given vectors α, β and integers m, n:

$$e^{m\alpha + n\beta} = (e^{\alpha})^m (e^{\beta})^n.$$

In logarithmic coordinates the region $\mathcal{C}$ where $\sum_k \theta_k a_{k,j} < 0$.
is the interior of the dual cone.
Therefore we have, in the interior of the dual cone, the formula which we need to invert in order to compute the number of integer points: Formula to be Invered

$$\prod_{i=1}^m \frac{1}{1 - e^{\alpha_i}} = \sum_{b \in \mathbb{Z}^n} S_A(b) e^b. \tag{5}$$

Even if this formulation is simple, the inversion problem is not at all simple, as we shall see.

7. Laplace transform

From now on we shall use the notations

- $A = (\alpha_1, \ldots, \alpha_m)$ the list of vectors.
- $C_A := \{\sum_i x_i \alpha_i, \ x_i \geq 0\}$, the cone they generate¿

We shall assume always that C_A is pointed and generates the ambient space V.

The dual cone $\hat{C}_A$ is pointed and generates the dual space V^*.

We fix on V a standard Lebesgue measure that in suitable coordinates one writes as $dx := dx_1 dx_2 \dots dx_n$.

The Laplace transform formally maps suitable functions on V to functions on V^* and is defined as: Laplace transform

$$Lf(y) := \int_V e^{-(y,x)} f(x) dx. \tag{6}$$

We apply the Laplace transform, in particular, to suitable functions with support in the cone C_A.

Remark 7.1.
- *In this case we define and compute the Laplace transform only for $y \in \hat{C}_A$.*
- *Only for these values of y the integral converges.*

We want apply the Laplace transform to the function $V_A(b)$ that gives the volume of the polytope $P_A(b)$.

Let us remark that this function is continuous and has support in the cone C_A.

In order to compute this transform let us go back to the functional formulation:

$$f_A : \mathbb{R}^m \to \mathbb{R}^n.$$
$$f_A(t_1, \dots t_m) = t_1\alpha_1 + \cdots + t_m\alpha_m.$$

We proceed in 3 steps

(1) We restrict f_A to the positive quadrant $\mathbb{R}^{m+}$ hence: $P_A(b) = f_A^{-1}(b)$.
(2) Apply now the theorem of Fubini to the function $e^{-(y,f_A(t))}\chi_+$, where χ_+ is the characteristic function of $\mathbb{R}^{m+}$.
(3) We must normalize the euclidean measures so that the one of $\mathbb{R}^m$ is the product of that of the basis $\mathbb{R}^n$ times that of the fiber.

Let us apply the theorem of Fubini to the function

$$\chi_+ e^{-(y,f_A(t_1,\dots t_m))} = \chi_+ e^{-\sum_{i=1}^m t_i(y,\alpha_i)}$$

and obtain:

Generating Function

$$\prod_{i=1}^m \frac{1}{(y,\alpha_i)} = \int_{\mathbb{R}^{m+}} e^{-(y,f(t_1,\dots t_m))} dt =$$

$$\int_{\mathbb{R}^n} (\int_{f^{-1}(x)} e^{-(y,x)}) dx = k \int_{\mathbb{R}^n} e^{-(y,x)} V_A(x) dx. \tag{7}$$

With k normalization constant to be computed. This analysis is certainly valid when we take y in the interior of the dual cone.

In other words the Laplace transform of the volume is defined in the interior of the dual cone and equals the rational function:

$$k^{-1}\prod_{i=1}^{m}\frac{1}{\sum_{k=1}^{n}a_{k,j}y_k}$$

IN CONCLUSION For the computation of the number of integral points and for the volume we must solve an *inversion problem*.

In order to do this let us observe that, with an orthogonal transformation U we can bring the matrix A in block form $AU = (B, 0)$, with B an $n \times n$ invertible matrix.

We think of the map f_A as a composition:

$$\mathbb{R}^m \xrightarrow{U} \mathbb{R}^m \xrightarrow{\pi} \mathbb{R}^n \xrightarrow{B} \mathbb{R}^n$$

π is the projection of matrix $(1_n, 0)$.

It follows that $\int_{\mathbb{R}^n} g(Bx)dx = |\det(B)| \int_{\mathbb{R}^n} g(x)dx$ hence that $k = |\det(B)|$. Finally $AA^t = (AU)(AU)^t = BB^t$ hence the final formula:

$$k^{-1} = |\det(B)| = \sqrt{|\det(AA^t)|}.$$

Final formula Thus the formula to invert, for the computation of the volume:

$$\int_{\mathbb{R}^n} e^{-(y,x)}V_A(x)dx = \sqrt{|\det(AA^t)|}\prod_{i=1}^{m}\frac{1}{\sum_{k=1}^{n}a_{k,i}y_k} \tag{8}$$

Part 4. The knapsack problem

8. The case of numbers: volume

Before attacking the general case let us discuss the case in which the vectors α_i are simply positive numbers, that is when $n = 1$:

We have thus positive integers $\underline{h} := (h_1, \ldots, h_m)$, and we want, given an integer n to compute:

1 The function with Laplace transform $\frac{|\underline{h}|}{\prod h_i}y^{-m}$ cf. (15).
2 The coefficient of x^n in the function

$$F_{\underline{h}}(x) := \prod_i \frac{1}{1 - x^{h_i}}. \tag{9}$$

The computation of 1. is rather easy:

Remarks 8.1. • *here the constant* $k = |\underline{h}| = \sqrt{\sum_i h_i^2}$.

- *The polyhedron $P_{\underline{h}}(b)$ associated to the numbers h_i, b is a simplex, given by:*

$$\{(x_1, \dots, x_m) \,|\, x_i \geq 0, \ \sum_i x_i h_i = b.\}$$

- *$P_{\underline{h}}(b)$ is the convex envelop of the vertices $P_i = (0, 0, \dots, 0, b/h_i, 0, \dots, 0)$.*

SUMMARIZING

We could compute the volume of $P_{\underline{h}}(b)$ directly but let us compute the *function whose Laplace trasform is y^{-m}.*

Let us use the following formulae:

$$\frac{d}{dy} Lf = L(-xf), \quad \int_0^\infty e^{-yx} dx = \frac{1}{y}. \tag{10}$$

from these we have that, if χ denotes the characteristic function of the half line $x \geq 0$ we have: Requested Formula

$$L((-x)^k \chi) = (-1)^k k! y^{-k-1}.$$

IN CONCLUSION:

Theorem 8.2. *The volume of $P_{\underline{h}}(b)$ is given by*

$$\boxed{\frac{|\underline{h}| b^{m-1}}{(m-1)! \prod h_i}}.$$

9. Decomposition of a number

Let us pass now to the more difficult case 2.

We must compute, as n varies, the coefficient of x^n in the function $\prod_i \frac{1}{1-x^{h_i}}$. In other words of x^{-1} in $\frac{x^{-n-1}}{\prod_i 1-x^{h_i}}$.

We have two possible strategies, essentially equivalent but to be analyzed separately from the algorithmic point of view.

1. Develop $F_{\underline{h}}(x)$ in partial fractions.

2. Compute the residue $\frac{1}{2\pi i} \oint \frac{x^{-n-1}}{\prod_i 1-x^{h_i}} dx$ around 0.

In both cases first we must expand in a suitable way the function $F_{\underline{h}}(x)$. Proceed with the following steps:

(1) Given k let us denote by $\zeta_k := e^{\frac{2\pi i}{k}}$, a $k-$th root of 1.
(2) Let us recall the identity:

$$1 - x^k = \prod_{i=0}^{k-1} (1 - \zeta_k^i x) = \prod_{i=0}^{k-1} (\zeta_k^i - x).$$

(3) Using the preceding identity we have:

$$F_{\underline{h}}(x) := \prod_{i=1}^{m}\prod_{j=1}^{h_i}\frac{1}{\zeta_{h_i}^j - x} = \prod_{i=0}^{m-1}\prod_{j=1}^{h_i}\frac{1}{1-\zeta_{h_i}^j x}. \tag{11}$$

In other words let m be the least common multiple of the numbers h_i and set $m = h_i k_i$. We set $\zeta = e^{2\pi i/m}$, we have $\zeta_{h_i} = \zeta^{k_i}$ therefore we have

Lemma 9.1.

$$\prod_i \frac{1}{1-x^{h_i}} = \prod_{i=0}^{m-1}\prod_{j=1}^{h_i}\frac{1}{1-\zeta^{k_i j}x} = \prod_{i=0}^{m-1}\frac{1}{(1-\zeta^i x)^{b_i}}$$

where the integers b_i can be computed easily from the numbers h_j.

- Thus given i we must count how many numbers k_i are divisors of i. This number is b_i.
- In particular the function $\prod_i \frac{x^{-n-1}}{1-x^{h_i}}$, $n \geq 0$ has poles in 0 and in the $m-$th roots of 1 (but not in ∞)

10. First method, development in partial fractions.

The classical method implies that there exist numbers c_i for which:

$$\prod_{i=0}^{m-1}\frac{1}{(1-\zeta^i x)^{b_i}} = \sum_{i=0}^{m-1}\frac{c_i}{(1-\zeta^i x)^{b_i}}. \tag{12}$$

In order to compute them we use for instance recursively the simple identity, (valid if $a \neq b$ are two numbers):

$$\frac{1}{(1-ax)(1-bx)} = \frac{1}{a-b}\Big[\frac{a}{(1-ax)} - \frac{b}{(1-bx)}\Big]$$

The second step is the simple identity:

$$\frac{1}{(1-t)^k} = \sum_{h=0}^{\infty}\binom{k-1+h}{h}t^h \tag{13}$$

hence we get:

$$\prod_{i=0}^{m-1}\frac{1}{(1-\zeta^i x)^{b_i}} = \sum_{i=0}^{m-1} c_i\Big(\sum_{h=0}^{\infty}\binom{b_i-1+h}{h}(\zeta^i x)^h\Big). \tag{14}$$

In other words we have a formula for the coefficient

$$S_{\underline{h}}(b) = \sum_{i=0}^{m-1} c_i(\zeta^{ib}) \binom{b_i - 1 + b}{b}.$$

Let us remark now that

$$\binom{b_i - 1 + b}{b} = \frac{(b+1)(b+2)\dots(b+b_i-1)}{(b_i-1)!}$$

is a polynomial of degree $b_i - 1$ in b, while the numbers ζ^{ib} depend only from the coset of b modulo m. Given an $0 \le a < m$ and restricting us to the numbers $b = mk + a$ we have:

Theorem 10.1. *The function $S_{\underline{h}}(mk + a)$ is a polynomial in the variable k of degree $\le \max(b_i)$, that can be computed.*

Sometimes we express the fact that $S_{\underline{h}}(b)$ is a polynomial on every coset saying that:
$S_{\underline{h}}(b)$ is a quasipolynomial or a periodic polynomial.

It remains a last small algorithmic problem to discuss. When we explicit the computations we get, for every coset a polynomial that takes integer values but that a priori is expressed with coefficients that are expressions in the root ζ. We must thus know how to manipulate such an expression, this is a simple problem that nevertheless requires a minimum of analysis with cyclotomic polynomials (appendix 39).

11. Second method, computation of residues

In this case the strategy is the following:

Shift the computation of the residue on the remaining poles, exploiting the fact that the sum of residues on the poles of a rational function is 0.

From the theory of residues we have:

$$\frac{1}{2\pi i} \oint \prod_i \frac{x^{-n-1}}{1 - x^{h_i}} dx = -\sum_{j=1}^{m} \frac{1}{2\pi i} \oint_{C_j} \prod_{i=1}^{m} \frac{x^{-n-1}}{(1 - \zeta^i x)^{b_i}} dx$$

where C_j is an small circle around ζ^{-j}.
Let us use the fact that the residue at ∞ is 0, given that $n \ge 0$.

(1) In order to compute the term $\frac{1}{2\pi i} \oint_{C_j} \prod_{i=1}^{m} \frac{x^{-n-1}}{(1-\zeta^i x)^{b_i}} dx$ we perform a change of coordinates $x = w + \zeta^{-j}$ obtaining:

(2)
$$\frac{1}{2\pi i} \oint_{C_j} \prod_{i=1}^{m} \frac{(w + \zeta^{-j})^{-n-1}}{(1 - \zeta^{i-j} - \zeta^i w)^{b_i}} dw.$$

(3) Now $\prod_{i=1,\ i\neq j}^{m} \frac{1}{(1-\zeta^{i-j}-\zeta^{i}w)^{b_i}}$ is holomorphic around 0 and we can explicitly expand in power series $\sum_{h=0}^{\infty} a_{j,h}w^h$ while:

$$(w+\zeta^{-j})^{-n-1} = \zeta^{j(n+1)}\sum_{k=0}^{\infty}(-1)^k\binom{n+k}{k}(\zeta^j w)^k.$$

Finally we have, for the $j-$th term:

$$-\frac{(-1)^{b_j}}{2\pi i}\oint_{C_j}\zeta^{j(n+1-b_j)}\Big(\sum_{k=0}^{\infty}(-1)^k\binom{n+k}{k}\zeta^{jk}w^k\Big)\Big(\sum_{h=0}^{\infty}a_{j,h}w^h\Big)w^{-b_j}dw$$
$$= -(-1)^{b_j}\zeta^{j(n+1-b_j)}\sum_{k+h=b_j-1}(-1)^k\zeta^{jk}\binom{n+k}{k}a_{j,h}.$$

This formula, added on all the j, provides again a explicit formula for $S_{\underline{h}}(b)$ and presents it again as a *quasipolynomial.*

Remark that, in order to develop these formulae it suffices to compute a finite number of coefficients $a_{j,h}$. Theorems due to Bell (1943).

The box spline There is an even more important function associated to our setting.

The *Box spline* is the function computing the volume of the polytopes $f^{-1}(b)\cap[0,1]^N$, cut by the cube $[0,1]^N$.

With a similar analysis we can see that its Laplace transform is:

$$\prod_{a\in X}\frac{1-e^{-a}}{a}.$$

The box spline is supported in the bounded polytope:

$$\boxed{B(X) := \{\sum_{i=1}^{N} t_i a_i \,|\, 0\le t_i\le 1,\ a_i\in X\}.}$$

By Fubini's theorem since the volume of the cube is 1 we have $\int_{\mathbb{R}^s} B_X(x)dx = 1$.

Next we use the simple property of
Laplace transform of a translated function:

(15)
$$\int_{\mathbb{R}^n} e^{-(y,x)}f(x-a)dx = \int_{\mathbb{R}^n} e^{-(y,x+a)}f(x)dx = e^{-(y,a)}\int_{\mathbb{R}^n} e^{-(y,x)}f(x)dx$$

$$L(f(x-a)) = e^{-a}Lf(x)$$

We see that, setting for $S\subset X$, $a_S := \sum_{a\in S} a$ that:

$$B_X(x) = \sum_{S\subset X}(-1)^{|S|}T_X(x-a_S). \tag{16}$$

Other properties:

If $X = \{Z, y\}$ we have (setting D_y to be directional derivative, and ∇_v the *difference operator* $\nabla_v f(x) := f(x) - f(x - v)$.)

$$D_y B_X(x) = \nabla_y B_X(x).$$

If the X are integral points we have the partition of 1.

$$\sum_{a \in \mathbb{Z}^s} B_X(x - a) = 1.$$

Example $X = \{1, 1, 1, 1\}$ then $T_X(x) = 0$ if $x \leq 0$ and $T_X(x) = 2/3x^3$ if $x \geq 0$.

$$B_X(x) = T_X(x) - 4T_x(x-1) + 6T_X(x-2) - 2T_X(x-3) + T_X(x-4).$$

Part 5. **Several variables**

We have now to pass to several variables. We follow the same steps of the 1-dimensional case. First, we analyze the computation of the volume looking at the Laplace transform. Next the problem of integral points trying to generalize the two methods: *partial fractions* and *residues.*

12. Laplace transform II

First we can extend the basic formulae (**??**) to more variables.

$$\frac{\partial}{\partial y_i} Lf = L(-x_i f), \quad \int_0^\infty \dots \int_0^\infty e^{-\sum_i y_i x_i} dx = \frac{1}{\prod_i y_i}. \tag{17}$$

From this, for every polynomial $P(x_1, \dots, x_n)$ we have:

$$P(-\frac{\partial}{\partial y_1}, \dots, -\frac{\partial}{\partial y_n}) Lf = L(P(x_1, \dots, x_n) f) \tag{18}$$

Let us take a basis $\underline{b} := (a_1, \dots, a_n)$ of $\mathbb{R}^n$ with $a_i := (a_{1,i}, a_{2,i}, \dots, a_{n,i})$.

Let A be the matrix with the a_i as columns, B its inverse.

Let $C_{\underline{b}} := \{\sum_{i=1}^n t_i a_i, \ t_i \geq 0\}$ the cone they generate and $\chi_{\underline{b}}$ its characteristic function.

We want compute its Laplace transform, (of $\chi_{\underline{b}}$*).*

We perform the change of coordinates $x_i = \sum_j a_{j,i} t_j, \ t_i = \sum_j b_{j,i} x_j$.

By construction, in the coordinates t_i the cone $C_{\underline{b}}$ is the positive quadrant.

We have thus:

Lemma 12.1.

$$L(\chi_{\underline{b}}) = \int e^{-\sum_i y_i x_i} \chi_{\underline{b}} dx = |\det(A)| \int_0^\infty \dots \int_0^\infty e^{-\sum_i y_i \sum_j a_{j,i} t_j} dt =$$
$$|\det(A)| \int_0^\infty \dots \int_0^\infty e^{-\sum_j (\sum_i y_i a_{j,i}) t_j} dt = |\det(A)| \prod \frac{1}{\sum_i a_{j,i} y_i}.$$

We set $\alpha_j(y) := \sum_i a_{j,i} y_i$.

We have from formula (17):

$$L(x_1^{h_1} \dots x_n^{h_n} \chi_{\underline{b}}) = (-1)^{\sum_i h_i} \frac{\partial^{h_1}}{\partial y_1} \dots \frac{\partial^{h_n}}{\partial y_n} \frac{1}{\prod_{i=1}^n \alpha_i(y)}$$

this last expression is a sum of terms of type $\frac{c_{k_1,\dots,k_n}}{\prod_{i=1}^n \alpha_i(y)^{k_i}}$ with $\sum_i k_i = \sum_i h_i$ and we see that:

Theorem 12.2. *The Laplace transform determines a linear isomorphism between, the space of polynomials in the variables x_i multiplied by the characteristic function $\chi_{\underline{b}}$ and the space of polynomials in the elements $\frac{1}{\alpha_j(y)} = \frac{1}{\sum_i a_{j,i} y_i}$ multiplied by $\prod \frac{1}{\alpha_j(y)} = \prod \frac{1}{\sum_i a_{j,i} y_i}$.*

Proof

(1) Let us use a change of variables $\alpha_i(y) = z_i$.
(2) The space of functions given in the Theorem, has as basis the monomials $\prod_{j=1}^n z_j^{-h_j}$, with all the exponents $h_j > 1$.
(3) With an easy induction one can see that this space can be obtained from the function $\prod \frac{1}{z_j}$ by applying, by iteration, operators of partial derivative.
(4) We can apply now the formula 17, hence the theorem follows. □

13. Reduction

We would like now to use this formula in order to invert the formula (15) and thus compute the volume function.

This cannot be done directly since the factors that appear in the denominator of (15) are not a basis.

It is necessary thus to be able to manipulate the expression (15) in order to reduce to apply (12.1). This can be obtained by a first version of the development in partial fractions in the case of more variables. Let us use the:

Lemma 13.1. *Let $\alpha_1(y), \dots, \alpha_k(y), \alpha_{k+1}(y)$ be linear forms (non zero) with $\alpha_1(y) = \sum_{j=2}^{k+1} c_j \alpha_j(y)$ then we have:*

$$\frac{1}{\prod_{j=1}^{k+1} \alpha_j(y)} = \sum_{j=2}^{k+1} c_j \frac{1}{\alpha_1(y)^2 \prod_{i=2}^{j-1} \alpha_i(y) \prod_{i=j+1}^{k+1} \alpha_i(y)} \tag{19}$$

Proof

$$\frac{1}{\prod_{j=1}^{k+1}\alpha_j(y)}=\frac{\alpha_1(y)}{\alpha_1(y)^2\prod_{j=2}^{k+1}\alpha_j(y)}=\sum_{j=2}^{k+1}c_j\frac{\alpha_j(y)}{\alpha_1(y)^2\prod_{j=2}^{k+1}\alpha_j(y)}.$$

Now we can prove the *Theorem of Reduction.*

Given linear forms $\alpha_1(y),\dots,\alpha_N(y)$, let d be the dimension of the vector space that they generate.

Theorem 13.2. *Every expression* $\frac{1}{\prod_{i=1}^{N}\alpha_i(y)^{h_i}}$ *can be expressed as linear combination of expressions* $\frac{1}{\prod_{j=1}^{d}\alpha_{i_j}(y)^{m_j}}$ *with* $\alpha_{i_1}(y),\dots,\alpha_{i_d}(y)$ *linearly independent and* $\sum_{j=1}^{d}m_j=\sum_{i=1}^{N}h_i$.

Proof Let us apply a recursion and an induction on the vector of the exponents $(h_1,\dots,h_N)$ in the following way.

(1) Using the given ordering we take the first linearly dependent elements that appear in the product with non zero exponents.
(2) Using Lemma (13.1) we can substitute the product of these terms with a sum in which developing, the vector of the exponents is increased in the lexicographic order maintaining nevertheless the same sum.
(3) In every term the space generated by the factors which appear, remains the same.
(4) Clearly this recursive procedure terminates after a finite number of steps, when all the summands are of the requested type.

14. No broken circuits

As a matter of fact the preceding Theorem can be made more precise, providing a canonical expression in partial fractions. In order to do this we need an idea that comes from the:

Theory of matroids.

Let us start thus from a list $\alpha_1,\dots,\alpha_N$ of non zero vectors (as for instance our linear functions), let us define *circuit* an ordered sublist $\alpha_{i_1},\dots,\alpha_{i_h}$ made of linearly independent elements.

We shall say that the circuit is *a broken circuit*, if there exists an integer $k\le h$ and an integer $i<i_k$ such that the vectors $\alpha_i,\alpha_{i_k},\dots,\alpha_{i_h}$ are linearly dependent.

If the circuit is a basis we shall speak of *no broken basis.*

The meaning of this notion is clear from the following:

Lemma 14.1. *Se* $\alpha_{i_1}(y),\dots,\alpha_{i_h}(y)$ *is a broken circuit then*

$$\frac{1}{\prod_{j=1}^{h}\alpha_{i_j}(y)}$$

is a linear combination of expressions $\frac{1}{\prod_{j=1}^{N}\alpha_j(y)^{h_j}}$ *with the vector of the exponents lexicographically bigger than the vector of the exponents of* $\frac{1}{\prod_{j=1}^{h}\alpha_{i_j}(y)}$.

Proof In the given hypotheses we have $\alpha_i(y) = c_k\alpha_{i_k}(y) + \cdots + c_h\alpha_{i_h}(y)$ with $i < i_k$. Let us substitute and simplify:

$$\frac{1}{\prod_{j=1}^{h}\alpha_{i_j}(y)} = \frac{\alpha_i(y)}{\alpha_i(y)\prod_{j=1}^{h}\alpha_{i_j}(y)} = \frac{c_k\alpha_{i_k}(y) + \cdots + c_h\alpha_{i_h}(y)}{\alpha_i(y)\prod_{j=1}^{h}\alpha_{i_j}(y)}$$

Simplifying in every term the numerator with the corresponding factor in the denominator we get the desired expression.

Now we can prove the:

Theorem 14.2. *Every expression* $\frac{1}{\prod_{j=1}^{N}\alpha_j(y)^{h_j}}$ *can be expressed, in a unique way, as linear combination of expressions* $\frac{1}{\prod_{j=1}^{d}\alpha_{i_j}(y)^{m_j}}$ *with* $\alpha_{i_1}(y), \dots, \alpha_{i_d}(y)$ *a no broken circuit (and* $\sum_{i=1}^{d} m_i = \sum_{i=1}^{N} h_i$*).*

Proof The fact that an expression of the given type can be expressed as linear combination of expression relative to no broken circuits can be proved by induction on the lexicographic order of the vector exponent as in Theorem 13.2, using repeatedly Lemma 14.1.

Uniqueness is a more delicate point and for now we cannot prove it, it follows from the analysis of Lecture 9. □

We want now to analyze an expression in partial fractions for every function of $R := \mathbb{C}[y_1, \dots, y_n, (\prod_{i=1}^{m}\alpha_i(y))^{-1}]$.

Let us proceed in the following way:

- For every no broken circuit $S := \alpha_{i_1}(y), \dots, \alpha_{i_d}(y)$, extracted from the list $\Delta := \{\alpha_1(y), \dots, \alpha_m(y)\}$, we choose coordinates $z_{d+1}, \dots, z_n$ such that $\alpha_{i_1}(y), \dots, \alpha_{i_d}(y), z_{d+1}, \dots, z_n$ be linear coordinates in space.
- In order to simplify the notations let us denote $\alpha_{i_k}(y) := z_k$, $k = 1, \dots, d$.
- Let $A_S = \mathbb{C}[z_{d+1}, \dots, z_n]$ be the corresponding ring of polynomials.
- We define

$$R_S := \{f \in R \,|\, f = \frac{g}{\prod_{j=1}^{d} z_j^{m_j}}, \quad g \in A_S,\ m_j > 0,\ \forall j\}. \tag{20}$$

Theorem 14.3. *(1) The space* R_S *has as basis the monomials:* $\prod_{i=1}^{n} z_i^{h_i} \,|\, h_i \geq 0, \forall i > d,\ h_i < 0,\ \forall i \leq d.$

(2) $R = \oplus_S R_S$ *as* S *varies among the no broken circuits.*

IN CONCLUSION This is the theorem describing expansion of functions in partial fractions.

We prove part of the theorem, completing the proof in Lecture 9.

(1) The elements $z_1, \dots, z_n$ are linear coordinates in space and R_S is contained in the ring of Laurent polynomials in these variables. These polynomials have as basis all the monomials in the variables with integer exponents. The proposed monomials are thus part of this basis and so linearly independent.
(2) From Theorem 14.2 follows immediately that every function in R can be written as linear combination of expressions $f = \frac{g}{\prod_{j=1}^{d} \alpha_{i_j}(y)^{m_j}} \mid g \in \mathbb{C}[y_1, \dots, y_n],\ m_j > 0,\ \forall j$ and S a no broken circuit.
(3) We write f as polynomial in the variables $\alpha_{i_1}(y), \dots, \alpha_{i_d}(y), z_{d+1}, \dots, z_n$, simplify the α_i that appear in the numerator and the denominator. Then prove with an easy induction that every element is sum of elements of the spaces R_S.

Remark 14.4. *The fact that the sum is direct will be proved using $D-$modules and reducing to prove that:*

Lemma 14.5. *The elements $fM_S := \frac{1}{\prod_{j=1}^{d} \alpha_{i_j}(y)}$ as $S = \alpha_{i_1}, \dots, \alpha_{i_n}$ in the no broken bases are linearly independent..*

Part 6. The volume

15. The volume function

Let us go back to the formula (15). The function to invert is: Laplace transform of the volume

$$\sqrt{|\det(AA^t)|} \prod_{i=1}^{m} \frac{1}{\sum_{k=1}^{n} a_{k,i} y_k} = \sqrt{|\det(AA^t)|} \prod_{i=1}^{m} \frac{1}{\alpha_i(y)}.$$

From the preceding analysis we can expand it (according to an explicit algorithm) as linear combination of terms $\frac{1}{\prod_{j=1}^{d} \alpha_{i_j}(y)^{m_j}}$ with $\alpha_{i_1}(y), \dots, \alpha_{i_d}(y)$ a no broken circuit.

From the discussion of section 6.1 we have that such a term is the Laplace transform of a polynomial function multiplied by the characteristic function of the cone $C_{\underline{b}}$, where $\underline{b}$ is the basis of $\mathbb{R}^n$ made of the vectors $\alpha_{i_j} := (a_{1,i_j}, a_{2,i_j}, \dots, a_{n,i_j})$ (the column i_j of the matrix A).

In order to invert the Laplace transform we shall use the theory of residues. Let us start from the identity 18 and Lemma 12.1:

$$P(-\frac{\partial}{\partial y_1}, \dots, -\frac{\partial}{\partial y_n}) \frac{|\det(A)|}{\prod_{i=1}^{n} \alpha_i(y)} = L(P(x_1, \dots, x_n)\chi_{\underline{b}})$$

Let us denote with NBB the set of no broken bases, we know that we can find polynomials $P_b(x_1, \dots, x_n)$ indexed by the no broken bases $b \in NBB$ and such that:

$$\prod_{i=1}^{m} \frac{1}{\alpha_i(y)} = \sum_{\underline{b} \in NBB} P_{\underline{b}}(-\frac{\partial}{\partial y_1}, \dots, -\frac{\partial}{\partial y_n}) \frac{|\det(A)|}{\prod_{i=1}^{n} \alpha_i(y)}. \tag{21}$$

From this we have that:
$\prod_{i=1}^{m} \frac{1}{\alpha_i(y)}$ *is the Laplace transform of:*

$$\sum_{\underline{b} \in NBB} P_{\underline{b}}(x_1, \dots, x_n)\chi_{\underline{b}}. \tag{22}$$

Finally we have proved that, from the explicit formula 15:

Theorem 15.1. *The volume of the polytope $P_A(b)$ has the same Laplace trasform of the linear combination $\sum_{\underline{b} \in NBB} P_{\underline{b}}(x_1, \dots, x_n)\chi_{\underline{b}}$ (formula (22)) of functions of the type a polynomial multiplied for the characteristic function of an cone $C_{\underline{b}}$, where $\underline{b}$ is a no broken basis.*

At this point there is a subtle point to be understood.

16. Formula for the volume

We would like to use the formulae 22 to determine the volume function. A first hypothesis is that the volume function is given by the formula (22). This a priori is true only almost everywhere, as a simple example shows.

- Let us take the vectors $a_1 = (1, 0), a_2 = (0, 1), a_3 = (1, 1) = a_1 + a_2$.
- The three bases that we extract generate three cones.
  One is the positive quadrant $C_{\{a_1,a_2\}}$ and the others give half of the quadrant $C_{\{a_1,a_1+a_2\}}, C_{\{a_1+a_2,a_2\}}$.
- We have clearly that $\chi_{a_1,a_2} = \chi_{a_1,a_1+a_2} + \chi_{a_1+a_2,a_2} - \chi_{a_1+a_2}$, where $\chi_{a_1+a_2}$ is the characteristic function of the half line for $a_1 + a_2$.
- For the Laplace transform we have instead:
$$\frac{1}{y_1 y_2} = \frac{1}{y_1(y_1 + y_2)} + \frac{1}{y_2(y_1 + y_2)}.$$

The point is that a function with preassigned Laplace transform is determined only up to values in a set of measure zero.

As a matter of fact with simple examples we can see that the function $\sum_{\underline{b} \in NBB} P_{\underline{b}}(x_1, \dots, x_n)\chi_{\underline{b}}$ can be discontinuous while it is easy to convince oneself that *the volume is a continuous function.*

We must thus understand how to modify the proposed formula.

In order to do this we must develop some geometry on the cone $C(A)$.

Geometry of polyhedra

We start with a general remark.

- Let be given N points P_i in n dimensional space and X be their convex envelop. We assume that the points are not contained in any proper linear subspace.

- If we choose in any possible way $k \leq n$ among these points that are independent they generate a simplex of dimension $k-1$, let Y be the union of all these simplices.
- Y is a closed subset of X of dimension $n-1$.
- The connected components of $X - Y$ are called *big cells* associated to the points.
- The small cells, which we will not use, are obtained by removing to X all the points laying on the subspaces generated by such sets of points.

Same geometry for the cone $C(A)$.

Let be given a list A of vectors v_i in n dimensional space, that generate a pointed cone $C(A)$.

- Let us take an hyperplane that intersects the half lines $\mathbb{R}^+ v_i$ in the points P_i (there exists one by hypothesis).
- The cone $C(A)$ is the projection from the origin of the convex envelop $X$of the points P_i
- Every simplex of dimension $\leq n-2$ projects to a simplicial cone of dimension $\leq n-1$.
- The union of such cones projects Y and the big cells of $C(A)$ are the projections of the big cells of X minus the origin.

Regular points The points of the big cells are also called regular, the others singular.

We shall say that X, Y represent $C(A)$ and its singular points *in section.*

Conclusion

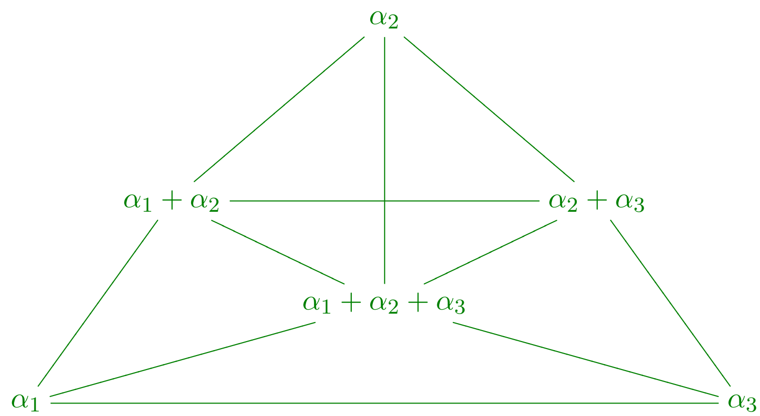
(1) Clearly the function $F(x) = \sum_{\underline{b} \in NBB} P_{\underline{b}}(x) \chi_{\underline{b}}$ given by the formula (22) is continuous and a polynomial on the big cells.
(2) The volume function on the other hand, for geometric reasons is a continuous function.
(3) It follows from elementary facts on the Laplace transform that the formula (22) coincides with the volume in the interior of the big cells.
(4) On the other hand on an big cell $\mathfrak{c}$ vanish all the functions $\chi_{\underline{b}}$ for which the cone $C_{\underline{b}}$ does not contain the cell $\mathfrak{c}$.

By continuity follows thus the general formula for the volume:

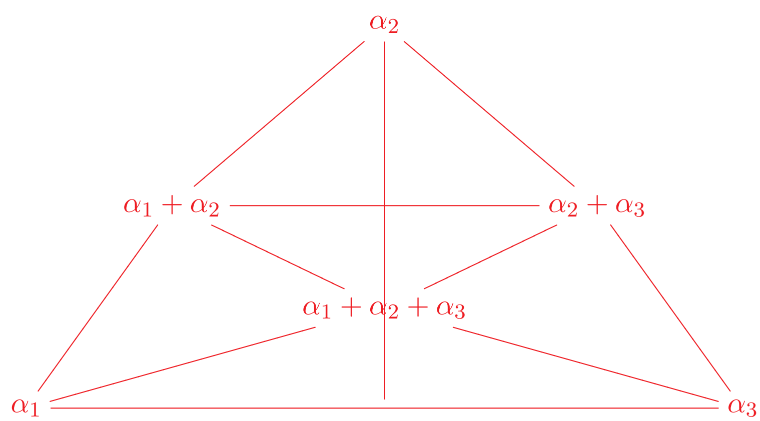
Theorem 16.1. *Given a point c in the closure of a big cell $\mathfrak{c}$ we have:*

$$V_A(c) = \sum_{\underline{b} \in NBB, \ | \ \mathfrak{c} \subset C_{\underline{b}}} P_{\underline{b}}(c)$$

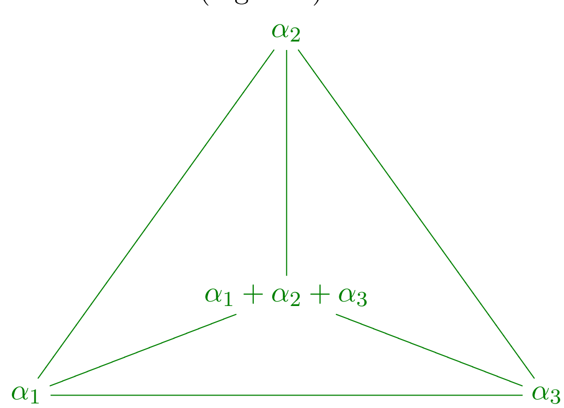
EXAMPLE Type A_3 in section (big cells):



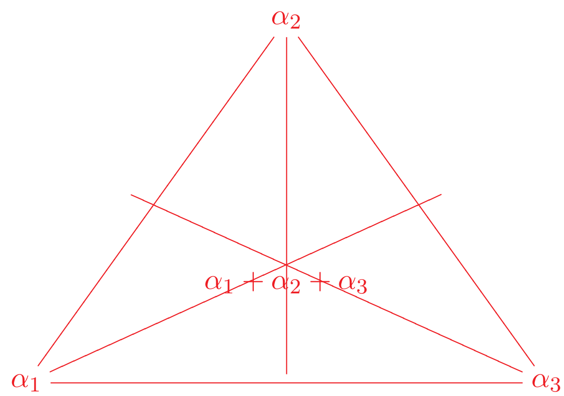
We have 7 big cells.
EXAMPLE Type A_3 in section (small cells):



We have 8 small cells.
EXAMPLE 2 in section (big cells):



We have 3 big cells.
EXAMPLE 2 in section (small cells):



We have 6 small cells.
In essence the given formula defines the notion of:
Jeffry–Kirwan residue

In order to make this formula effective we need two algorithms.

- The first an algorithm that allows us to identify, given $p = (a_1, \dots, a_n)$:
  - a big cell C with $p \in \bar{c}$ and
  - the set of the $\underline{b} \in NBB, \mid c \subset C(\underline{b})$.
- The second is the computation of the polynomials $P_{\underline{b}}(c)$.

In order to do this we shall see in the next paragraphs that such polynomials can be computed, up to sign as suitable residues that will be defined presently and denoted with $res_{\underline{b}}(e^{\sum_i a_i y_i} \frac{\sqrt{|\det(AA^t)|}}{\prod_{i=1}^N \alpha_i(y)})$.

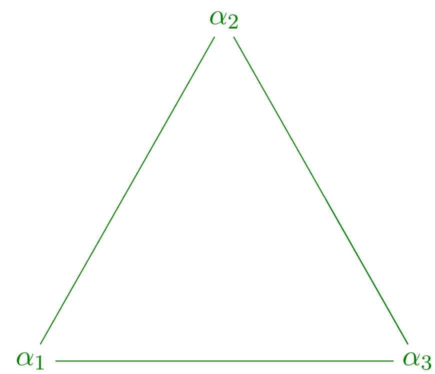
A theorem, proved in [12]proves that the set Y can be costructed only through the simplices formed with no broken circuits. Example A_3 ordered as:

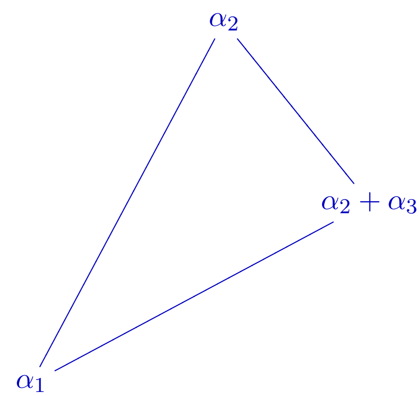
$$\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3.$$

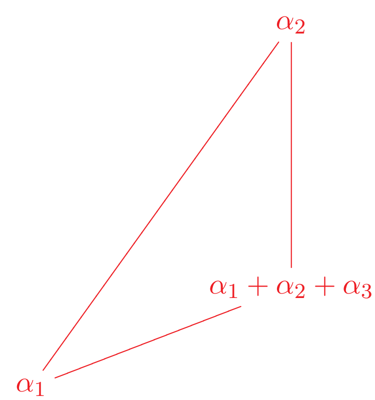
The n.b.b contain all necessarily α_1, the other two elements are given by 6 possibilities:

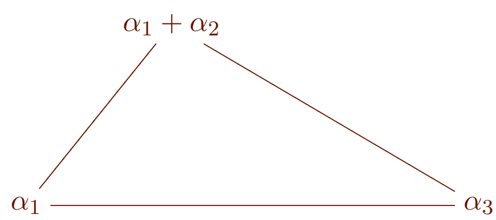
(1) α_2 e uno degli elementi $\alpha_3, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3$
(2) $\alpha_3, \alpha_1 + \alpha_2$ e $\alpha_3, \alpha_1 + \alpha_2 + \alpha_3$
(3) $\alpha_1 + \alpha_2, \alpha_2 + \alpha_3$

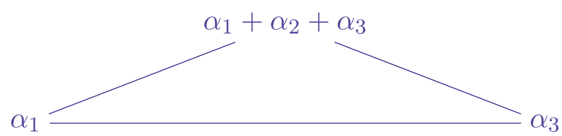
Let us visualize the simplices generated by the 6 n.b.b:

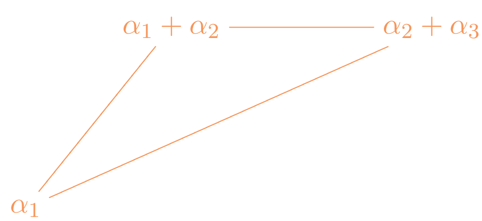

α_2
α_1
α_3


α_2
$\alpha_2 + \alpha_3$
α_1

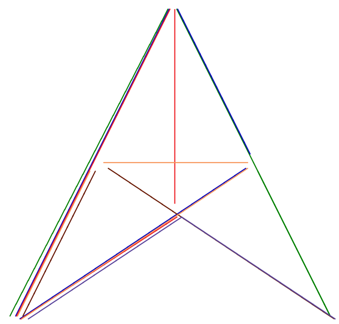

α_2
$\alpha_1 + \alpha_2 + \alpha_3$
α_1







Let us visualize the decomposition into big cells, obtained overlapping the cones generated by no broken bases.



In order to make the preceding discussion more efficient from the algorithmic point of view, and to connect it with the theory of multidimensional residues, we must develop two further constructions:

The total residue

and

The Jeffry–Kirwan residue.

In order to give a clear definition it is appropriate to introduce a more geometric language.

17. Hyperplane arrangements

Let us denote with $\Delta := \{\alpha_1(y), \dots, \alpha_N(y)\}$ the list of the linear forms $\alpha_j(y) = \sum_i a_{j,i} y_i$, given by the columns α_i of A.

Every linear form $\alpha_j(y)$ defines as *zero locus* an hyperplane H_j.

Definition 1. *This list of hyperplanes and all the subspaces that are obtained from them by intersection is called a*

Hyperplane arrangement.

NOTE: Even if the forms are with real coefficients we shall always consider all the complex points.

The complement of the union of such hyperplanes (in complex space) will be denoted with $\mathcal{A}_\Delta$. The geometric point we want to stress is that: A rational function in the variables y_i is well defined (has no poles) on the open set $\mathcal{A}_\Delta$ if and only if its denominator is a product of powers of the linear forms $\alpha_j(y)$.

The set of such functions in algebraic geometry is called *the coordinate ring of* $\mathcal{A}_\Delta$.

It is this ring that we must now discuss.

Let us denote it with the symbol R_Δ.

Remark also that, denoting $d = \prod_{i=1}^N \alpha_i(y)$

the elements of R_Δ can be also written as f/d^k with f a polynomial. In other notations:

$$R_\Delta := \mathbb{C}[y_1, \dots, y_n, d^{-1}]. \tag{23}$$

This ring is also described as the ring of an hypersurface in $n+1$ dimensioni space of equation $1 - zd = 0$:

$$R_\Delta := \mathbb{C}[y_1, \dots, y_n, z]/(1 - zd). \tag{24}$$

The principal combinatoricsl object, associated to an arrangement of hyperplanes is the list of all the subspacesobtained through intersection of such hyperplanes.

This is a *partially ordered* set, by inclusion (shortly a *poset*).

As poset it has special properties:

(1) Given two elements a, b there exists their minimum $\min(a, b)$, (the intersection of the two subspaces).
(2) Every maximal ordered chain has length n (the dimension of the space).
(3) Every element has a rank $r(a)$ such that in every increasing maximal chain in which appears a, it appears in the position $r(a) + 1$.

We speak of a *ranked poset*.

EXAMPLE The configurations of numbers .

Example 17.1. *In n dimensional space of coordinates z_i let us consider the hyperplanes $z_i - z_j = 0, i < j$.*

Such a hyperplane consists of points in which two coordinates z_i, z_j are equal.

An intersection of a set of such hyperplanes defines thus a subspace in which various subsets (disjoint) of the indices have equal coordinates. In other words, such a subspace, corresponds to a partition $\mathcal{P} := \{A_1, \dots, A_k\}$ of the set $\{1, 2, \dots, n\}$ in disjoint subsets A_i.

The dimension of the subspace is the number of sets of the partition.

A subspace, corresponding to a partition $\mathcal{P}$ contains the subspace corresponding to a partition $\mathcal{Q}$ if and only if the partition $\mathcal{P}$ can be obtained from the partition $\mathcal{Q}$ by *refining* that is subdividing in parts the parts of $\mathcal{Q}$.

The minimum subspace consists of the line made of the vectors with all the coordinates equal.

Remark 17.2. *Usually it is convenient to reduce to:*

central arrangements

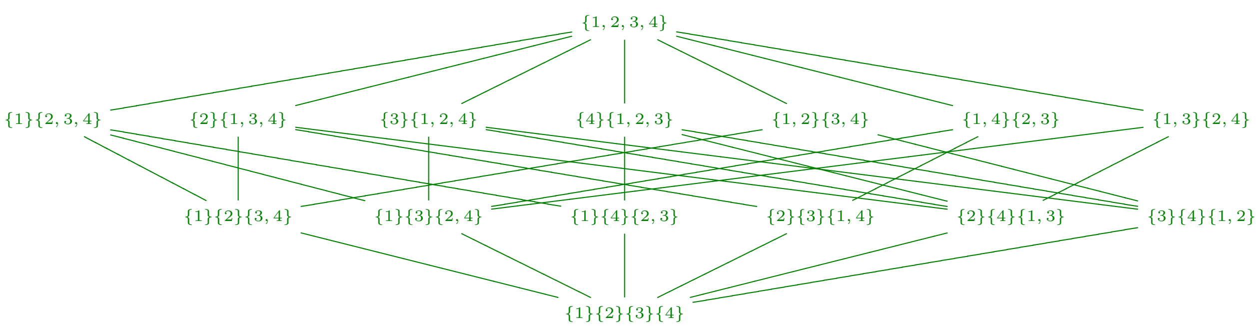
that is in which the intersection of all the hyperplanes is reduced to 0 (in an equivalent way the linear equations of the hyperplanes generate the dual space).

In the case of configurations of numbers, it is convenient to restrict ourselves to the $n-1$ dimensional space where the sum of the coordinates is 0 in order to obtain a central arrangement.

This example belongs to an important class of examples, those of root systems and of reflection groups.

Important examples in various sectors of Mathematics as Lie theory and singularities.

EXAMPLE The poset of partitions of $\{1,2,3,4\}$:



One of the important characteristics of this theory consists in the fact that, many geometric, topological and algebraic properties of the arrangement depend only from the combinatorics of the poset, that is they are:

combinatorial properties.

Perhaps the most interesting is the *cohomology* of $\mathcal{A}_\Delta$, given by the Theory of Orlik–Solomon [26].

18. The total residue

The total residue is essentially a concept of cohomology, but let us start to introduce it in an elementary fashion.

We define the subspace $\partial(R_\Delta)$ as the space generated by all the partial derivatives that is by the elements $\dfrac{\partial f}{\partial y_i},\ f \in R_\Delta,\ i = 1, \dots, n.$

Example one variable: Let $R_\Delta = \mathbb{C}[y, y^{-1}] = \{\sum_{i=-h}^{k} a_i y^i,\ h, k \in \mathbb{N}\}$ be the space of Laurent polynomials.

The space of the derivatives consists of all the polynomials without y^{-1}.

Similarly for the Laurent polynomials in the variables $y_i, y_i^{-1},\ i = 1, \dots, n$ the space of the derivatives consists of all the polynomials without the term $\prod_{i=1}^{n} y_i^{-1}$.

Definition 2. *Let H_Δ be the space spanned by the elements $M_{\underline{b}} := \frac{1}{\prod_{h=1}^{n} \alpha_{i_h}(y)}$, as $\underline{b} := \{\alpha_{i_1}, \dots, \alpha_{i_n}\}$ varies among the bases extracted from the list Δ.*

The fundamental Theorem is the following:

Theorem 18.1. *(1) We have the decomposition in direct sum:*

$$R_\Delta = H_\Delta \oplus \partial(R_\Delta)$$

(2) One basis of H_Δ is given by the elements $M_{\underline{b}}$, as $\underline{b}$ varies in the no broken bases.

In the following we shall also work with a larger algebra of functions. The algebra S_Δ made of the (meromorphic) functions fd^{-k} where f is any function holomorphic around 0.

Corollary 18.2. *It is not difficult to convince ourselves (assuming the preceding Theorem). That we have again:*

$$S_\Delta = H_\Delta \oplus \partial(S_\Delta)$$

We can prove this Theorem as corollary of theorem 14.3. In the next lecture we shall formulate it in a more intrinsic way, introducing the language of *differential forms.*

Proof of the theorem

In order to prove this theorem we need to go back to the decomposition $R_\Delta = \oplus_S R_S$ given by theorem 14.3 where 20:

$$R_S := \{f \in R \,|\, f = \frac{g}{\prod_{j=1}^d z_j^{m_j}}, \quad g \in A_S,\ m_j > 0,\ \forall j\}.$$

with the coordinates z_i depending on the no broken basis S.

Proof of the theorem

The fundamental remark to be made, is that: The vector spaces R_S are all stable with respect to all the partial derivatives.

Thus, with the notations of Theorem 14.3, let $\prod_{i=1}^n z_i^{h_i} \in R_S \,|\, h_i \geq 0, \forall i > d$, $h_i < 0 \forall i \leq d$.

We can take as basis of the operators of partial derivative, the derivatives in the variables z_i and we have:

$$\frac{\partial \prod_{i=1}^n z_i^{h_i}}{\partial z_j} = h_j z_1^{h_1} \dots z_{j-1}^{h_{j-1}} z_j^{h_j - 1} z_{j+1}^{h_{j+1}} \dots z_n^{h_n},$$

that is 0 or a monomial in R_S.

Now let us observe that:

Proposition 18.3. *If the cardinality d of S is $< n$ then every element of R_S is a sum of derivatives.*

A monomial $\prod_{i=1}^n z_i^{h_i} \in R_S$ has thus $h_n \geq 0$ hence

$$\prod_{i=1}^n z_i^{h_i} = \frac{\partial}{\partial z_n}\Big((h_n + 1)^{-1} \prod_{i=1}^{n-1} z_i^{h_i} z_n^{h_n+1}\Big).$$

Let us take now the case $d = n$ now the monomials to be considered are of the type $\prod_{i=1}^n z_i^{-h_i}$, $h_i > 0$, we take a derivative and we have:

$$\frac{\partial}{\partial z_j}\prod_{i=1}^{n} z_i^{-h_i} = -h_j z_1^{h_1}\dots z_{j-1}^{-h_{j-1}} z_j^{-h_j-1} z_{j+1}^{-h_{j+1}}\dots z_n^{-h_n}.$$

We see thus that:

we get, as derivatives, all the monomials in which at least one of the exponents $h_i > 2$. The only remaining monomial $(\prod_{i=1}^n z_i)^{-1}$ is not sum of derivatives.

19. The residues

We can now define, in an elementary fashion, the total residue $Tres$. From the preceding theorem, a function $f \in R_\Delta$ (o also in S_Δ) can be decomposed in a unique way as $f = a + b$, $a \in H_\Delta$, $b \in \partial(R_\Delta)$.

Definition 3. *We set*

$$Tres(f) := a.$$

The total residue of f*.*

In other words $Tres$ is the projection on the summand H_Δ of the decomposition.

Let us denote with NBB the set of no broken bases, and with

$$\alpha_{\underline{b}} := \frac{\det(A)}{\prod_{i=1}^n \alpha_i(y)},\ \ \epsilon_{\underline{b}} := \frac{|\det(A)|}{\det(A)} = \pm 1,\quad \underline{b} := \{\alpha_i(y),\ i = 1, \dots, n\}.$$

We know that the elements $\alpha_{\underline{b}}$ form a basis of H_Δ and thus:

we can define the numbers $res_{\underline{b}}(f)$ through the formula:

$$Tres(f) = \sum_{\underline{b}\in NBB} res_{\underline{b}}(f)\alpha_{\underline{b}}. \tag{25}$$

The application we have in mind is to formula (21). We want to prove the:

Theorem 19.1.

$$P_{\underline{b}}(a_1, \dots, a_n) = \epsilon_{\underline{b}} res_{\underline{b}}\Big(\frac{e^{\sum_i a_i x_i}}{\prod_{i=1}^m \alpha_i(y)}\Big).$$

Once we prove this Theorem, the delicate point will be to develop a *method for computing* $res_{\underline{b}}(\frac{e^{\sum_i a_i x_i}}{\prod_{i=1}^m \alpha_i(y)})$.

In order to prove this Theorem we must develop some properties of $Tres$.

The first property of $Tres$, which follows from the definition is that, given a function f and i we have $Tres(\frac{\partial f}{\partial y_i}) = 0$, hence for two functions f, g:

$$Tres\Big(\frac{\partial f}{\partial y_i} g\Big) = -Tres\Big(f\frac{\partial g}{\partial y_i}\Big).$$

In other words for a polynomial P:

$$Tres(P(\frac{\partial}{\partial y_i})(f)g) = Tres(fP(-\frac{\partial}{\partial y_i})(g)). \tag{26}$$

We shall use the preceding relation (29) in particular for the function $f = e^{\sum_i a_i y_i}$ for which we have:

$$P(\frac{\partial}{\partial y_i})e^{\sum_i a_i y_i} = P(a_i)e^{\sum_i a_i y_i}, \tag{27}$$

The second property, simple to verify, of $Tres$ is that, given a basis $\underline{b} := \{\alpha_i(y),\ i = 1, \dots, n\}$ extracted from Δ and a function f holomorphic around to 0 we have:

$$Tres(\frac{f}{\prod_{i=1}^n \alpha_i(y)}) = \frac{f(0)}{\prod_{i=1}^n \alpha_i(y)}. \tag{28}$$

Going on:

$$Tres(e^{\sum_i a_i y_i} P(-\frac{\partial}{\partial y_i})\frac{1}{\prod_{i=1}^n \alpha_i(y)}) = \tag{29}$$

$$Tres(\frac{1}{\prod_{i=1}^n \alpha_i(y)} P(\frac{\partial}{\partial y_i})(e^{\sum_i a_i y_i})) =$$

$$\frac{P(a_i)}{\prod_{i=1}^n \alpha_i(y)} = \frac{P(a_i)}{|\det(A)|} L(\chi_{\underline{b}}).$$

Let us take thus a function written in the form

$$f := \sum_{\underline{b} \in NBB} P_{\underline{b}}(-\frac{\partial}{\partial y_i})\alpha_{\underline{b}} = L(\sum_{\underline{b} \in NBB} P_{\underline{b}}(x_i)\epsilon_{\underline{b}}\chi_{\underline{b}})$$

Theorem 19.2. *We deduce that:*

$$Tres(e^{\sum_i a_i y_i} f) = \sum_{\underline{b} \in NBB} P_{\underline{b}}(a_i)\alpha_{\underline{b}}.$$

Remark 19.3. *This allows us to recover the polynomials $P_{\underline{b}}(x_i)$ knowing the value of $Tres(e^{\sum_i a_i y_i} f)$. Thus:*

$$P_{\underline{b}}(a_i) = res_{\underline{b}}(e^{\sum_i a_i y_i} f). \tag{30}$$

Let us apply the method to the volume
In particular, for the volume function, we must compute:

$$Tres(\frac{e^{\sum_i a_i x_i}}{\prod_{i=1}^m \alpha_i(y)}) =$$

$$\sum_{\underline{b}\in NBB} Tres(e^{\sum_i a_i x_i} P_{\underline{b}}(-\frac{\partial}{\partial y_1},\dots,-\frac{\partial}{\partial y_n})\frac{|\det(A)|}{\prod_{i=1}^n \alpha_i(y)}) = \sum_{\underline{b}\in NBB} P_{\underline{b}}(a_1,\dots,a_n)\frac{|\det(A)|}{\prod_{i=1}^n \alpha_i(y)} \tag{31}$$

Preliminary inversion formula hence the function:

$$V(a_1,\dots,a_n) := \sum_{\underline{b}\in NBB} res_{\underline{b}}(e^{\sum_i a_i y_i}\frac{\sqrt{|\det(AA^t)|}}{\prod_{i=1}^N \alpha_i(y)})\epsilon_{\underline{b}}\chi_{\underline{b}}. \tag{32}$$

Let us repeat the discussion.

Clearly, while the volume function is continuous everywhere, on the function $V(a_1,\dots,a_n)$ (cf. 32), we can only say that it is continuous on the regular points, that is on the big cells.

From elementary facts of the Laplace transform (proved in the Appendix) we can deduce that $Vol_A(b) = V(b)$ if b is regular.

We can now now conclude.

The volume formula Let $b := (b_1,\dots,b_n) \in C(A)$. Given a big cell c with $b \in c$ we have clearly:

$$V(b) = \sum_{\underline{b}\in NBB,\ c\subset C(\underline{b})} res_{\underline{b}}(e^{\sum_i b_i y_i}\frac{\sqrt{|\det(AA^t)|}}{\prod_{i=1}^N \alpha_i(y)})$$

On the other hand, this function is continuous on the closure of c and thus finally we have:

Theorem 19.4. *If $b \in \overline{c}$ (closure of c) we have:*

$$Vol_A(b_1,\dots,b_n) = \sum_{\underline{b}\in NBB,\ c\subset C(\underline{b})} res_{\underline{b}}(e^{\sum_i b_i y_i}\frac{\sqrt{|\det(AA^t)|}}{\prod_{i=1}^N \alpha_i(y)}). \tag{33}$$

NOTE

- This formula provides a volume formula on the closure of an cell.
- At a point, in the closure of two cells, we have two different formulae that coincide on the intersection of the closure of such cells.

Part 7. Differential forms

20. Differential forms

The theory of differential forms appears in differential geometry and in algebra, we give an idea of its essential points.

The theory starts from the concept of differential df of a function, this one can develop in various ways and we refer to the textbooks for a discussion.

The fundamental point is that, given coordinates x_i (themselves functions) we have the formula:

$$df(x_1, \ldots, x_n) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

In local coordinates a 1-differential form is an expression

$$\psi := \sum_{i=1}^{n} f_i(x) dx_i.$$

In general such a ψ is not the differential of a function.

In order for this to hold (locally), it is necessary and sufficient that hold the: compatibility rules:

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}, \quad \forall i, j.$$

This, in a geometric language, this is what we learn in physics looking at the: theory of the *potential of a conservative field*.

21. Grassmann calculus

The second important point is:

Grassmann or exterior calculus.

This is essentially an algebraic concept. We introduce a product, the *exterior product* (in that it produces new forms starting from the given ones) denoted with the symbol $\wedge$.

We assume that such a product is associative and distributive and that holds the: fundamental rule:

$$df \wedge dg = -dg \wedge df$$

(this type of rule appears now often in Physics, in the theory of Fermions and supersimmetry).

With this product, starting from the 1-forms we can build, for every $k \leq n$ the $k-$forms of type:

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} f_{i_1, i_2, \ldots, i_k}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}.$$

For two forms ϕ, ψ of degrees h, k we have the commutation rule $\phi \wedge \psi = (-1)^{hk} \psi \wedge \phi$.

Example 21.1. *Given n functions $f_i(x_1, \ldots, x_n)$ in n variables, we have*

$$df_1 \wedge \cdots \wedge df_n = J(f_1, \ldots, f_n) dx_1 \wedge \cdots \wedge dx_n$$

With $J(f_1, \dots, f_n) = \det(\frac{\partial f_i}{\partial x_j})$ *il*
Jacobian determinant.

22. Differential calculus

The second essential concept is of analytic type.

We have of define the *differential* $d\psi$ of a form ψ.
This operator is completely characterized by the following properties:

(1) d is an linear operator from $k-$forms to $k+1$-forms for every k.
(2) If f is a function, df is the differential already defined.
(3) For two forms ϕ, ψ, of degrees h, k, holds (Leibniz rule):

$$d(\phi \wedge \psi) = d\phi \wedge \psi + (-1)^h \phi \wedge d\psi.$$

From these rules one can see easily that:

Remark 22.1.

$$d \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} f_{i_1, i_2, \dots, i_k}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} =$$

$$\sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \sum_{i=1}^{n} \frac{\partial f_{i_1, i_2, \dots, i_k}(x)}{\partial x_i} dx_i \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}.$$

NOTE:

- $\psi := dx_i \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} = 0$ if $i = i_j$ for some j.
- If instead $i_j < i < i_{j+1}$, ψ can be rewritten as

$$(-1)^j dx_{i_1} \wedge \dots dx_{i_j} \wedge dx_i \wedge dx_{i_{j+1}} \cdots \wedge dx_{i_k} = 0$$

-

  Example 22.2.

$$d \sum_{i=1}^{n} f_i(x) dx_i = \sum_{i<j} (\frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}) dx_i \wedge dx_j$$

- Note that the integrability condition for the 1-form $\psi := \sum_{i=1}^{n} f_i(x) dx_i$ is thus $d\psi = 0$!

As a matter of fact we have the: fundamental properties of differential calculus:

(1) $d^2 = 0$.
(2) A k-form ψ is (locally) of type $d\phi$ (with ϕ a $k-1$ form) if and only if $d\psi = 0$.

We have insisted on the (local) aspect, since the theory of *residues* is essentially tied to the global lack of integrability.

We use the following: terminology:

- A form ψ is said to be *closed* if $d\psi = 0$.
- It is said to be *exact* if $\psi = d\phi$.

Therefore exact implies closed, the inverse implication is only local.

The fundamental example is the form $1/zdz$ The form $1/zdz$ is a complex form, defined on all the non zero complex numbers $\mathbb{C}^*$.

We have, as a matter of fact:

$$z^{-1}dz = d\log(z),$$

now it is necessary to understand that $\log(z)$ cannot be defined everywhere on the set $\mathbb{C}^*$, we can only choose locally a determination.

(in classical terms, it is a *polydromous function*)

We can understand this thinking that, the function whose differential is the given 1-form, can be obtained by integration on paths.

When we integrate the form $z^{-1}dz = d\log(z)$ on a closed path around 0, we may obtain a non zero value (a residue) and pass from one determination of the logarithm to another.

Clearly for forms of higher degree the theory is more complex. It has to be discussed with the ideas of homology and cohomology.

The theory of differential forms acquires its full meaning when we develop it(without reference to coordinates) for a *differentiable manifold*.

Moreover in our case, we want to use complex variables $z = x + iy$ and think of $dz = dx + idy$.

We will not need of insist on this important point.

23. Integral calculus

The last point, of fundamental nature, of the theory is:

Integration of forms

and

Stokes Theorem.

First we must understand the behavior of forms under differentiable maps.

If $F : M \to N$ is a differentiable map of manifolds and ψ a k form on N we can define the form $F^*(\psi)$ as a $k-$form on M using the following rules:

(1) If $\psi = f$ is a function, we have $F^*f(x) = f(F(x))$.
(2) $d(F^*(\psi)) = F^*(d\psi)$.
(3) $F^*(\phi \wedge \psi) = F^*(\phi) \wedge F^*(\psi)$.

In coordinates, $(x_1, \dots, x_m)$ on M and $(y_1, \dots, y_n)$ on N, with $y_i = F_i((x_1, \dots, x_m)$ we have: Substitution formula

$$F^*(f(y_1, \dots, y_n)dy_{i_1} \wedge \cdots \wedge dy_{i_k}) =$$

$$f(F_1(x_1, \dots, x_m), \dots, F_n(x_1, \dots, x_m))dF_{i_1} \wedge \cdots \wedge dF_{i_k}.$$

One $k-$form should be integrated on a $k-$dimensional oriented object. (a 1-form on a curve, a 2-form on a surface etc.)

Definition 4. *Locally in (oriented) coordinates* $(x_1, \dots, x_m)$ *the integral of* $f(x_1, \dots, f_k)dx_1 \wedge \cdots \wedge dx_k$ *is the usual multiple integral* $\int f(x_1, \dots, f_k)dx_1 \dots dx_k$.

Global integration can be developed through the use of *partitions of unity*.
The usual theorem, describing the coordinate change in multiple integrals, implies that the definition is well posed.

Typically, given a manifold M and a k form, we integrate on an $k-$manifold choosing a *parametrization* $i : N \to M$ with N a $k-$manifold.
It is not necessary for the definition, to assume that i embeds N in M.
By way of definition, we integrate on N the form $i^*(\psi)$.
We has thus, taking an oriented manifold M of dimension k, with boundary ∂M (of dimension $k-1$) and a $k-1$-form ψ, the fundamental rule: Stokes Theorem

$$\int_M d\psi = \int_{\partial(M)} \psi.$$

An important consequence is thus the following:

Remark 23.1. *When we integrate a closed $k-$formψ on an $k-$manifold, boundary of an $k+1$ manifold we get 0.*

In the case of $d\log(z)$,integrated on a circle around to the origin we have a non zero integral, because, while it is true that such a circle is the boundary of a disk, this disk contains the point 0. 0 is not in $\mathbb{C}^*$ where the form is defined.

Therefore in $\mathbb{C}^*$ this circle is not a boundary. It is not convenient to

integrate forms only on manifolds, it is convenient to integrate them on suitable *chains of integration*.

24. Chains, homology

The definition of chain is formal but very useful. We can give in very general way starting from the standard simplex:

$$\Delta_n := \{(x_0, x_1, \dots, x_n) \,|\, \sum x_i = 1,\ x_i \geq 0,\ \forall i\}.$$

For every topological space X we can define:

- A *singular simplex* is a continuous function $\sigma : \Delta_n \to X$.
- A *singular chain* is a formal linear combination $\sum_{i=1}^k c_i\sigma_i$ of singular simplices.

Note: The coefficients in our treatment will be real numbers (but one can take coefficients in a ring, as the integers o the integers mod-m).

If X is a differentiable manifold we can speak of differentiable simplices or chains (usually C^∞).

Therefore, given a differentiable manifold M, a $k-$form ψ on M and a differentiable chain c of dimension k we can integrate ψ on c.

The notions of boundary and cicle

At this point we introduce the notion of *boundary* δ of a singular chain as follows.

- Every face $\Delta_n^i := \{(x_0, x_1, \ldots, x_n) \in \Delta_n \,|\, x_i = 0\}$ is an $(n-1)-$dimensional simplex.
- Restricting to such a face an $n-$dimensional singular simplex, we obtain an $(n-1)-$dimensional simplex $\epsilon_i(\sigma)$.
- We define thus

$$\delta(\sigma) = \sum_{i=0}^{n} (-1)^i \epsilon_i(\sigma), \ \delta(\sum_{i=1}^{k} c_i \sigma_i) = \sum_{i=1}^{k} c_i \delta(\sigma_i)$$

Definition 5.
- *We shall say that a chain c is a boundary or exact if it is of the form $\delta(e)$.*
- *c is a cycle or closed if $\delta(c) = 0$.*

We easily see that $\delta^2 = 0$:
a boundary is thus necessarily a cycle.
All these informations can be organized as follows:

Definition 6. *Denoting $B_k(X) \subset Z_k(X)$ the space of boundaries, respectively of singular cycles we define:*

$$H_k(X) := Z_k(X)/B_k(X),$$

the $k-$th group of singular homology.

Differentiable chains

When we restrict to differentiable chains we have, for the chains and the forms Stokes Theorem $\int_c d\psi = \int_{\delta(c)} \psi$.

Remark 24.1. *If X is a differentiable manifold we have first, that the homology can be computed using only differentiable chains, therefore we shall use such chains.*

Definition 7. *We define with $B^k(X) \subset Z^k(X)$ the spaces of exact and closed $k-$forms respectively and:*

$$H^k(X) := Z^k(X)/B^k(X),$$

$k-$th group of De Rham cohomology.

From these definition and from the theorem of Stokes follows that, integration on chains determines a duality pairing:

$$H^k(X) \times H_k(X) \to \mathbb{R}, \quad (\psi, x) \mapsto \int_c \psi.$$

The fundamental result is: Theorem of De Rham states that, if X is a compact manifold the two vector spaces $H^k(X), H_k(X)$ are finite dimensional and in perfect duality.

We want to apply this Theory to the case of the (affine algebraic) variety $\mathcal{A}_\Delta$, which is not compact, and to the algebraic differential forms, that is with coefficients in the coordinate ring R_Δ.

In this case holds the analogue of the De Rham' Theorem and it is due to Grothendieck.

We have a perfect duality between cohomology computed with the algebraic differential forms and the singular homology with complex coefficients.

Let us denote with

$$\Omega^k_\Delta := \{ \sum_{1\leq i_1 < i_2 < \cdots < i_k \leq n} f_{i_1,\dots,i_k}(x) dx_{i_1} \wedge \cdots \wedge dx_{i_k} \mid f_{i_1,\dots,i_k}(x) \in R_\Delta \}$$

the space of algebraic differential forms.

In particular let us look in maximal degree. We see immediately the interpretation of the spaces $\partial(R_\Delta), H_\Delta$. We have:

$$d(\Omega^{n-1}_\Delta) = \partial(R_\Delta) dx_1 \wedge \cdots \wedge dx_n.$$

We can now thus identify the space H_Δ to the space $H^n(\mathcal{A}_\Delta)$.

Moreover we see immediately that, given n independent forms $\alpha_i(y) = \sum_{j=1}^n a_{i,j} y_j$ with $A = (a_{i,j})$ invertible we have the formula:

$$\det(A) \prod_{i=1}^n \frac{1}{\alpha_i(y)} dy_1 \wedge \cdots \wedge dy_n = d\log(\alpha_1(y)) \wedge \cdots \wedge d\log(\alpha_n(y)).$$

Since $\det(A) \prod_{i=1}^n \frac{1}{\alpha_i(y)} dy_1 \wedge \cdots \wedge dy_n = \alpha_{\underline{b}} dy_1 \wedge \cdots \wedge dy_n$ we deduce that:

Theorem 24.2. *The cohomology classes of the forms*

$$\omega_{\underline{b}} := d\log(\alpha_1(y)) \wedge \cdots \wedge d\log(\alpha_n(y))$$

as $\alpha_1(y), \dots, \alpha_n(y)$ varies in the no broken bases, form a basis of the cohomology $H^n(\mathcal{A}_\Delta)$.

It is not difficult to verify that we get the same cohomology if, instead of taking forms with coefficients in R_Δ we take coefficients of type f/d^k with f any function holomorphic around 0.

total residue $Tres$ and of the residues $res_{\underline{b}}$.

Now these operators are defined on algebraic differential forms of degree n instead that on functions (having identified a function f with the form $f dy_1 \wedge \cdots \wedge dy_n$).

intrinsic definition of $Tres$ and of the residues $res_{\underline{b}}$. $Tres(f dy_1 \wedge \cdots \wedge dy_n)$ is the cohomology class of $f dy_1 \wedge \cdots \wedge dy_n$.

$$Tres(\psi) = \sum_{\underline{b}=\{\alpha_1,\dots,\alpha_n\}\in NBB} res_{\underline{b}}\psi \, d\log(\alpha_1(y)) \wedge \cdots \wedge d\log(\alpha_n(y))$$

Part 8. **Computing residues**

25. Residues

For the moment our definition of residues $res_{\underline{b}}\psi$ is purely algebraic. We need to give to it a geometric content that will allow us of compute them.

This is a subtle point that we attack first in an essentially local and elementary way and next we will give an idea of its true geometric meaning, giving a more general meaning to the notion of residue in several dimensions.

We explain the strategy leaving the details to the next Lecture.

Given a basis $\underline{b}$ we perform a *non linear change of coordinates*.

- In fact given $\underline{b}$ we perform the change in two steps.
  - First, we choose a system of linear coordinates y_i dependent on $\underline{b}$.
  - Next we build coordinates $z_1, \dots, z_n$ in which, every y_i is expressed as an appropriate product (monomial) of a subset of the coordinates $z_1, \dots, z_n$. This depends on the combinatorics associated to $\underline{b}$ through the theory of nested sets.
- In this new system of coordinates, every function $\alpha_i(y)$ becomes a polynomial of the form:

$$\alpha_i(y_1(z), \dots, y_n(z)) = \prod_{j=1}^{n} z_j^{h_{i,j}} Q_i(z), \ | Q_i(0) \neq 0, \ h_{i,j} = 0, 1. \tag{34}$$

- It follows that every denominator that appears in R_Δ becomes:

$$\frac{1}{\prod_{i=1}^{N} \alpha_i^{k_i}} =$$

$$\prod_{j=1}^{n} z_j^{-\sum_{i=1}^{N} h_{i,j}k_i} \prod_{i=1}^{N} Q_i(z)^{-h_{i,j}k_i} = \prod_{j=1}^{n} z_j^{-\sum_{i=1}^{N} h_{i,j}k_i} T(z), \ | T(0) \neq 0.$$

- Finally, while before around 0 the poles of the function form a very complex geometric structure, in these new coordinates we have *resolved the complication* and the poles are only on the coordinate hyperplanes.

- The price we pay is that we must develop several non linear changes of coordinates/ In technical words we modify the ambient space *blowing up* certain varieties.
- With the new coordinates, an algebraic $n-$form can be written in the form

$$(z_1 \dots z_n)^{-k} f(z_1, \dots, z_n) dz_1 \wedge \dots \wedge dz_n$$

  with $f(z_1, \dots, z_n)$ holomorphic around to 0. We can compute its residue around allo 0 as the coefficient of $(z_1 \dots z_n)^{-1}$.

  Let us denote this residue res_z for the moment.
- This residue is an integral $\frac{1}{(2\pi i)^n} \int_T \psi$ where $T = \{(z_1, \dots, z_n) \mid |z_i| = \epsilon\}$ is a small torus around 0.

  Of course in the old coordinates the definition of T is more complex and does not appear immediately as a torus.
- The fundamental fact, that allows us of conclude the discussion is the following:

$$res_z = res_{\underline{b}}$$

- Since the residue depends only on the cohomology class, this can be proved verifying that, given a no broken basis $\underline{c} := \{\alpha_1(y), \dots, \alpha_n(y)\}$:

$$res_z \ d\log(\alpha_1(y)) \wedge \dots \wedge d\log(\alpha_n(y)) = \begin{cases} 1 & \text{se} \quad \underline{c} = \underline{b} \\ 0 & \text{se} \quad \underline{c} \neq \underline{b} \end{cases}$$

All the steps of the preceding discussion are constructive and thus provide a computational algorithm.

In order to build the coordinate changes, we must develop a new theory of algebraic combinatorial type.

Wonderful models

- This theory has been developed in [10] with the purpose of building the so called *wonderful models* of configurations of hyperplanes.
- We have of build an algebraic variety X_Δ with a proper morphism $\pi : X_\Delta \to \mathbb{C}^n$ in such a way that $\pi^{-1}(A_\Delta)$ is isomorphic through π to A_Δ and the complement in X_Δ of $\pi^{-1}(A_\Delta)$ is *a divisor with normal crossings.*

We will not need these notions and thus we will not make them precise.

26. Irreducibles and "nested sets"

Before we introduce the concepts of this lecture we recall some basic definitions.

Direct sums

- We say that U is *a direct sum* of the U_i when the expression of a vector of U as sum of the u_i is unique that is, if $0 = u_1 + u_2 + \cdots + u_k,\ u_i \in U_i$, it must be necessarily $u_i = 0,\ \forall i$.
- In this case we write:

$$U = U_1 \oplus U_2 \oplus \cdots \oplus U_k.$$

Under these hypotheses we shall use two simple properties: Propriety of direct sum

- $$\dim(U) = \sum_{i=1}^{k} \dim(U_i)$$
- If $W_i \subset U_i$ are subspaces, also the W_i form a direct sum.

27. Irreducibles and decompositions

The notions that we are about to give are of combinatorial nature (cf. [16]) but we develop them in the following context:

Given a list of non zero vectors $\Delta := \{v_1, \ldots, v_N\}, \quad v_i \in \mathbb{C}^n$.
Given a subset $A \subset \Delta$ let us define:

Complete sets

(1) $\langle A \rangle$ the vector space generated by A.

(2) $\overline{A} := \Delta \cap \langle A \rangle$ the *completion* of A.

We shall say thus that A is *complete* if $A = \overline{A}$.

In our applications, we think of the vectors Δ as linear equations on the dual space, that we shall call V.

Given a set A, set $A^{\perp}$ to be the subspace of V defined by the vanishing of the equations in A.

Remark 27.1. *By duality, the completion $\overline{A}$ is the set of all the equations in Δ that vanish on $A^{\perp}$.*

Finally: *the complete subsets of Δ are in 1-1 correspondence (inverting the inclusion ordering), with the subspaces of the arrangement defined by Δ.*
Decompositions and irreducibles

3 Given a complete set $A \subset \Delta$, a decomposition is a decomposition $A = A_1 \cup A_2$ in non empty sets, such that:

$$\langle A \rangle = \langle A_1 \rangle \oplus \langle A_2 \rangle.$$

Clearly the two sets A_1, A_2 are necessarily complete.

4 We shall say that a complete set A is irreducible if it does not have a non trivial decomposition.

Of course Δ is complete by definition, it can be or not be irreducible. We can reduce always to discuss the irreducible case.

If $A = A_1 \cup A_2$ is a decomposition of a complete set and $B \subset A$ is still complete we have:

$B = B_1 \cup B_2,$ *where* $B_1 = A_1 \cap B,\ B_2 = A_2 \cap B.$

Given that clearly $\langle B\rangle = \langle B_1\rangle \oplus \langle B_2\rangle$, we have that:

Corollary 27.2. *$B = B_1 \cup B_2$ is a decomposition, unless one of the two sets is empty.*

We deduce immediately:

Lemma 27.3. *If $A = A_1 \cup A_2$ is a decomposition and $B \subset A$ is irreducible, then $B \subset A_1$ or $B \subset A_2$*

By induction we prove:

Theorem 27.4. *Every set A can be decomposed as $A = A_1 \cup A_2 \cup \cdots \cup A_k$ with the A_i Irreducible and:*

$$\langle A\rangle = \langle A_1\rangle \oplus \langle A_2\rangle \oplus \cdots \oplus \langle A_k\rangle.$$

This decomposition is clearly *unique up to order*:

if $A = B_1 \cup B_2 \cup \cdots \cup B_h$ is a second decomposition, from the preceding Lemma, every B_i is contained in an A_j and viceversa.

This implies that the sets are the same up to order.

$A = A_1 \cup A_2 \cup \cdots \cup A_k$ is called the *decomposition in Irreducibles of* A.

Example 27.5. *We work out these concepts for the configuration space 17. In this case the vectors are the equations $z_i - z_j$.*

- We have already seen that the complete sets correspond to the subspaces of this configuration. These subspaces correspond to partitions $\mathcal{P} := S_1, \dots, S_k$ of $\{1, 2, \dots, n\}$.
- To such partition corresponds the complete set formed by the elements $z_i - z_j$ such that i, j is in the same part of $\mathcal{P}$.

At this point we leave as exercise to verify that:

Exercise

(1) Irreducible sets correspond to partitions with a unique part with at least 2 elements.
(2) Therefore the irreducibles are in 1-1 correspondence with the subsets $C \subset \{1, 2, \dots, n\}$ with at least 2 elements.
(3) To the set C corresponds $I_C := \{z_i - z_j \,|\, i, j \in C\}$.
(4) If A corresponds to a partition $S_1, \dots, S_k$ its decomposition in Irreducibles is made of the elements I_{S_i}, as S_i varies on the subsets of the partition with at least 2 elements.

28. Nested sets

In the next step we define the notion of *nested set*.

We say that two sets A, B are *comparable* if one is contained in the other:

Definition 8. *We shall say that a family $\mathcal{S}$ of Irreducibles A_i is nested if the following property hold:*

Given elements $A_1, \ldots, A_i \in \mathcal{S}$ mutually incomparable we have:

$C := A_1 \cup A_2 \cup \cdots \cup A_i$ is complete and $C := A_1 \cup A_2 \cup \cdots \cup A_i$ is its decomposition into irreducibles.

We are in particular interested in maximal nested sets. We shall denote them by MNS (maximal nested set).

We shall use two simple inductive ways to construct nested sets:

EXERCISE

(1) Given:
- A nested set$\mathcal{S}$.
- An irreducible A minimal in $\mathcal{S}$.
- A nested set $\mathcal{P}$ contained in A we have:

$\mathcal{S} \cup \mathcal{P}$ *is nested.*

(2) Given:
- A nested set$\mathcal{S}$.
- A complete set A containing $\mathcal{S}$.
- Let $A = A_1 \cup \cdots \cup A_k$ be its decomposition into irreducibles, we have:

$\mathcal{S} \cup \{A_1, \ldots, A_k\}$ *is nested.*

Example: configuration space. We know that the Irreducibles are in 1-1 correspondence with the subsets $C \subset \{1, 2, \ldots, n\}$ with at least 2 elements.

We must understand when a family of such subsets is nested.

We leave the answer to the reader:

EXERCISE Verify that, in this case, the condition is that two subsets of the family are comparable or disjoint.

REMARK If $A_1, \ldots, A_k$ is nested we have that $\cup_i A_i$ is complete. In fact this union can be obtained, taking the maximal elements, that are necessarily non comparable and then applying the definition of nested.

Let thus $\Delta \subset V$ with $\langle \Delta \rangle = V$.

Theorem 28.1. *Let $\mathcal{S} := \{A_1, \ldots, A_k\}$ a MNS in Δ.*

Given $A \in \mathcal{S}$ let $B_1, \ldots, B_s$ be the elements of $\mathcal{S}$ contained properly in A, and maximal with this property.

(1) $C := B_1 \cup \cdots \cup B_s$ is complete and decomposed by the B_i.

(2) $\dim\langle A \rangle = \dim\langle C \rangle + 1$.

(3) $k = n = \dim(V)$.

Theorem 28.2. *(1) $C := B_1 \cup \cdots \cup B_s$ is complete and decomposed by the B_i.*
(2) $\dim\langle A\rangle = \dim\langle C\rangle + 1$.
(3) $k = n = \dim(V)$.

Proof. 1. is the definition of Nested since the B_i, being maximal are necessarily non comparable.

2. Let us consider $\langle C\rangle = \oplus_{i=1}^{s}\langle B_i\rangle \subset \langle A\rangle$.
It cannot be $\langle C\rangle = \langle A\rangle$ otherwise, being by definition of nested, C complete we must have $A = C$.

This is absurd since A is irreducible and the B_i are properly contained in A.

Therefore there exists an element $a \in A \,|\, a \notin C$. Let us denote by $A' := \Delta \cap \langle C, a\rangle$.
We have $C \subsetneq A' \subset A$. We claim that $A = A'$.

Otherwise, as one can easily see, adding all the Irreducibles that decompose A' to the family $\mathcal{S}$, we obtain a nested family that contains properly $\mathcal{S}$. This contradicts the maximality of $\mathcal{S}$.
Clearly $A = A'$ implies that $\langle A\rangle = \langle C\rangle \oplus \mathbb{C}a$ and thus $\dim\langle A\rangle = \dim\langle C\rangle + 1$.

3. We prove it by induction.

If $n = 1$ there is nothing to prove, there is a unique set complete and irreducible, Δ.

First, decompose $\Delta = \cup_{i=1}^{h}\Delta_h$ into irreducibles.

We have that a MNS in Δ is the union of MNS in each Δ_i. Then $n = \dim(\langle\Delta\rangle) = \sum_{i=1}^{h}\dim(\langle\Delta_h\rangle)$.

We can now reduce ourselves to the case Δ irreducible. In this case we have that $\Delta \in \mathcal{S}$ for every MNS $\mathcal{S}$.

Let $B_1, \dots, B_s$ be the elements of $\mathcal{S}$ properly contained in Δ and maximal with this property.

The set $\mathcal{S}$ consists of Δ and of the subsets $\mathcal{S}_i := \{A \in \mathcal{S} \,|\, A \subset B_i\}$.

Clearly $\mathcal{S}_i$ is a MNS relative to the set B_i (otherwise we could add an element to $\mathcal{S}_i$ and to $\mathcal{S}$ contradicting the maximalty of $\mathcal{S}$).

By induction $\mathcal{S}_i$ has $\dim\langle B_i\rangle$ elements and thus by 2) The claim follows. □

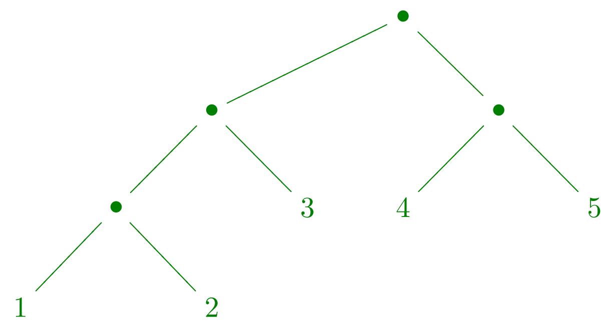
Example of configuration space
One can see easily (exercise) that a maximal nested set $\mathcal{S}$, of subsets of $\{1, \dots, n\}$ is formed by $n - 1$ elements and has the following properties:

Given $A \in \mathcal{S}$ a set with a elements.
We have one of the two following possibilities for the elements $B_1, \dots, B_s$ of $\mathcal{S}$ properly contained in A and maximal with this property.

- $s = 1$ and B_1 has $a - 1$ elements.
- $s = 2$ and $A = B_1 \cup B_2$.

We can present such a MNS in a convenient way as a *planar binary graph*.

Example 28.3.



Every internal vertex of the graph corresponds to the set of numbers that appear on its *leaves*, in the example the MNS is:

$$\{1, 2\}, \{1, 2, 3\}, \{4, 5\}, \{1, 2, 3, 4, 5\}$$

Given a MNS $\mathcal{S}$ let us define a map:

$$p_{\mathcal{S}} : \Delta \to \mathcal{S}$$

as follows.

- If an element $a \in \Delta$ appears in two elements $A, B \in \mathcal{S}$ the two elements must be necessarily comparable, thus there exists a minimum among the two.
- Since $\Delta \in \mathcal{S}$ there exists a minimum element $A \in \mathcal{S}$ with $a \in A$.
- We define then $A := p_{\mathcal{S}}(a)$.

Now a new definition:

Definition 9. *We shall say that a basis $\underline{b} := \{a_1, \ldots, a_n\} \subset \Delta$ of $\mathbb{C}^n$, is adapted to the MNS $\mathcal{S}$ if the map $a_i \mapsto p_{\mathcal{S}}(a_i)$ is a bijective map between $\underline{b}$ and $\mathcal{S}$.*

Such basis exists always, it suffices to take, as in the proof of 28.1, for every $A \in \mathcal{S}$ an element $a \in A - \cup_i B_i$, where the B_i are the elements of $\mathcal{S}$ properly contained in A.

29. Non linear coordinates

Now we pass to the fundamental geometric construction.

Given a MNS $\mathcal{S}$ and a basis $\underline{b} := \{a_1, \ldots, a_n\}$ adapted to $\mathcal{S}$ let us *consider the a_i as a system of linear coordinates on V.*

If $p_{\mathcal{S}}(a_i) = A$ we denote also $a_i := a_A$. We build now new coordinates z_A, $A \in \mathcal{S}$ using the monomial expressions:

$$a_A := \prod_{B\in\mathcal{S},\ A\subset B} z_B. \tag{35}$$

The residues The residue in 0, of a form ψ, in the coordinates z_A associated to $\mathcal{S}$ and $\underline{b}$, will be denoted by the symbol $res_{\mathcal{S},\underline{b}}\psi$ (or $res_{\mathcal{S}}\psi$ if $\underline{b}$ is clear from the context).

We see now, how to write an element $a \in \Delta$ in these coordinates.

Given $A \in \mathcal{S}$ let $\mathcal{S}_A := \{B \subset A,\ B \in \mathcal{S}\}$.

Clearly $\mathcal{S}_A$ is a MNS for A (in place of Δ).

The elements a_B with $B \in \mathcal{S}_A$ form a basis, adapted to $\mathcal{S}_A$.

Let $p_{\mathcal{S}}(a) = A$, since the elements a_B with $B \subset A,\ B \in \mathcal{S}$ form a basis adapted to $\mathcal{S}_A$ we must have $a = \sum_{B\in\mathcal{S}_A} c_B a_B,\ c_B \in \mathbb{C}$.

Let us substitute now the expressions (35):

$$a_A := \sum_{B\in\mathcal{S}_A} c_B \prod_{C\in\mathcal{S},\ B\subset C} z_C =$$

$$\prod_{B\in\mathcal{S},\ A\subset B} z_B (c_A + \sum_{B\in\mathcal{S}_A, B\neq A} c_B \prod_{C\in\mathcal{S},\ B\subset C\subsetneq A} z_C). \tag{36}$$

Since A is the minimum set of $\mathcal{S}$ containing a we must have $c_A \neq 0$.

Let us remark that, in our program, we have succeeded to define non linear changes of coordinates that satisfy the condition of (34).

We must now study the other conditions. First we must tie these concepts to that of no broken basis.

We start with a remark.

Given any basis $\underline{b} := \{a_1, \dots, a_n\} \subset \Delta$, we shall build a MNS $\mathcal{S}_{\underline{b}}$ to which it is adapted, in the following way.

- We define the complete sets $A_i := \Delta \cap \langle\{a_i, \dots, a_n\}\ \rangle = \overline{\{a_i, \dots, a_n\}}$. Clearly $A_1 = \Delta \supset A_2 \supset \cdots \supset A_n$.
- For each i consider all the Irreducibles in the decomposition of A_i. Clearly for different i we can obtain also several times the same irreducible, in any case we have:

Theorem 29.1. *The family $\mathcal{S}_{\underline{b}}$ of all the (distinct) irreducibles that appear in the decompositions of the sets A_i form a MNS to which the basis $\underline{b}$ is adapted.*

Proof

By induction. Decompose $A_1 = B_1 \cup B_2 \cup \cdots \cup B_k$ into irreducibles, by construction:

$$n = \dim\langle A_1\rangle = \sum_{i=1}^{k} \dim\langle B_i\rangle.$$

Therefore we can have $\dim\langle A_2 \cap B_i\rangle < \dim\langle B_i\rangle$ for at most one index i.

We have that $A_2 = (A_2 \cap B_1) \cup (A_2 \cap B_2) \cup \cdots \cup (A_2 \cap B_k)$ is a decomposition of A_2, not necessarily into irreducibles.

Since $\dim\langle A_2\rangle = n - 1$ we have:

$$n - 1 = \dim\langle A_2\rangle = \sum_{i=1}^{k} \dim\langle A_2 \cap B_i\rangle.$$

Therefore can be $\dim\langle A_2 \cap B_i\rangle < \dim\langle B_i\rangle$ for in the more an index i. In other words we must have that $A_2 \cap B_i = B_i$ for all the i, except for one index i_0. For such an index necessarily $a_1 \in B_{i_0}$.

By induction, the family of all the (distinct) irreducibles that appear in the decompositions of the sets $A_i,\ i \geq 2$ form a MNS for $\langle A_2\rangle$, with adapted basis $\{a_2, \dots, a_n\}$. To this set we must thus only add B_{i_0}.

We have thus a nested set with n elements, hence maximal.

Clearly the basis is adapted and the required properties hold. □

Given an ordering of Δ, we have defined the residues $res_{\underline{b}}$, as $\underline{b}$ varies in the no broken bases.

Our goal is to prove that:

$$res_{\underline{b}}\psi = res_{\mathcal{S}_{\underline{b}}}\psi.$$

Assume now that the basis $\underline{b} := \{a_1, \dots, a_n\} \subset \Delta$ is no broken, we get:

Lemma 29.2. *a_i is the minimum element of $p_{\mathcal{S}_{\underline{b}}}(a_i)$ for every i.*

Proof By the definition of $S_{\underline{b}}$, if $a_i \in A \in S_{\underline{b}}$ we must have that A decomposes one of the sets $A_k = \langle a_k, \dots, a_n\rangle \cap \Delta$. Necessarily it must be $k \leq i$.

On the other hand a_i belongs to one of the irreducibles of A_i, that therefore is contained in each irreducible B of $A_k,\ k < i$ that contains a_i.

By definition of no broken basis a_i is the minimum element of $A_i = \langle a_i, \dots, a_n\rangle \cap \Delta$. □

This property suggests us to define:

Definition 10. *A MNS $\mathcal{S}$ is said to be proper, if the elements $a_S := \min a,\ a \in S \,|\, S \in \mathcal{S}$ form a basis.*

Remark 29.3. *If $\mathcal{S}$ is proper, $p_{\mathcal{S}}(a_S) = S$. Thus the elements $a_S := \min a,\ a \in S \,|\, S \in \mathcal{S}$ form a basis adapted to $\mathcal{S}$.*

Proof Let $U \in \mathcal{S}$ with $a_S \in U$.

If $U \subset S$, we have necessarily that $a_S = \min a \in U$ and thus $a_S = a_U$.

Since the elements a_S are a basis, this implies that $S = U$. □

If $\mathcal{S}$ is proper, we order its subsets $S_1, \dots, S_n$ using the increasing order of the elements a_S. We denote then $a_i := a_{S_i}$, and we have:

Theorem 29.4. *The basis* $\underline{b} := \{a_1, \dots, a_n\}$ *is no broken.*

In this way we establish a 1-1 correspondence between no broken bases and proper MNS.

Proof

From the remark at page 44, setting $A_i := \cup_{j \geq i} S_j$, we have that A_i is complete and decomposed by the S_j which are maximal.

Clearly, by definition, a_i is the minimum of A_i. It suffices toprove that $A_i = \langle a_i, \dots, a_n \rangle \cap \Delta$.

That is, since A_i complete, that $\langle A_i \rangle = \langle a_i, \dots, a_n \rangle$.

We prove it by induction. If $i = 1$ since $\mathcal{S}$ is maximal we have $A_1 = \Delta$ and a_1 is the minimum element in Δ.

Since $a_1 \notin A_2$ we must have $A_2 \neq \Delta$ and, since A_2 is complete $\dim(\langle A_2 \rangle) < n$.

Since clearly $\langle A_2 \rangle \supset \langle a_2, \dots, a_n \rangle$ we must have $\langle A_2 \rangle = \langle a_2, \dots, a_n \rangle$.

At this point it follows that, by induction, $\langle a_2, \dots, a_n \rangle$ is a no broken basis in A_2 and thus, since a_1 is minimum in Δ, that $\langle a_1, a_2, \dots, a_n \rangle$ is a no broken basis.

From the proof, it follows that the two constructions, of MNS associated to a no broken basis and of no broken basis associated to a proper MNS, are inverse of each other and thus the 1-1 correspondence is established. □

Example configuration space.

In this example we identify Δ to the set of pairs $(i, j),\ i < j$ of numbers $\leq n$.

We fix an total ordering on Δ by imposing:

$(i, j) \leq (h, k)$ se $k - h < j - i$ e se $k - h = j - i$, se $i \leq h$.

Lemma 29.5. *In this case a proper MNS* $\mathcal{S}$ *is a MNS of subsets of* $\{1, 2, \dots, n\}$. *Each of these subsets is formed by at least two elements, with the following property. Taking for every subset* A *the pair* $c(A) := (i, j),\ i := \min(A),\ j := \max(A)$,*formed by the minimum and the maximum in* A *we obtain distinct pairs.*

Proof Let $z_A := z_i - z_j$ with $(i, j) = c(A)$. To say that $\mathcal{S}$ is proper means that the $n - 1$ elements z_A are linearly independent. In other words that, chosen k, the vectors z_k, z_A as A varies in $\mathcal{S}$, generate the space spanned by the z_i.

We argue by induction starting from $A = \{1, 2, \dots, n\} \in \mathcal{S}$ to which we associate $z_A = z_1 - z_n$.

For the maximal subsets of $\mathcal{S}$, properly contained in A, we have two possibilities.

Either they reduce to an unique element B, with $n - 1$ elements, or to two disjoint elements B_1, B_2, with union A.

In the first case we must have, (since $\mathcal{S}$ is proper) either $B = \{2, \dots, n\}$, or $B = \{1, 2, \dots, n - 1\}$.

By induction, adding z_n in the first case, and z_1 in the second, we deduce easily the propriety that they span.

If instead $A = B_1 \cup B_2$, always using the fact that $\mathcal{S}$ is proper, we can assume that $1 \in B_1,\ n \in B_2$.

By induction, the space generated by $z_C,\ C \subset B$ and z_1 coincides with the space generated by the $z_i,\ i \in B_1$, similarly for B_2 and z_n.

If thus, we add z_1 to all the $z_A,\ A \in \mathcal{S}$, we have also $z_n = z_1 - (z_1 - z_n)$ and thus all the z_i. □

We want to establish a 1-1 correspondence between such proper MNS and the permutations of n elements that fix n, (proving in particular that their number is $(n-1)!$.

It is better to formulate this claim in a more abstract way.

Give a proper MNS in a totally ordered S set. We want associare to it a word, without repetitions, in all the elements of S terminating with the maximal element of S.

Let us consider thus a MNS $\mathcal{S}$ given by a sequence $\{S_1, \dots, S_{n-1}\}$ of subsets of $\{1, \dots, n\}$, with the preceding properties.

We may assume that $S_1 = (1, 2, \dots, n)$. We know that $\mathcal{S}' := \mathcal{S} - \{S_1\}$ has 1 or 2 maximal elements.

We treat first the case of a unique maximal element. We can assume it is S_2.

Since $\mathcal{S}$ is proper we cannot have $1, n \in S_2$. If $1 \notin S_2$, we must have $S_2 = (2, \dots, n)$ otherwise $n \notin S_2$, and $S_2 = (1, 2, \dots, n-1)$.

In the first case we associate a word $p(\mathcal{S}')$, formed with the elements $2, \dots, n$ and that terminates with n. We define then $p(\mathcal{S}) := 1p(\mathcal{S}')$.

We obtain in this way all the words in $1, 2, \dots, n-1, n$ that, start with 1 and end with n.

In the second case, let us consider S_2. In this case though, we use the opposite ordering $(n-1, \dots, 2, 1)$. Still by induction we have a word $p(\mathcal{S}')$ in $1, \dots, n-1$ that ends with 1. We set $p(\mathcal{S}) = p(\mathcal{S}')n$.

We obtain all the words in $1, 2, \dots, n-1, n$ that end with 1,n.

It remains to determine the words in which 1 appears in the interior of the word, preceding n.

We assume now to have two maximal elements (each with at least 2 elements) that we denote S_2 and S_3, whose disjoint union is $\{1, \dots, n\}$. Let us denote with $\mathcal{S}_i,\ i = 1, 2$ the corresponding induced MNS.

Moreover, since $\mathcal{S}$ is proper, we can assume that $1 \in S_2$, $n \in S_3$.

By induction we have two words $p(\mathcal{S}_2)$ for S_2, relative to the opposite order to the natural one. By construction $p(\mathcal{S}_2)$ terminates with 1. $p(\mathcal{S}_3)$ for S_3 relative to the usual ordering and that terminates with n. We set $p(\mathcal{S}) = p(\mathcal{S}_2)p(\mathcal{S}_3)$. Precisely a word of the remaining type.

It isessentially clear that this construction can be inverted and that, in this way we have established a 1-1 correspondence. In particular we have $(n-1)!$ proper MNS that can be recovered recursively starting from the permutations. □

Example 29.6. *1. Let us take the MNS formed by the subsets*

$$\{i, i+1, \dots, n\},\ i = 1, \dots, n-1.$$

Its associated permutation is the identity

2. Let us take the MNS formed by the subsets

$$\{1, 2, \dots, i\},\ i = 2, \dots, n.$$

Its associated permutation is $n-1, n-2, \dots, 1, n$.

3. The example 28.3 is proper and its associated permutation is: $3, 2, 1, 4, 5$.

30. Residues and cycles

We can now complete our analysis computing the residues.

For convenience, if $\underline{b} := \{b_1, \dots, b_n\} \subset \Delta$ is a basis, we denote with:

$$\omega_{\underline{b}} := d\log(b_1) \wedge \cdots \wedge d\log(b_n)$$

Theorem 30.1. *Given two no broken bases* $\omega_{\underline{b}}, \omega_{\underline{c}}$ *we have:*

$$res_{\mathcal{S}_{\underline{b}}}\omega_{\underline{c}} = \begin{cases} 1 & se \quad \underline{b} = \underline{c} \\ 0 & se \quad \underline{b} \neq \underline{c} \end{cases}$$

Proof We prove first that $res_{\mathcal{S}_{\underline{b}}}\omega_{\underline{b}} = 1$.

By definition $b_i = \prod_{A_i \subset B} z_B$ da cui $d\log(a_i) = \sum_{A_i \subset B} d\log(z_B)$.

When we expand the product we get a sum of products of type $d\log(z_{B_1}) \wedge d\log(z_{B_2}) \wedge \cdots \wedge d\log(z_{B_n})$ with $A_i \subset B_i$.

Now, the unique non decreasing and injective map, of $\mathcal{S}_{\underline{b}}$ in itself is the identity.

Therefore in this sum, all the monomials vanish except for the monomial $d\log(z_1) \wedge d\log(z_2) \wedge \cdots \wedge d\log(z_n)$, that has residue 1.

Let us pass now ito the second case $\underline{b} \neq \underline{c}$. This follows immediately from the following Lemma.

Lemma 30.2. *1. If* $\underline{b} \neq \underline{c}$*, the basis* $\underline{c}$ *is not adapted to* $\mathcal{S}_{\underline{b}}$.

2. If a basis $\underline{c}$ *is not adapted to a MNS* $\mathcal{S} = \{S_1, \dots, S_n\}$ *we have* $res_{\mathcal{S}}\omega_{\underline{c}} = 0$.

Proof 1. Let $\underline{c} = \{c_1, \dots, c_n\}$ be adapted to $\mathcal{S}_{\underline{b}}$, we want prove that we have $\underline{b} = \underline{c}$.

We know that $c_1 = a_1 = b_1$, is the minimum element of Δ.

Let A be the irreducible component of Δ containing a_1. This, by definition, is an element of $\mathcal{S}_{\underline{b}}$. We have thus $A = S_1$.

We claim that $p_{\mathcal{S}_{\underline{b}}}(a_1) = A$.
This follows from the fact that $a_1 = b_1$ and $\mathcal{S}_{\underline{b}}$ is proper.

Set $\Delta' := S_2 \cup \cdots \cup S_n$, Δ' is complete. The set $S_2, \ldots, S_n$ coincides with the proper MNS $\mathcal{S}_{\underline{b}'}$ associated to the no broken basis $\underline{b}' := \{b_2, \ldots, b_n\}$ of $\langle \Delta' \rangle$.

Moreover clearly, $\underline{c}' = \{c_2, \ldots, c_n\}$ is adapted to $\mathcal{S}_{\underline{b}'}$. Therefore $\underline{b}' = \underline{c}'$ by induction. Hence $\underline{b} = \underline{c}$.

2. Given $a \in \Delta$, with $p_{\mathcal{S}}(a) = A$ we have from fomula 36:

$$a = \prod_{B \in \mathcal{S},\ A \subset B} z_B (c_A + \sum_{B \in \mathcal{S}_A, B \neq A} c_B \prod_{C \in \mathcal{S},\ B \subset C \subsetneq A} z_C),\ c_A \neq 0.$$

Therefore $d \log(a)$ is the sum of the two closed 1-forms

$$\gamma_A := \sum_{B \in \mathcal{S},\ A \subset B} d \log z_B, \quad \psi_A := d \log(c_A + \sum_{B \in \mathcal{S}_A, B \neq A} c_B \prod_{C \in \mathcal{S},\ B \subset C \subsetneq A} z_C). \tag{37}$$

Since $c_A \neq 0$ there exists, in a neighborhood of 0, a holomorphic determination $\log(c_A + \sum_{B \in \mathcal{S}_A, B \neq A} c_B \prod_{C \in \mathcal{S},\ B \subset C \subsetneq A} z_C)$. Therefore ψ_A is exact and holomorphic in a neighborhood of 0.

Let us apply the formula 37 to a_i setting $p_{\mathcal{S}}(a_i) = A_i, \psi_i := \psi_{A_i}$ and $\gamma_{A_i} = \gamma_i$.

We develop the product of the forms $\gamma_i + \psi_i$ in order to obtain the form $\omega_{\underline{c}}$. We obtain various terms, those that contain a factor ψ_i are exact since:
i) the product of closed forms is a closed form and
ii) a closed form times an exact form, is exact.
The residue of an exact form is 0. It remains the term $\gamma_1 \wedge \cdots \wedge \gamma_n$ this equals zero. In fact the basis $\underline{c}$ is not adapted to $\mathcal{S}$, thus we have $A_i = A_j$ for two distinct indices and thus two repeated factors in the product. □

CONCLUSION We have proved for no broken bases the desired formula:

$$res_{\underline{b}} = res_{\mathcal{S}_{\underline{b}}}.$$

This last residue $res_{\mathcal{S}_{\underline{b}}} \psi$ is the coefficient of $\prod z_A^{-1} dz_1 \wedge \ldots dz_n$, in the development of the form ψ in the coordinates z_i associated to $\mathcal{S}$.

This residue can be computed with an explicit algorithm.

Remark 30.3. *The formula of Theorem 30.1 proves immediately that the forms $\omega_{\underline{b}}$ are linearly independent and completes the proof of Theorem 18.1.*

It remains to discuss a last algorithmic point.
In order to compute the Jeffry–Kirwan residue, in a given a point $p \in C(A)$, it is necessary to determine a big cell c for which $p \in \overline{c}$.

In general, the determination of the big cells is a very complex problem. For our computations it suffices much less.

Let us take thus simply a point q internal to $C(A)$ and not laying on any hyperplane generated by $n-1$ vectors of Δ. This is not difficult to do, and let us consider the segment qp.

This segment intersects these hyperplanes in a finite number of points, thus we can determine an ϵ sufficiently small for which all the points $tp + (1-t)q, 0 < t < \epsilon$ are regular.

If we take one of these points q_0 it lays in a cell for which p is in the closure.

At this point, for every no broken basis, we must verify in simple way if q_0 lays or not in the cone generated by the basis.

Part 9. $D-$**modules Parte I**

Lezione. 10 $D-$modules Part I

In this lezione we prove the linear independence of the monomials asserted in Theorem 14.3, using a general method with its origins in the theory of D-modules.

31. The Weyl Algebra

We will need only some basic results from the theory of D-modules, for which a good reference is [8].

Let us start with the algebra of differential operators with polynomial coefficients, sometimes called the Weyl algebra:

$$W(n) := \mathbb{C}[x_1, \dots, x_n, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}]. \tag{38}$$

It is easy to see that each element of $W(n)$ can be written in a unique manner as a linear combination of elements:

$$x_1^{h_1} \dots x_n^{h_n} \frac{\partial^{k_1}}{\partial x_1} \dots \frac{\partial^{k_n}}{\partial x_n}, \quad h_i, k_j \in \mathbb{N}.$$

The algebra $W(n)$ is not commutative; its generators satisfy the canonical commutation relations (of quantum mechanics):

$$[x_i, \frac{\partial}{\partial x_i}] = x_i \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_i} x_i = -1, \quad [x_i, \frac{\partial}{\partial x_j}] = 0, \ \forall i \neq j, \tag{39}$$

$$[x_i, x_j] = [\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0, \ \forall i, j.$$

Remark: We use the notion of the *commutator* of two elements $[a, b] := ab - ba$ in an arbitrary algebra.

We will use some simple properties of the commutator:

Lemma 31.1.

$$[a, bc] = [a, b]c + b[a, c], \quad \textit{Leibnitz rule}$$

If $[a, b] = 1$ *then* $[a, b^n] = nb^{n-1}, \ \forall n \geq 1.$

Proof $[a, b]c + b[a, c] = (ab - ba)c + b(ac - ca) = abc - bca = [a, bc]$.

For the second identity, the case $n = 1$ is true by hypothesis, then we use induction and the Leibnitz rule:

$$[a, b^n] = [a, b]b^{n-1} + b[a, b^{n-1}] = b^{n-1} + b(n-1)b^{n-2} = nb^{n-1}.$$

Let us apply this lemma to $a = \frac{\partial}{\partial x_i}, \ b = x_i$, and then again to $a = -x_i, \ b = \frac{\partial}{\partial x_i}$, obtaining:

$$[\frac{\partial}{\partial x_i}, x_i^n] = nx_i^{n-1}, \ [x_i, \frac{\partial}{\partial x_i}^n] = -n\frac{\partial}{\partial x_i}^{n-1}.$$

Remark 31.2. *It is not necessary to consider all variables and derivatives. Below, it shall be useful to consider other orderings of variables and derivatives, and even variables and derivatives alternatively. Using the commutation rules, these are bases obtained from each other by triangular changes of basis.*

As an exercise, the reader can deduce the previous remark, as well as the simple fact: *(algebraic Fourier transform)*:

EXERCISE For each i, there exists an automorphism $f_i : W(n) \to W(n)$ defined by:

$$f_i(x_j) = x_j, \ f_i(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial x_j}, \ \forall j \neq i, \quad f_i(\frac{\partial}{\partial x_i}) = -x_i, \ f_i(x_i) = \frac{\partial}{\partial x_i}$$

The algebra $W(n)$ operates (through differential operators) on the ring of polynomials $\mathbb{C}[x_1, \ldots, x_n]$ and it is immediate to see that: if $f \in \mathbb{C}[x_1, \ldots, x_n], \ f \neq 0$, then $W(n)$ operates on the ring $\mathbb{C}[x_1, \ldots, x_n, f^{-1}]$ of rational functions with a power of f as the denominator.

These modules are important in the general theory (the Theory of Bernstein polynomials) and have very special properties.

In our context we wish to study the case in which $f = \prod_{i=1}^m \alpha_i(x)$ is a product of linear forms.

These and some of their sub- and quotient modules are the only $D-$modules that we will use.

We begin with an example. Consider the ring of Laurent polynomials $R := \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

Let us define the subspace R_k consisting of all the Laurent polynomials in which at most k of the variables appear with a strictly negative exponent.

From the chain rule, it follows that R_k is a sub-module.

For $k \leq n$ consider the module R_k/R_{k-1}, which has as basis the (equivalence classs of) monomials in which exactly k of the variables have a strictly negative exponent.

Moreover, for each choice of subset S with of the variables, with cardinality of S equal to k, we have the sub-module $(R_k/R_{k-1})_S \subset R_k/R_{k-1}$ which has as basis the classs of the monomials in which exactly the k variables in S have a strictly negative exponent.

Proposition 31.3. *R_k/R_{k-1} is the direct sum of modules$(R_k/R_{k-1})_S$, as S runs over the $\binom{n}{k}$ subsets of $\{x_1, \dots, x_n\}$ with k elements.*

Let us analyse one of these modules, say, $S = \{x_1, \dots, x_k\}$.

Now, R_S has as basis the monomials $\prod_{i=1}^k x_i^{-h_i} \prod_{i=k+1}^n x_i^{h_i} (x_1 \dots x_k)^{-1}$, $h_i \geq 0$.

Consider the element $e_S := \frac{1}{x_1 \dots x_k}$ modulo R_{k-1}. We have:

$$\prod_{i=1}^k x_i^{-h_i} \prod_{i=k+1}^n x_i^{h_i} (x_1 \dots x_k)^{-1} = \prod_{i=k+1}^n (-1)^{h_i} h_i! \prod_{i=k+1}^n x_i^{h_i} \prod_{i=1}^k \frac{\partial^{h_i}}{\partial x_i} (e_S). \tag{40}$$

$x_i e_S = \frac{x_i}{x_1 \dots x_k} \cong 0$ modulo R_{k-1}, $\forall i = 1, \dots, k$. $\dfrac{\partial}{\partial x_i} e_S = 0$, $\forall i = k+1, \dots, n$.

We deduce:

Theorem 31.4. *1. The module R_S is generated by the element e_S (a cyclic module).*

2. The annihilator in $W(n)$ of e_S is the left ideal generated by elements x_i, $\forall i = 1, \dots, k$; $\dfrac{\partial}{\partial x_i}$, $\forall i = k+1, \dots, n$.

3. R_S is an irreducibile module with basis the set of elements

$\prod_{i=k+1}^n x_i^{h_i} \prod_{i=1}^k \dfrac{\partial^{k_i}}{\partial x_i} (e_S)$, $h_i, k_i \geq 0$.

Proof 1. Follows immediately from the formula 40.

2. Let I_S denote the left ideal generated by the elements x_i, $\forall i = 1, \dots, k$; $\dfrac{\partial}{\partial x_i}$, $\forall i = k+1, \dots, n$. Clearly I_S annihilates e_S. Consider the module $M_S := W(n)/I_S$ and the homomorphism $\pi : M_S \to R_S$ which sends the class of an operator $p \in W(n)$ to the element $p(e_S)$ (which is well-defined since $I_S e_S = 0$). We need to prove that π is an isomorphism. For this, it suffices to prove that M_S is linearly generated by the classes of the elements:

$$\prod_{i=k+1}^n x_i^{h_i} \prod_{i=1}^k \frac{\partial^{k_i}}{\partial x_i}, \ h_i, k_i \geq 0. \tag{41}$$

From the remark 31.2 it follows that the algebra $W(n)$ has as a basis the set of monomials:

$$M = \prod_{i=k+1}^{n} x_i^{h_i} \prod_{i=1}^{k} \frac{\partial^{k_i}}{\partial x_i} \prod_{i=1}^{k} x_i^{h_i} \prod_{i=k+1}^{n} \frac{\partial^{k_i}}{\partial x_i}$$

Since the elements $x_1, \dots, x_k, \frac{\partial}{\partial x_{k+1}}, \dots, \frac{\partial}{\partial x_n}$ commute it is clear that, if one of the exponents $h_i,\ i = 1, \dots, k$ or one of the exponents $k_i,\ i = k+1, \dots, n$ is bigger than 0, then the monomial M is in the ideal I_S. Thus only the desired monomials remain.

3. Let us prove that R_S is irreducible. We have already seen that it has as basis the classes of elements $\prod_{i=k+1}^{n} x_i^{h_i} \prod_{i=1}^{k} \frac{\partial^{h_i}}{\partial x_i}(e_S),\ h_i \geq 0$.

To prove that a module M over a ring R is irreducible it suffices to prove that, given any element $m \in M,\ m \neq 0$, m generates M, that is, $Rm = M$.

Let us denote by N the submodule Rm, which we wish to prove equals M. Consider $m \in M$, a linear combination, with nonzero coefficients, of the elements of the form $\prod_{i=k+1}^{n} x_i^{h_i} \prod_{i=1}^{k} \frac{\partial^{h_i}}{\partial x_i}(e_S),\ h_i \geq 0$. It suffices to prove that the sub-module N generated by m contains the class of 1. Let us proceed by induction on the maximum of the exponents h_i which occur in the above monomials. If this maximum is 0, we have $m = ce_S$, with c a nonzero costant, and the conclusion is evident. Else, assume to start with that this maximum h occurs with one of the variables, say, x_{k+1}^h. By the definition of the module, we have $\frac{\partial}{\partial x_{k+1}} m \in N$.

By applying the commutation rules $\frac{\partial}{\partial x_{k+1}} x_{k+1}^h = x_{k+1}^h \frac{\partial}{\partial x_{k+1}} - h x_{k+1}^{h-1}$ we see that the first term generates only terms which end with the operator $\frac{\partial}{\partial x_{k+1}}$ thus they equal 0 modulo I_S, in the second term the leading degree in x_{k+1} has been lowered and the element remains nonzero if $h > 0$. By induction thus e N contains an element in which x_{k+1} does not appear. We reason in a similar way for the other terms. We pass now to the maximum degree in $\frac{\partial}{\partial x_1}$ say m. If $h > 0$ we use the commutation: $x_{k+1} \frac{\partial^h}{\partial x_{k+1}} = \frac{\partial^h}{\partial x_{k+1}} x_{k+1} - h \frac{\partial^{hs-1}}{\partial x_{k+1}}$ and proceed as before arriving finally to see that $e_S \in N$ and thus $N = M$. □

let us see what happens if, instead of considering the module generated by $\frac{1}{x_1 \dots x_k}$ modulo R_{k-1} we study which is the module generated inside the algebra R.

We leave as an exercise the verification that such a module is the set of rational fractions $\mathbb{C}[x_1, \dots, x_n][(x_1 \dots x_k)^{-1}]$ which have in the denominator only the variables from among $x_1, \dots, x_k$. Such a module is not irreducible, but has a *finite composition series* in which the irreducible factors are the modules M_U as U runs over the 2^k subsets of S.

EXERCISE Using the automorphisms of pages 54 give another proof of the irreducibility of M_S by reduction to the case of polynomials.

It is important to show that two modules M_A, M_B are not isomorphic when $A \neq B$ are two different sets of variables..

For this it is useful to find some invariants which will distinguish the two. In our case we use a geometric invariant - the *characteristîc variety*.

For a finitely generated module we define it as follows:

In the Weyl algebra $W(n)$ let us consider for each $k \geq 0$ the subspace $W(n)_k$ consisting of those operators which are of degree $\leq k$ in the derivatives.

The commutation rules, together with the definitions imply that:

$$(42) \qquad W(n)_0 \subset W(n)_1 \subset \dots W(n)_k \subset \dots, \quad W(n) = \cup_{i=0}^{\infty} W(n)_k,$$

$$W(n)_h W(n)_k \subset W(n)_{h+k}.$$

These are the defining properties of a *filtered algebra*.

Definition 11. *A filtration of an algebra R is a sequence of subspaces $R_k, \; k = -1, \dots, \infty$ such that:*

$$(43) \qquad R = \cup_{k=0}^{\infty} R_k, \quad R_h R_k \subset R_{h+k}, \quad R_{-1} = 0.$$

The concept of a filtered algebra compares with the much more restrictive notion of a *graded algebra*.

Definition 12. *A graded algebra is an algebra R with a sequence of subspaces $R_k \subset R; \quad k = 0, \dots, \infty$ such that:*

$$(44) \qquad R = \oplus_{k=0}^{\infty} R_k, \quad R_h R_k \subset R_{h+k}.$$

An element of R_k is said to be *homogeneous of degree k.*

The product of two homogeneous elements of degree h, k is homogeneous of degree $h + k$.

In the case of the algebra $W(n)$ the above filtration is also known as the *Bernstein filtration*.

To a filtered algebra is associated its *associated graded algebra*.

Let $R = \cup R_k$ be a filtered algebra. We wish to associate to it a graded algebra.

Intuitively the idea is that, given un element of R_h, we wish to concentrate on the degree h and neglect elements in $R_{(h-1)}$.

More concretely, we replace R with the direct sum:

$$Gr(R) := \oplus_{h=0}^{\infty} R_h / R_{h-1}$$

Given elements $a \in R_h,\ b \in R_k$ when we take their product we consider it only modulo R_{h+k-1}; we see that this gives a well-defined product

$$R_h/R_{h-1} \times R_k/R_{k-1} \to R_{h+k}/R_{h+k-1}.$$

This way we define the associated graded algebra.

The class in R_h/R_{h-1} of an element $a \in R_h$ is called the *symbol of a*. This is consistent with the terminology used in the theory of differential opetrators.

The graded algebra associated to a filtration of an algebra R can be viewed as a *simplification* or *degeneration* of R.

Often it is possible to recover many properties of R from those of the (simpler) graded algebra.

A particularly important case arises when R is non-commutativa but $gr(R)$ is commutativa, which happens if and only if for any two elements a, b of degree (in the filtration) h, k we have $[a, b] \in R_{h+k-1}$.

This occurs in particular in the case of the Bernstein filtration!

Lemma 31.5. *The associated graded algebra of $W(n)$ relative to the Bernstein filtration is a ring of polynomials in the variables x_i and ξ_i, the classes of $\frac{\partial}{\partial x_i}$.*

In essence, because the x_i have degree 0 and the $\frac{\partial}{\partial x_i}$ have degree 1, both $x_i \frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial x_i} x_i$ have degree 1.

Their difference $x_i \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_i} x_i = -1$ has degree 0 and therefore these clasees commute in the graded algebra.

The fact that these classes polynomially generate the graded algebra is a simple exercise, which follows from the canonical form of the operators in $W(n)$. □

Consider now a module M over a filtered algebra R.

Definition 13. *A filtration on M, compatible with the filtration of R, is an increasing sequence of subspaces $0 = M_{-1} \subset M_0 \subset M_1 \subset M_k \subset \dots$ which satisfy:*

$$\cup_k M_k = M, \quad R_h M_k \subset M_{h+k},\ \forall h, k.$$

Lemma 31.6. *The graded vector space $gr(M) := \oplus_{h=0}^{\infty} M_h/M_{h-1}$ is in a natural way a module over the graded algebra.*

Proof We leave this as an exercise.

If M is finitely generated module over R, we can define on M a compatible filtration by choosing generators $m_1, \dots, m_d$ and setting $M_k := \sum_{i=1}^{d} R_k m_i$. This filtration depends on the generators m_i but:

Lemma 31.7. *Let M^1_k, M^2_k be filtrations corresponding to two sets of generators $m_1, \dots, m_d;\ n_1, \dots, n_e$. There exist two non-negative integers a, b such that for each k we have:*

$$M^1_k \subset M^2_{k+a}, \quad M^2_k \subset M^1_{k+b}$$

Proof Let a be an integer such that $n_i \in M^1_a, i = 1, \dots, e$ and b an integer such that $m_i \in M^2_b, i = 1, \dots, d$. The assertion follows immediately from the definition of the filtrations □

When the graded algebra is commutative, it is useful to consider the ideal in $Gr(R)$ which annihilates $gr(M)$.

In the case of the Weyl algebra this is an ideal in a polynomial ring, and the variety of its zeros is the *characteristiĉc variety*.

Before turning to this, let us recall some elementary facts about affine varieties.

(The theory outlined below can obviously be developed in much greater generality). Given a set of polynomials $f_i := f_i(x_1, \dots, x_m),\ i = 1, \dots, k$ in m variables with complex coefficients,

Definition 14. *the affine variety defined by the polynomials f_i, is the set:*

$$V(f_1, \dots, f_k) := \{(a_1, \dots, a_m) \in \mathbb{C}^m \mid f_i(a_1, \dots, a_m) = 0,\ \forall i = 1, \dots, k.\}$$

The points of $V(f_1, \dots, f_k)$ are therefore the *solutions* of the system of (in general) nonlinear equations $f_i(x_1, \dots, x_m) = 0$.

We are thus considering a (quite complicated) generalisation of the theory of systems of linear equations.

To start with, note that given polynomials, $f_i,\ i = 1, \dots, k$ and $g_j,\ j = 1, \dots, h$ we have :

$$V(f_1, \dots, f_k) \cap V(g_1, \dots, g_h) = V(f_1, \dots, f_k, g_1, \dots, g_h),$$

$$V(f_1, \dots, f_k) \cup V(g_1, \dots, g_h) = V(f_i g_j), i = 1, \dots, k, j = 1, \dots, h.$$

Given the variety $V := V(f_1, \dots, f_k)$ it is important to study the set of all polynomials which vanish on V.

In other words, this is the set of nonlinear equations which can be deduced from the polynomials $f_1, \dots, f_k$.

Denote by I_V this set. Clearly, I_V has the following properties:

If $f, g \in I_V$ we have $f + g \in I_V$; if $f \in I_V$ and g is any polynomila, we have $fg \in I_V$.

These are the properties that, in any commutative ring, define an *ideal*.

It is clear that given polynomials g_i we have that $\sum_{i=1}^k g_i f_i$ vanishes on $V(f_1, \dots, f_k)$; in other words, $\sum_{i=1}^k g_i f_i \in I_V$.

It is evident that the set of elements $\sum_{i=1}^k g_i f_i$, letting the g_i vary, is an ideal and that each ideal containing the polynomials f_i contains these elements.

Consequently, the set of elements $\sum_{i=1}^k g_i f_i$ is called *the ideal generated by elements f_i* is denoted $(f_1, \ldots, f_k)$.

It is important to realise that in general it is not true that $(f_1, \ldots, f_k) = I_{V(f_1,\ldots,f_k)}$; in fact if g is a polynomial such that (for some positive integer k) $g^k = \sum_{i=1}^k g_i f_i$ this ensures that g vanishes on $V(f_1, \ldots, f_k)$.

The fundamental theorem due to Hilbert and known as the *Hilbert Nullstellensatz* says:

Theorem 31.8. *I_V coincides with the set of elements g for which there exists a positive integer k such that $g^k \in (f_1, \ldots, f_k)$.*

Theorem 31.9. *Let M be a fintely generated module over the Weyl algebra. Its characteristic variety does not depend on the generators chosen. This variety is therefore an invariant of the module.*

Proof Let us take two sets of generators $m_1, \ldots, m_d;\ n_1, \ldots, n_e$, denote by $Gr^1(M),\ Gr^2(M)$ the two associated graded modules.

By definition an element $x \in R_k,\ x \notin R_{k-1}$ gives, modulo the grading, an element in the annihilator of $Gr^1(M)$ (respectively of $Gr^2(M)$) if for each h we have that $xM_h^1 \subset M_{h+k-1}^1$ (respectively if for each h we have $xM_h^2 \subset M_{h+k-1}^2$).

In the first case, for each positive integer p we have that x^p gives an element with degree kp. By induction $x^p M_h^1 \subset M_{h+pk-p}^1$, so that:

$$x^p M_h^2 \subset x^p M_{h+b}^1 \subset M_{h+b+pk-p}^1 \subset M_{h+b+a+pk-p}^2.$$

Taking $p = a + b + 1$ we see then that x^{a+b+1} gives an element in the annihilator of $Gr^2(M)$.

In other words, denoting by I_1, I_2 the two annihilator ideals we have that each element of I_1 raised to a suitable power is in I_2 and similarly for I_2.

This clearly implies that I_1, I_2 have the same characteristic variety.

For the module R_S considered above the ideal associated to the filtration corresponding to the generators e_S is the ideal generated by elements $x_1, \ldots, x_k, \xi_{k+1}, \ldots, \xi_n$, which therefore defines a subspace.

We can give a description independent of the choice of linear co-ordinates. If we make the linear change of variables $y_i = \sum_j a_{i,j} x_j$ we see that $\dfrac{\partial f}{\partial x_i} = \sum_{j=1}^n \dfrac{\partial f}{\partial y_j}\dfrac{\partial y_j}{\partial x_i} = \sum_{j=1}^n a_{j,i}\dfrac{\partial f}{\partial y_j}$ so that $\dfrac{\partial}{\partial y_i} = \sum_{j=1}^n b_{i,j}\dfrac{\partial}{\partial x_i}$ where the matrix $b_{i,j}$ is the i nverse transpose of the matrix $a_{i,j}$.

In more intrinsic terms, if the x_i are coordinates in the vector space V, the ξ_i are coordinates in the dual space V^*.

Remark 31.10. *The charateristc varieties are contained $V \times V^*$, or in more global geometric terms in the cotangent bundle of V.*

In the case of R_S: the variety with equations $x_1 = \cdots = x_k = \xi_{k+1} = \cdots = \xi_n = 0$ is the *conormal bundle* of the subspace given by $x_i = 0, i = 1, \ldots, k$.

The definition of the conormal bundle of a submanifold V of a differentiable manifold W is the following.

The cotangent bundle of W is the set of couples q, p with $q \in W$ and p a linear form on the tangent space $T_q(W)$ of W at q.

The conormal bundle of a submanifold V of W is:
the set of couples q, p with $q \in V$ and p a linear form on the tangent space $T_q(W)$ of W in q,which vanishes on the tangent space of V at q.

Remark 31.11. *To appreiate this definition, we need to study a little Hamiltonian formalism and symplectic geometry.*

In this context conormal bundles are the simplest examples of Lagrangian submanifolds.

We are now finally in a position to prove the linear independence of the elements in 14.3.

We need some elementary facts about modules which can be found in any book on abstract algebra.

Definition 15. *Recall that a module M is said to be finite length if there exists a chain of sub-modules:*

$$M = M_0 \supset M_1 \supset \cdots \supset M_k = 0$$

such that M_i/M_{i+1} is irreducible for each i.

A module is said to be semisimple if it is the direct sum of irriducible modules.

Fact A semisimple module of finite length is a direct sum of a finite number of irreducible modules.

The basic theorems assert:

- *Theorem of Jordan Hölder.* Given two maximal chains $M = M_0 \supset M_1 \supset \cdots \supset M_k = 0$, $M = M'_0 \supset M'_1 \supset \cdots \supset M'_h = 0$ then $h = k$ and the modules M_i/M_{i+1} which appear in the first list coincide up to possible re-ordering, with modules M'_i/M'_{i+1}.
- If a module M is a direct sum of irreducibles, one has a canonical decomposition into *isotypical components.*
- An isotypical component of M is the sum of all its sub-modules which are isomorphic to a given irreducible module. M can be decomposed uniquely as a direct sum $M = \oplus_{\alpha \in A} M_\alpha$ where A is the list of irriducibles N_α occuring as submoduled of M.
- Every M_α can be decomposed, but non-canonically, and in general in infinitely many ways, as direct sum of modules isomorphic to N_α.
- If a module M is a sum, not necessarily direct, of irriducibles N_i each irreducible submodule of M is isomorphic to one of the N_i.

We are interested in the case of the algebra R of rational functions in the variables y_i with denominators which are products of the form $\alpha_i(y)$ in Δ.

From Theorem 13.2, we know that each such function can be expanded into terms such that the distinct forms that occur in the denominator are linearly independent.

Let R_k denote the subspace of R consisting of functions in which at most k distinct forms appear in the denominator. It is evident that these subspaces R_k are sub-modules. We wish to analyse in detail the modules $M_\Delta^k := R_k/R_{k-1}$.

From what we have already proved, 14.3, it follows that M_Δ^k is a sum (we do not know yet whether direct or not) of the images of the subspaces $R_{S,W}$ as W runs over the subspaces of dimension k generated by the forms belonging to Δ and$S$ among the non-broken circuits with k elements that generate W.

Moreover, it follows easily from our analysis that the images in M_Δ^k of the subspaces $R_{S,W}$ coincide with the irreducible modules R_S, We need therefore to prove that the sum of the modules R_S as we vary $W := \langle S\rangle$ and S among the circuits non-broken relative to W is direct.

In this case two modules R_S, R_T are isomorphic if and only if S and T generate the same space of linear forms, or rather, correspond to the same space W. For each such W we have therefore a isotypical component $M_\Delta^k(W)$ and $M_\Delta^k = \oplus_W M_\Delta^k(W)$.

If we wish to decompose an isotypical component we can decompose it using the S which are non=-split on W. The final result, which also yields the linear independence 14.3 is:

Theorem 31.12.

$$R_k/R_{k-1} = \oplus_W \oplus_S R_S$$

where W runs over the subspaces of dimension k generated by elements of Δ and S runs over the subspaces with k elements of Δ which generate W and which are non-broken on W.

Before we turn to the proof of this Theorem, we need several lemmas.

Denote by A_S the sub-module generated by e_S in R. We know that once we choose a system of coordinates $z_1, \dots, z_n$ such that $S = z_1, \dots, z_k$, we have $A_S = \mathbb{C}[z_1^{\pm 1}, \dots, z_k^{\pm 1}, z_{k+1}, \dots, z_n]$.

Let A_S^0 be the subspace of A_S consisting of elements in which not all the variables $z_i, i = 1, \dots, k$ appear in the denominator. Clearly A_S^0 is a submodule and A_S/A_S^0 is isomorphic to R_S.

Lemma 31.13. *Every proper submodule of A_S is contained in A_S^0.*

Proof It suffices to prove that, given an element $f \in A_S$ not contained in A_S^0, the submodule N generated by it is all of A_S.

If $f \notin A_S^0$ it contains a term in which all the variables $z_i, i = 1, \dots, k$ appear with a negative exponent. Consider first all the terms in which z_1 appears with an exponent $-h$ with h as large as possible. After multiplying by z_1^{h-1}, these terms appear with z_1 in degree -1; let us write them as

$z_1^{-1}g$ with z_1 not occuring in in g, now the others f_0 have z_1 in non-negative degree. Apply $\frac{\partial}{\partial z_1}^k$ with k sufficiently big, we can obtain:

$$\frac{\partial f_0}{\partial z_1}^k = 0, \quad \frac{\partial z_1^{-1} g}{\partial z_1}^k = (-1)^k k! z_1^{-k-1} g \in N.$$

Moltiplying by $[(-1)^k k!]^{-1} z_1^k$ we get $z_1^{-1} g \in N$.

Repeat the procedure on g with the other variables to obtain in N an element of the type $f(z_{k+1}, \dots, z_n)(z_1 \dots z_k)^{-1} \in N$.

Now, differentiating suitably with respect to the variables $z_{k+1}, \dots, z_n$ we also get $(z_1 \dots z_k)^{-1} \in N$. Since $(z_1 \dots z_k)^{-1}$ generate A_S we have $N = A_S$. □

Lemma 31.14. *Given a cyclic module over the algebra R generated by an element c, take d copies Rc_i, $i = 1, \dots, d$, and in the direct sum of the Rc_i, consider a linear combination $p := \sum_{i=1}^d \lambda_i c_i$ with the λ_i constant and not all zero. Then the submodule generated by p is isomorphic to Rc.*

Proof If $a \in R$ annihilates p, we have $a\lambda_i c_i = 0$ for each i. Since at least one λ_i is nonzero, we have $ac = 0$. Conversely, $ac = 0$ implies $ap = 0$ and therefore $Rp = R/I_c = Rc$ where I_c is the annihilator of c. □

Lemma 31.15. *Consider a nonzero linear combination $p = \sum_i \lambda_i e_{S_i}$ of elements e_{S_i} where the S_i are non-broken circuits which generate a fixed subspace W of dimension k. Then $p \notin R_{k-1}$.*

Proof Let us make two reductions. First, we can suppose that W, is the entire vector space in other words, that $k = n$.

In this case we know that R_{k-1} is contained in the space generated by the partial derivatives, equivalently (in terms of differential forms) that the forms $f dy_1 \wedge \dots \wedge dy_n$ with $f \in R_{k-1}$ are exact.

It suffices to show that the form $p dy_1 \wedge \dots \wedge dy_n$ is not exact. For this, it is enough to observe that at least one of its residues (relative to one of the bases S_i) is nonzero.

From the Theorem (25) and its corollories we have $res_{S_i} p dy_1 \wedge \dots \wedge dy_n = \lambda_i d_i$ where d_i is the determinant of the matrix which expresses the basis S_i in terms of the basis $y_1, \dots, y_n$.

The assertion now follows from our hypothesis.

Let us now take for the set Δ_1 all unions of elements of the S_i. Fix a set of adapted co-ordinates $z_1, \dots, z_n$ such that $W = \langle z_1, \dots, z_k \rangle$ in such way that all the linear forms $\alpha(y) \in \Delta_1$ are linear combimations of only the first k variables.

We can now make a reduction to the previous case.

Letting $d_1 := \prod_{\alpha(y) \in \Delta_1} \alpha(y)$, we have a homomorphism:

$$\pi : R = \mathbb{C}[z_1, \dots, z_n][d_1^{-1}] \to \mathbb{C}[z_1, \dots, z_k][d_1^{-1}] = R', \ \pi(z_i) = z_i, \\ \forall i \leq k, \quad \pi(z_i) = 0, \forall i > k.$$

In such a homomorphism if p could be written as an element of R_{k-1} it would be in R' which contradicts the previous case.

We now turn to the general situation. Consider $\mathbb{C}[z_1, \dots, z_n][d_1^{-1}] \subset \mathbb{C}[z_1, \dots, z_n][d^{-1}]$. If by contradiction the element $p \in \mathbb{C}[z_1, \dots, z_n][d^{-1}]_{k-1}$ the submodule it generates is contained in $\mathbb{C}[z_1, \dots, z_n][d^{-1}]_{k-1}$. In particular, a copy of the irreducible module R_S ahould appear in each composition series of $\mathbb{C}[z_1, \dots, z_n][d^{-1}]_{k-1}$. This is absurd since by induction the modules forming a composition series of $\mathbb{C}[z_1, \dots, z_n][d^{-1}]_{k-1}$ have as characteristic variety the conormal bundle of a subspace of codimension $\leq k-1$. □

Lemma 31.16. *In the module R_S, if an element u satisfies the equations*

$$x_i u = 0,\ i = 1, \dots, k;\ \frac{\partial}{\partial x_i} u = 0,\ i = k+1, \dots, n \tag{45}$$

, then u is a multiple of e_S.

Proof 1. Consider a element of the form

$$f := \sum c_{h_1, \dots, h_n} \prod_{i=k+1}^{n} x_i^{h_i} \prod_{i=1}^{k} \frac{\partial^{h_i}}{\partial x_i}(e_S),\ h_i \geq 0.$$

If one of the variables $x_i, i = k+1, \dots, n$ occurs with exponent > 0 it is immediate that $\frac{\partial f}{\partial x_i} \neq 0$. On the other hand, if a derivative $\frac{\partial}{\partial x_i},\ i = 1, \dots, k$ appears with exponent > 0 we see that $x_i f \neq 0$. □

We are now in a position to prove the theorem.

We have seen that R_k / R_{k-1} is the sum of the modules R_S as S, W vary.

Now for different W we have non-isomorphic modules since the characteristic varieties are different. Therefore we can concentrate on an isotypical component, and we need to show that the sum of the R_S with S non-broken with respect to W is direct.

Otherwise, from the theory of semisimple modules, we can extract from this sum a direct sum not containing all the summands R_S.

It follows that in this isotypical component, the space of elements satisfying the quations (45) has a dimension strictly inferior to the number of irreducible R_S associated to son-split S. On the other hand, by Lemma (31.15), the elements e_S are linearly independent modulo R_{k-1}. We have thus arrived at a contradiction.

Lezione 11. The arithmetic case

32. Tori and characters

Let us now treat the case of vectors with integer co-ordinates and the problem of counting points in the interior of the corresponding polytopes.

We have seen that this is also a problem of inversion, albeit in the discrete situation.

It is convenient to introduce terminology that is well-known in the theory of algebraic groups.

Definition 16. *(1) The set $T_n := (\mathbb{C}^*)^n$ of $n-$tuples of non-zero complex numbers is called the standard $n-$dimensional torus and is an (algebraic) group with co-ordinatewise multiplication as the group law.*

(2) A character of T_n is an algebraic homomorphism $\chi : T_n \to \mathbb{C}^*$.

It is not difficult to prove that such a character is necessarily of the form $\chi : (x_1, \dots, x_n) \mapsto \prod_{i=1}^n x_i^{h_i},\ h_i \in \mathbb{Z}$.

- We could even take this as the definition of a character.
- The product of two characters is again a character $\prod_{i=1}^n x_i^{h_i} \prod_{i=1}^n x_i^{k_i} = \prod_{i=1}^n x_i^{h_i+k_i}$.
- It is convenient to write $\prod_{i=1}^n x_i^{h_i} := x^{\underline{h}},\ \underline{h} := (h_1, \dots, h_n) \in \mathbb{Z}^n$.
- So $x^{\underline{h}+\underline{k}} = x^{\underline{h}} x^{\underline{k}}$ and the characters, with respect to this product, form a group isomorphic to the group $\mathbb{Z}^n$.
- We have thus: The algebraic functions on T_n are the Laurent polynomials in the variables x_i, x_i^{-1}. the charaters $x^{\underline{h}},\ \underline{h} \in \mathbb{Z}^n$ form a basis of this ring (this is the algebraic part of classic Fourier analysis).

When there is no possibility of confusion we will write $T = T_n$. Let us denote the ring of algebraic functions on T:

$$A_T := \mathbb{C}[x_i^{\pm 1}], \tag{46}$$

In a torus the analogue of a linear change of co-ordinates is a change of basis for the characters.

- Suppose given a matrix $A := (a_{i,j})$ with integer entries, such that its inverse $(b_{i,j})$ also is one such (in other words, such that $\det(A) = \pm 1$).
- We can define new co-ordinates with characters

$$y_i = \prod_{j=1}^n x_j^{a_{i,j}},\ \ x_i = \prod_{j=1}^n y_j^{b_{i,j}}$$

this change of co-ordinates corresponds to an integral change of basis in the group of characters. $\mathbb{Z}^n$.

Given m charaters $\chi_i := x^{\underline{h}_i}$ the set $X := \{p \in T \,|\, \chi_i(p) = 1\}$ is an algebraic subgroup T_n which depends only on the subgroup $\Lambda \subset \mathbb{Z}^n$ which they generate.

In fact if $\chi(p) = 1, \psi(p) = 1$ we have $\chi\psi(p) = 1$; in other words, the set of characters which take the value 1 on X is a subgroup.

To proceed further we use the following fact (which can be proved by elementary means, cf. Appendix 3):

Theorem 32.1. *given a subgroup $\Lambda \subset \mathbb{Z}^n$ there exists a basis of $\mathbb{Z}^n$ such that Λ is generated by k elements of the type*
$(d_1, 0, \dots, 0),\ (0, d_2, 0, \dots, 0), \dots, (0, 0, \dots, d_k, 0, \dots, 0), d_i \in \mathbb{N}^+$.

Since in these co-ordinates, denoting by a_i, the charaters which generate Λ are $a_i^{d_i},\ i = 1, \dots, k$ we have:

In these coordinates X consists of $n-$tuples of numbers $(a_1, \dots, a_n)$ for which $a_i^{d_i} = 1, i = 1, \dots, k$.

The equation $x^d = 1$ defines the d roots of unity $e^{\frac{2\pi i k}{d}},\ 0 \leq k < d$. Therefore:

X has $d_1 d_2 \dots d_k$ connected components given by:

$$(\zeta_1, \dots, \zeta_k, a_{k+1}, \dots, a_n)|\ \zeta_i^{d_i} = 1.$$

and the component X_0 containing the element 1 is the *subtorus* of dimension $n-k$, with the first k coordinates equal to 1.

- It is now not difficult to convince ourself that the characters which take the value 1 on X_0 are the ones which have tha last $n-k$ coordinates equal to 0.
- These charatters form a subgroup $\overline{\Lambda}$ with $\Lambda \subset \overline{\Lambda}$ and $\overline{\Lambda}/\Lambda = \oplus_{i=1}^k \mathbb{Z}/(d_i)$.
- $\overline{\Lambda}$ can also be characterised as the set of elements $a \in \mathbb{Z}^n$ for which there exists a nonzero integer k with $ka \in \Lambda$ (the torsion elements modulo Λ).
- Moreover, $\mathbb{Z}^n/\overline{\Lambda} = \mathbb{Z}^{n-k}$.

In this context we it is useful to think of a torus as a periodic analogue of a vector space and a subtorus as an analogue of a subspace.

Well, we immediately note a point of difference - the intersection of two subtori is not necessarily connected.

An important case to consider arises when we have n linearly independent vectors $\underline{h}_i$ with integer co-ordinates, which we can think of as the rows of an integer matrix A.

In this case $X := \{p \in T \,|\, \chi_i(p) = 1\}$ is a finite subgroup with $|\det(A)|$ elements.

This is important when we consider the problem of integral points, associated with a list of integer vectors a_i.

In this case, each time we extract from this list a basis)as vector space) this gives rise to an integral matrix, with determinant some integer with absolute value m and determines also the m points of the previously described subgroup.

All these points of such subgroups will contribute to the formulae which we wish to derive.

The formulae 4 and 5 can be rewritten in the language of tori as the identity

$$\prod_{i=1}^{m} \frac{1}{1 - x^{\alpha_i}} = \sum_{b \in \mathbb{Z}^n} S_A(b) x^b. \tag{47}$$

We can at once deal with a simple case, where we take the $a_i,\ i = 1, \dots, n$ (rows of a matrix A) which are linearly independent, and which, according to the above discussion, generate a subgroup of index $|\det(A)|$.

Inverting:

$$\prod_{i=1}^{n} \frac{1}{(1 - x^{a_i})^{b_i}} = \prod_{i=1}^{n} \sum_{h=0}^{\infty} \binom{b_i - 1 + h}{h} x^{h a_i} = \tag{48}$$

$$\sum_{h_1, h_2, \dots, h_n} \prod_{i=1}^{n} \binom{b_i - 1 + h_i}{h_i} x^{\sum_{i=0}^{n} h_i a_i}$$

Thus the value of $S_A(b)$ in this case is given by the polynomial $\prod_{i=1}^{n} \binom{b_i - 1 + h_i}{h_i}$ on the vectors of the positive cone generated by the basis a_i and with integer coordinates with respect to the elements a_i.

If $m := |\det(A)| > 1$, these vectors do not generate all of $\mathbb{Z}^n$, but only a subgroup Λ of index m.

$S_A(b)$ is 0 on the other lateral classes Λ in $\mathbb{Z}^n$.

We have thus at hand a simple example, in many dimensions, of a quasi-periodic polynomial.

To explore the general case, we can proceed by looking (as in the case of the volume) for a suitable expansion in partial fractions.

We will see that the program that we carried out for the case of hyperplanes can be generalised but with some significant complications.

First of all, let us generalise the notion of arrangement.

33. Toric Arrangements

Definition 17. *Given a finite set $\Delta = \{a_1, \dots, a_m\}$, of vectors with integer co-ordinates let us define the associated toric arrangement as the collection of connected components of all intersections of the subvarieties $H_i \subset T$ defined by the equations $1 - x^{a_i} = 0$.*

The varieties H_i are the loci where the generating functions have poles; so we see that we need to follow a strategy analogous to the case of hyperplanes, cancelling the poles and computing appropriate residues.

First, note that among these connected components are the points of intersection of the n hyperplanes H_i relative to the n linearly independent elements a_i.

These points will be called *points of the arrangement* and the set of these points is denoted Π_Δ.

For each $p \in \Pi_\Delta$ denote also:

$$\Delta_p := \{a \in \Delta \,|\, x^a(p) = 1.$$

EXAMPLE $\Delta = \{(1,0),(0,1),(1,2)\}$ i points $\Pi_\Delta := \{(1,1),(1,-1)\}$ si ha $\Delta_{(1,-1)} = \{(1,0),(1,2)\}$.

Here is the simple geometric idea that we will use. In a neighbourhood of each point of the arrangement, in logarithmic co-ordinates, the divisors of $1-x^a$ (for $a \in \Delta_p$) form a hyperplane arrangement. For this configuration, the theory carried out in [25] furnishes the required wonderful model. We will content ourself with an elementary approach as in the case of hyperplanes.

What we will prove is the following:

- For each $p \in \Pi_\Delta$ consider the set Δ_p with the ordering induced from Δ. We have then the non-broken bases of Δ_p, which we will denote NB_p.
- For each pair $(p, \underline{b})$, $p \in \Pi_\Delta$, $\underline{b} \in NB_p$ we will define a residue $res_{(p,\underline{b})}\psi$ of the n-form by a suitable change of co-ordinates around p.
- We will finally prove that the function $S_A(b)$ can be computed as the sum of some of these residues.

Let us proceed systematically:

Fixing a basis of Λ, which corresponds to a system of coordinates x_i, consider the invariant form

$$\omega_T = d\log x_1 \wedge d\log x_2 \wedge \ldots \wedge d\log x_n. \tag{49}$$

This form depends only on the orientation of Λ; in other words, a change of basis has determinant ± 1 and the corresponding form ω_T remains invariant or changs sign.

Consider now algebraic functions on the torus with poles along H_i.

These can be written in the form $p(x_1,\ldots,x_n)[\prod_{i=1}^n x_i \prod_{j=1}^m (1-x^{a_j})]^{-k}$, with $p(x_1,\ldots,x_n)$ a polynomial.

Let us denote this algebra:

$$R_\Delta := A_T[\prod_{j=1}^m (1-x^{a_j})^{-1}]. \tag{50}$$

As first step let us define the residues $res_{(p,\underline{b})}\psi$ where $p \in \Pi_\Delta$ is a point in the arrangement and $\underline{b} \in NB_p$ a non-broken basis of Δ_p.

The form ψ is of type $f\omega_T$ with $f \in R_\Delta$.

To start with, take $p = (\zeta_1,\ldots,\zeta_n)$ in coordinates x_i, where the ζ_i are roots of unity depending on p.

Choose local logarithmic co-ordinates around p, putting $x_i = \zeta_i e^{\theta_i}$. If $b = (b_1, \dots, b_n) \in \Delta$ we have

$$x^b = \prod_{i=1}^{n} \zeta_i^{b_i} e^{\sum_{i=1}^n b_i\theta_i}.$$

To say that $b \in \Delta_p$ is equivalent to saying $\prod_{i=1}^n \zeta_i^{b_i} = 1$.

We introduce the notation $\zeta^b := \prod_{i=1}^n \zeta_i^{b_i}, \quad (b, \theta) := \sum_{i=1}^n b_i\theta_i$.

Therefore the function $1 - x^b = 1 - \zeta^b e^{(b,\theta)}$ vanishes at the point with co-ordinate $\theta_i = 0$ if and only if $b \in \Delta_p$.

In this case we have $1 - x^b = -(b, \theta)(1 + \sum_{k=1}^\infty \frac{1}{(k+1)!}(b, \theta)^k)$; in other words $(1-x^b)^{-1}$ equals $(\sum_{i=1}^n b_i\theta_i)^{-1} = (b, \theta)^{-1}$ for a holomorphic function, whose series expansion can be made explicit.

As for the form ω_T we have $d\log x_i = d\log(\zeta_i e^{\theta_i}) = d\theta_i$ and therefore $\omega_T = d\theta_1 \wedge d\theta_2 \wedge \dots \wedge d\theta_n$.

We deduce that a form $\psi = f\omega_T$ with $f \in R_\Delta$ is, in a neighbourhood of 0 with logarithmic co-ordinate θ_i, a holomorphic form moltiplied by a possible polar part of the type $\prod_{b\in\Delta_p}(b, \theta)^{-m_b}$.

For such a form we have defined (cf. page 52) the notion of a residue, $res_{\mathcal{S}_{\underline{b}}}\psi$ with respect to a non-broken basis $\underline{b}$.

This residue is the coefficient of $\prod z_i^{-1} dz_1 \wedge \dots dz_n$, in the development of the form ψ. The co-ordinates z_i are the ones associated to the nested set $\mathcal{S}_{\underline{b}}$ which decomposes $\underline{b}$.

Let us then define $res_{(p,\underline{b})}\psi := res_{\mathcal{S}_{\underline{b}}}\psi$.

We can at once note at least one algebraic property of this residue:

Proposition 33.1. *Given a function $f \in R_\Delta$, $\psi := f\omega_T$ the residue $res_{(p,\underline{b})}x^c\psi$ as a function of c is a periodic polynomial .*

Proof In adapted coordinates z_i the form ψ has an expression

$\psi = (\prod_i z_i)^{-m} g(z_1, \dots, z_n) dz_1 \wedge \dots \wedge dz_n$ with $g(z_1, \dots, z_n)$ holomorphic. Also, for x^c, $c = (c_1, \dots, c_n)$ we have $x^c = \zeta^c e^{(c,\theta)}$. In the coordinates z_i (c, θ) has an expression of the type $\sum_i c_i p_i(z)$ where $p_i(z)$ are polynomials independent of c. Let us calculate the coefficients of $\prod z_i^{-1}$ in the series expansion of

$$\zeta^c e^{\sum_i c_i p_i(z)} (\prod_i z_i)^{-m} g(z_1, \dots, z_n).$$

Expanding in a series $e^{\sum_i c_i p_i(z)} = \sum_{h_1,\dots,h_n} p_{h_1,\dots,h_n}(c_1, \dots, c_n) z_1^{h_1} \dots z_n^{h_n}$ where $p_{h_1,\dots,h_n}(c_1, \dots, c_n)$ is a polynomial in c_i.

Also, $g(z_1, \dots, z_n) = \sum_{h_1,\dots,h_n} a_{h_1,\dots,h_n} z_1^{h_1} \dots z_n^{h_n}$ with $a_{h_1,\dots,h_n}$ costants so that the coefficient of $\prod z_i^{-1}$ is:

$$\zeta^c \sum_{(h_1,\dots,h_n)} a_{m-1-h_1,\dots,m-1-h_n} p_{h_1,\dots,h_n}(c_1, \dots, c_n),$$

clearly a periodic polynomial.

Another useful observation is the following:

Let $\underline{b} = (a_1, \dots, a_n)$ be a non-broken basis of Δ_p. Consider the form $\Omega_{\underline{b}} := d\log(1 - x^{a_1}) \wedge \cdots \wedge d\log(1 - x^{a_n})$.

In the coordinate θ_i we have $1 - x^{a_i} = -(a_i, \theta) f_i(\theta)$ with $f_i(0) \neq 0$. Therefore $d\log(1 - x^{a_i}) = d\log(a_i, \theta) + \gamma_i$ with γ_i exact.

Proposition 33.2. *In the cohomology of the arrangement of hyperplanes given by characters of Δ_p in a neighbourhood of p so the class of $\Omega_{\underline{b}}$ coincides with the class of the form $\omega_{\underline{b}} = d\log(a_1, \theta) \wedge \cdots \wedge d\log(a_n, \theta)$.*

Definition 18. *Given a cell $\mathfrak{c}$ let us define the Jeffrey–Kirwan residue, relative to $\mathfrak{c}$, of a function $f \in R_\Delta$:*

$$JK(\mathfrak{c}, f) = (-1)^r \sum_p \sum_{\underline{b}} \epsilon_{\underline{b}} res_{(p,\underline{b})}(f\omega_T), \tag{51}$$

where $p \in \Pi_\Delta$ varies among the points of the arrangement and $\underline{b} \in NB_p$ among the non-broken bases of Δ_p for which $\mathfrak{c} \subset C(S)$.

$\epsilon_{\underline{b}}$ is 1, if $\underline{b}$ is equioriented with respect to the given orientation of space, and -1 otherwise.

Remarks 33.3. *1. Note that $JK(\mathfrak{c}, f)$ does not depend on the orientation.*

2. From proposition (33.2) it follows that $res_{(p,\underline{b})}(f\omega_T)$ is the coefficient in cohomology of the class of $\Omega_{\underline{b}}$.

The main Theorem is then:

Theorem 33.4. *1. Given a vector $b \in C(A) \cap \mathbb{Z}^n$, the integer $S_A(b)$ coincides with $JK(\mathfrak{c}, \frac{x^b}{\prod_{i=1}^m (1-x^{a_i})})$ where $\mathfrak{c}$ is a big cell with b in the closure of $\mathfrak{c}$.*

2. The residues $res_{(p,\underline{b})}(\frac{x^b}{\prod_{i=1}^m (1-x^{a_i})}\omega_T)$ can be calculated using formulae analogous to the ones worked out for the volume.

In order to prove this theorem we will have to develop the theory of partial fractions and the theory of residues. This last one will be reduced to known facts due to the definition given at page 69.

Lezione 12. The function $S_A(b)$

To proceed to the study of the generating function $S_A(b)$ we need a suitable expansion in partial fractions.

This expansion works not in the co-ordinate ring R_Δ but rather (in geometric language) in the co-ordinate ring of a torus which covers it.

This torus T' is defined by means of its characters which we take to be the fractional lattice $m^{-1}\mathbb{Z}^n$, with m a carefully chosen integer.

Now the coordinate x_i of the original torus T can be expressed as y_i^m where the y_i are coordinates of the torus T' which covers T.

We begin with a series of identities.

34. Some identities

Lemma 34.1.

$$\frac{n}{1-x^n} = \sum_{i=0}^{n} \frac{1}{1-\zeta^i x}, \quad \zeta := e^{2\pi i/n}. \tag{52}$$

Proof Take an auxiliary variable t and differentiate with respect to t

$$\frac{nt^{n-1}}{t^n - x^n}dt = d\log(t^n - x^n) = d\log(\prod_{i=0}^{n-1}(t-\zeta^i x))$$

$$= \sum_{i=0}^{n-1} d\log(t-\zeta^i x)) = \sum_{i=0}^{n-1} \frac{1}{(t-\zeta^i x)}dt.$$

Then set $t = 1$ in the coefficients of dt. □

Lemma 34.2.

$$1 - \prod_{i=1}^{r} z_i = \sum_{\emptyset \subsetneq I \subset \{1,\dots,r\}} (-1)^{|I|+1} \prod_{i\in I}(1-z_i) \tag{53}$$

$$1 - \prod_{i=1}^{n} x_i = \sum_{I \subsetneq \{1,2,\dots,n\}} \prod_{i\in I} x_i \prod_{j\notin I}(1-x_j). \tag{54}$$

Proof This is proved by induction on r. The case $r = 1$ is clear. In general, using the inductive hypothesis we have:

For the first:

$$\sum_{\emptyset \subsetneq I \subset \{1,\dots,r\}} (-1)^{|I|+1} \prod_{i\in I}(1-z_i) = \sum_{\emptyset \subsetneq I \subset \{1,\dots,r-1\}} (-1)^{|I|+1} \prod_{i\in I}(1-z_i) -$$

$$\sum_{I \subset \{1,\dots,r-1\}} (-1)^{|I|+1} \prod_{i\in I}(1-z_i)(1-z_r) = 1 - \prod_{i=1}^{r-1} z_i$$

$$+(1-(1-\prod_{i=1}^{r-1} z_i))(1-z_r) =$$

$$= 1 - \prod_{i=1}^{r} z_i$$

□

As for the second:
Split the sum in 3 terms: $I = \{1, \dots, n-1\}$, $I \subsetneq \{1, \dots, n-1\}$ and $n \in I$. We get

$$\prod_{i=1}^{n-1} x_i(1-x_n) + (1 - \prod_{i=1}^{n-1} x_i)(1-x_n) + x_n(1 - \prod_{i=1}^{n-1} x_i) = 1 - \prod_{i=1}^{n} x_i.$$

A variant of this formula is the following:

Lemma 34.3. *Let* $t = \prod_{i=1}^h z_i \prod_{i=h+1}^r z_i^{-1}$. *Then*

$$1 - t = \sum_{\emptyset \subsetneq I \subset \{1,\dots,h\}} (-1)^{|I|+1} \prod_{i \in I} (1 - z_i) - t \sum_{\emptyset \subsetneq I \subset \{h+1,\dots,r\}} (-1)^{|I|+1} \prod_{i \in I} (1 - z_i) \tag{55}$$

Proof This is immediate from the previous Lemma once we note that:

$$1 - t = 1 - \prod_{i=1}^h z_i - t(1 - \prod_{i=h+1}^r z_i).$$

□

Lastly:

Lemma 34.4. *(1) Let* $t = \prod_{i=1}^h z_i \prod_{i=h+1}^r z_i^{-1}$ *with* $0 \le h \le r$. *Then*

$$\frac{1}{\prod_{i=1}^r (1 - z_i)} = \sum_{\emptyset \subsetneq I \subset \{1,\dots,h\}} \frac{(-1)^{|I|+1}}{(1-t) \prod_{i \notin I} (1 - z_i)} - \sum_{\emptyset \subsetneq I \subset \{h+1,\dots,r\}} \frac{(-1)^{|I|+1} t}{(1-t) \prod_{i \notin I} (1 - z_i)}. \tag{56}$$

If $a \in \mathbb{C}^*$ *and* $a \neq 1$

If $a \in \mathbb{C}^*$ *and* $a \neq 1$

$$\frac{1}{\prod_{i=1}^r (1 - z_i)(1 - at)} = -\frac{1}{(a-1) \prod_{i=1}^r (1 - z_i)}$$

$$\frac{a}{a-1}\Big(\sum_{\emptyset \subsetneq I \subset \{1,\dots,h\}} \frac{(-1)^{|I|+1}}{(1-at) \prod_{i \notin I} (1 - z_i)} - \sum_{\emptyset \subsetneq I \subset \{h+1,\dots,r\}} \frac{(-1)^{|I|+1} t}{(1-at) \prod_{i \notin I} (1 - z_i)} \Big). \tag{57}$$

Proof the first relation follows from (55) once we divide by

$$(1-t)(1-z_1)(1-z_2) \cdots (1-z_r).$$

For the second, write:

$$\frac{a(1-t)}{(a-1)(1-at)} = \frac{1}{(1-at)} + \frac{1}{a-1}$$

and then multiply the (56) by $\frac{a(1-t)}{(a-1)(1-at)}$ to obtain:

$$\frac{1}{\prod_{i=1}^{r}(1-z_i)}\left(\frac{1}{(1-at)} + \frac{1}{a-1}\right) =$$

$$\frac{a}{(a-1)}\Big(\sum_{\emptyset\subsetneq I\subset\{1,\dots,h\}} \frac{(-1)^{|I|+1}}{(1-at)\prod_{i\notin I}(1-z_i)} - \tag{58}$$

$$\sum_{\emptyset\subsetneq I\subset\{h+1,\dots,r\}} \frac{(-1)^{|I|+1}t}{(1-at)\prod_{i\notin I}(1-z_i)}\Big).$$

□

35. Rational exponents

To analyse our function, we present an expansion in partial fractions closely following 13.2.

Let us consider, in addition to vectors with integer coordinates, also those whose co-ordinates are rational.

Definition 19. *Such a vector $b \in \mathbb{Q}$ will be said to be compatibile with Δ if there exists $n \in \mathbb{N}$ with $nb \in \Delta$. A pair ζ, b or rather the monomial ζx^b, is compatibile with Δ if there exists a $n \in \mathbb{N}$ with*

$$\zeta^n = 1, nb \in \Delta \iff (\zeta x^b)^n = x^a,\ a \in \Delta.$$

To state the main result we need to generalize, to this context, the notion of a non-broken circuit.

A non-broken circuit is a sequence $(\zeta_1, b_1), \dots (\zeta_k, b_k)$ of admissible elements such that:

(2) (1) $b_j = \frac{a_{i_j}}{n_j},\ i_1 < i_2 < \dots, < i_k$.
(2) I b_j are linearly independent.
(3) There does not exist $e \le k$, un $a_s \in \Delta,\ s < i_e$ and integers $m > 0,\ p_j,\ j = e, \dots, k$ with:

$$(x^a)^m = \prod_{j=e}^{k}(\zeta_j x^{b_j})^{p_j}.$$

More precisely, $ma = \sum_{j=e}^{k} p_j b_j,\ \prod_{j=e}^{k}(\zeta_j)^{p_j} = 1$.

Let

$$\mathcal{S} := \{(\zeta_1, b_1), \dots, (\zeta_M, b_M)\}$$

a sequence of elements which are compatibile with Δ.

Let, for each $j = 1, \dots, M$, $c_j \in \Gamma_{\mathbb{Q}}$ such that there exist $0 \leq n_j \leq m_j$, $m_j > 0$, with $m_j c_j = n_j b_j$. Put $c := c_1 + \cdots + c_M$. Then:

Theorem 35.1. *In R the element*

$$\frac{x^c}{\prod_{i=1}^M (1 - \zeta_i x^{b_i})}$$

can be written as a linear combination with constant coefficients of elements of the form

$$\frac{1}{(1 - \eta_1 x^{d_1})^{h_1} \cdots (1 - \eta_r x^{d_r})^{h_r}}$$

with$h_1, \dots, h_r \geq 0$ and $\{(\eta_1, d_1), \dots (\eta_r, d_r)\}$ a non-broken circuit consisting of pairs compatible with Δ.

Proof Let $D_{\mathcal{S}} := \prod_{i=1}^M (1 - \zeta_i x^{b_i})$.

We first reduce to the case $c = 1$. In fact, for each j take $d_j \in \Gamma_{\mathbb{Q}}$ with $m_j d_j = b_j$.

We have $c_j = n_j d_j$.

Let η_j be a m_j-eme root of ζ_j, and write:

$$1 - \zeta_j x^{b_j} = 1 - (\eta_j x^{c_j})^{m_j} = \prod_{s=0}^{m_j - 1} (1 - \exp(2\pi i s/m_j) \eta_j x^{c_j}) \tag{59}$$

Note that for s, the pair $(\exp(2\pi i s/m_j)\eta_j, c_j)$ is compatibile with Δ.

Now substitute in our sequence $\mathcal{S}$ the pair (ζ_j, b_j) with the sequence $\{(\exp(2\pi i s/m_j)\eta_j, c_j)\}$, $s = 0, \dots, m_j - 1$.

We then get a new sequence

$$\mathcal{S}' = \{(\mu_1, d_1,), \dots, (\mu_N, d_N)\}$$

with $N = m_1 + m_2 + \cdots + m_M$ elements and with $D_{\mathcal{S}'} = D_{\mathcal{S}}$.

In this equivalence, we have substituted (ζ_j, b_j) with a sequence of m_j pairs each with the second coordinate c_j.

Since $c_j = n_j b_j$ and $n_j \leq m_j$, we have $c = \varepsilon_1 d_1 + \cdots + \varepsilon_N d_N$ with $\varepsilon_i \in \{0, 1\}$ for each i.

Conclusion

- In conclusion, $\frac{x^c}{D_{\mathcal{S}}}$ is a product of factors of the type $\frac{1}{1 - \mu_k x^{d_k}}$ or $\frac{x^{d_k}}{1 - \mu_k x^{d_k}}$
- Clearly:

$$\frac{x^{d_k}}{1 - \mu_k x^{d_k}} = \frac{1 - (1 - \mu_k x^{d_k})}{\mu_k (1 - \mu_k x^{d_k})} = \mu_k^{-1} (\frac{1}{1 - \mu_k x^{d_k}} - 1)$$

Expanding the product we get a linear combination of terms of the desired kind.

The case $c = 0$ Now we are reduced to the case $c = 0$, let us prove the assertion $1/D_{\mathcal{S}}$.

If the sequence of pairs that appear in the denominator is a non-broken circuit there is nothing to show.

the first condition is not satisfied

Else, suppose the first condition of the definition of non-broken circuit is not met, i.e.,: there is a$a := a_s \in \Delta$, and distinct element $(\zeta_{i_1}, b_{i_1}) < \cdots < (\zeta_{i_t}, b_{i_t})$ in $\mathcal{S}$ with $a_s < b_{i_1}$ e:

$$(x^a)^m(\zeta_{i_1}x^{b_{i_1}})^{n_1}\cdots(\zeta_{i_t}x^{b_{i_t}})^{n_t} = 1 \tag{60}$$

for suitable non-zero integers $m, n_1, \dots n_t$.

In particular we have, since the elements a and b_j are characters:

$$ma + n_1 b_{i_1} + \cdots + n_t b_{i_t} = 0 \tag{61}$$

Let $p = |mn_1 \dots n_t|$, and define

$$f_0 := \frac{|m|a}{p}, f_1 := \frac{|n_1|b_{i_1}}{p}, \dots, f_t := \frac{|n_t|b_{i_t}}{p}$$

the relation (61) becomes:

$$\varepsilon_0 f_0 + \cdots + \varepsilon_t f_t = 0. \tag{62}$$

with $\varepsilon_j \in \{1, -1\}$.

Apply, for each $1 \leq s \leq t$, the formula 34.1, substituting in the place of x the monomial $\eta_s x^{f_s}$, and with $n = p/|n_s|$. Where η_s is a $n-$th root ζ_{i_s} .

In particular, $(\eta_s x^{f_s})^{p/|n_s|} = \zeta_{i_s} x^{b_{i_s}}$ yielding:

$$\frac{1}{1 - \zeta_{i_s} x^{b_{i_s}}} = n^{-1} \sum_{i=0}^{n} \frac{1}{1 - \eta_s^{i+1} x^{f_s}}. \tag{63}$$

Substituting in $1/D_{\mathcal{S}}$ each factor $\frac{1}{1-\zeta_{i_s}x^{b_{i_s}}}$ with the precedente sum.

we obtain an expression for $1/D_{\mathcal{S}}$ as a linear combination of p^t terms each of the form $1/D_{\mathcal{S}'}$ where $\mathcal{S}'$ is gotten from $\mathcal{S}$ by replacing each pair (ζ_{i_s}, b_{i_s}) with the pair (η_s, f_s) (As before, η_s is a $p/|n_s|$-eme root of ζ_{i_s}.

In particular, $\mathcal{S}'$ has the same cardinality as $\mathcal{S}$ and the sequence of elements in Δ corresponding to the elements in $\mathcal{S}'$ and $\mathcal{S}$ coincide.

Fix one of these sets $\mathcal{S}'$. This determines uniquely a $p/|m|$-eme root η_0 of unity for which:

$$(\eta_0 x^{f_0})^{\varepsilon_0}(\eta_1 x^{f_1})^{\varepsilon_1}\cdots(\eta_t x^{f_t})^{\varepsilon_t} = 1. \tag{64}$$

We can now apply the formula (56) pf the Lemma 1.

We get $1/D_{\mathcal{S}'}$ espressed as: a linear combination of elements of the form $1/D_{\mathcal{S}''}$ where:

(1) either the cardinality of $\mathcal{S}''$ is strictly smaller than that of $\mathcal{S}'$,
(2) or these are equal but $\mathcal{S}''$ is obtaied from $\mathcal{S}'$ by removing a pair (η_s, f_s) and inserting a preceding pair (η_0, f_0).

Iterating this procedure we obtain the result.

The second condition is not satisfied Suppose the second condition of the definition of a non-broken circuit is not satisfied, i.e.: there exist distinct elements $(\zeta_{i_0}, b_{i_0}) < \cdots < (\zeta_{i_t}, b_{i_t})$ in $\mathcal{S}$ with $b_{i_0}, b_{i_1}, \ldots, b_{i_t}$ linearly dipendent. i.e.,

$$n_0 b_{i_0} + \cdots n_t b_{i_t} = 0 \tag{65}$$

for suitable integers $n_0, \ldots, n_t$.

We can also suppose that

$$\zeta_{i_0}^{n_0} \cdots \zeta_{i_t}^{n_t} \neq 1 \tag{66}$$

Otherwise we can repeat the above discussion with $a = b_{i_0}$.

As before, put $p = |n_0 \ldots n_t|$, and let $d_h := \frac{b_{i_h}}{p/|n_h|}$ per each $h = 0, \ldots, t$. The relation (65) implies

$$\varepsilon_0 d_0 + \cdots + \varepsilon_t d_t = 0, \quad \epsilon_i = \pm 1. \tag{67}$$

Again apply Lemma (34.1). Substite for x the element $\eta_s x^{d_s}$, $n = p/|n_s|$, with η_s a n-th root of ζ_{i_s}, for each $0 \le s \le t$. Substituting in $1/D_{\mathcal{S}}$, we get an expansion of $1/D_{\mathcal{S}}$ as linear combination of p^{t+1} terms, each of the form: $1/D_{\mathcal{S}'}$, where $\mathcal{S}'$ is obtained from $\mathcal{S}$ by replacing each pair (ζ_{i_s}, b_{i_s}) by a pair (η_s, d_s), η_s is a $p/|n_s|$-eme root of ζ_{i_s}.

From the relation (66) we deduce that:

$$(\eta_0 x^{d_0})^{\varepsilon_0} \cdots (\eta_t x^{d_t})^{\varepsilon_t} = \alpha x^0 = \alpha \neq 1. \tag{68}$$

We can now use the formula (57) of Lemma 1 (with $t = (\eta_0 d_0)^{-\varepsilon_0}$) and express $1/D_{\mathcal{S}'}$ as linear combination of elements of the form $1/D_{\mathcal{S}''}$ where the cardinality of $\mathcal{S}''$ is strictly smaller than that of $\mathcal{S}'$.

A simple induction completes the proof of the proposition. □

The inversion formula

Let us complete the proof of the inversion formula, in the arithmetic case, closely following the proof given in [30].

the situation is the following:

- we have a linear form ϕ with $\langle \phi, \alpha_i \rangle > 0,\ \forall i$.
- setting $v_i := \frac{\alpha_i}{\langle \phi, \alpha_i \rangle}$, the vectors $\Psi := \{v_i\}$ generate the $r-$dimensional vector space V and lie in the affine hyperplane Π, given by the equation $\langle \phi, x \rangle = 1$.
- The intersection of the cone $C(\Psi) = C(\Delta)$ with Π is the convex polytope Σ, the envelope (hull?) of the vectors v_i.
- Each cone, generated by $k+1$ independent vectors in Ψ (or of Δ), intersects Π in a k-dimensional simplex.

Finally, we have a configuration of cones obtained by projecting a configuration of simplices and there is a simple dictionary between the properties of the cones and those of the simplices.

It is clear that Σ is an union of simplices with vertices which are vectors independent of Ψ.

It is natural to call *regular* a point in Σ which is not contained in any $r-2$-dimensional simplex (or in the corresponding cone).

The connected components of the set of regular points are called in [6] the *grandi cells*.

Let us add to the set of cells the cells of the boundary, obtaining a stratification of Σ in cells and a stratification $\mathfrak{S}$ of $C(\Delta)$ in polyhedral cones.

Recall that from [12], Theorem 5.5 we have.

Proposition 35.2. *Two elements of $C(\Delta)$ are in the same cone of $\mathfrak{S}$ if and only if they are contained in the same set of simplicial cones generated by non-broken bases.*

Remark 35.3. *There are two interesting aspects of this statement:*

- *while $\mathfrak{S}$ is intrinsically defined, the non-broken bases depend on a total ordering of Δ.*
- *The second is the proof that the big cells are convex.*

Fix once and for all, an orientation of the vector space $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ and take the invariant r-form ω_T defined in (49).

We now work on a torus U which covers T:

The most important case is the torus $T_{1/m}$ that has as character group $m^{-1}\Lambda$.

In coordinates:

- given a coordinate basis x_i for T we can choose a coordinate basis y_i for U with $x_i = y_i^m$.
- Given a character $a \in \Lambda$ we have $1 - x^a = 1 - (y^a)^m = \prod_{i=0}^{m-1}(1 - \zeta^i y^a),\ \zeta = e^{2\pi i/m}$.

The definition (32) of Jeffrey–Kirwan residue: relative to a cell $\mathfrak{c}$, applies also to functions $f \in R_U$ as follows:

$$JK(\mathfrak{c}, f) = (-1)^r \sum_P \sum_{\underline{b}} \epsilon_{\underline{b}} res_{\underline{b},P}(f\omega_U),$$

where:

- P runs over points oef the arrangementin U that are the inverse images of points of the arrangement in T
- and $\underline{b}$ on the non-broken bases in Δ_P for which $\mathfrak{c} \subset C(\underline{b})$.

Remarks 35.4.
- *Observe that $JK(\mathfrak{c}, f)$ does not depend on the choice of orientation.*
- *It follows from proposition 33.1 that $JK(\mathfrak{c}, \chi^{-1} f)$, as function of χ, is a periodic polynomial on the cell $\mathfrak{c}$.*

Comparison of residues
We need to compare two notions of of residue.

Lemma 35.5. *If $\pi : U \to T$ is a covering of degree n:*

$$\pi^*(\omega_T) = n\omega_U$$

Proof From the theory of elementary divisors there exists an oriented basis $\mu_1, \dots, \mu_r$ of the character group M of U and positive integers $n_1, \dots, n_r$ such that $n = \prod_i n_i$ and $\mu_1^{n_1}, \dots, \mu_r^{n_r}$ is a basis of the character group Λ of T.

the assertion is clear using these two bases.

□

- Let $\psi = f\omega_T,\ f \in R_\Delta$ be a form on T.
- Let $P \in U$ be a point and $Q = \pi(P)$,
- Suppose given a non-broken basis of characters $\underline{b} := (a_1, \dots, a_n),\ a_i \in \Delta_P \subset \Delta \subset \Lambda$.

Since $\Lambda \subset M$ we have also that $\Delta_P = \Delta_Q$ makes sense and we have:

Corollary 35.6.

$$res_{\underline{b},P}\pi^*(\psi) = res_{\underline{b},Q}\psi$$

Proof Compose with π the map that describes a neighbourhood of P in logaritmic co-ordinates.

From proposition (33.2), the numbers $res_{\underline{b},Q}\psi$ are coefficients of the class of ψ respect to the classes of the form $\Omega_{\underline{b}}(T) = d\log(1 - x^{a_1}) \wedge \cdots \wedge d\log(1 - x^{a_n})$

(Note that we have incorporated T in the notation).

It only remains to observe that $\pi^*(\Omega_{\underline{b}})(T) = \Omega_{\underline{b}}(U)$. □

Lemma 35.7. *Let $\pi : U \to T$ be a finite cover of T of degee m. Let $f \in R_T$; then $f \circ \pi \in R_U$, and*

$$JK(\mathfrak{c}, f) = JK(\mathfrak{c}, f \circ \pi) \tag{69}$$

Proof A non-broken circuit $\underline{b}$ associated to a point P in T is also associated to each point in $\pi^{-1}(p)$ in U.

This way, we obtain all the points in U associated to $\underline{b}$.

From the definition of local residue from the Lemma 35.5, we deduce that, if $Q \in \pi^{-1}(P)$

$$res_{\underline{b},Q}(f\omega_U) = m^{-1} res_{\underline{b},P}(f\omega_T).$$

Since $m = |\pi^{-1}(P)|$ all follows. □

Consider a non-broken basis $\underline{b} := \{a_1, \dots, a_n\} \subset \Delta$, compatible elements $\{(\zeta_1, b_1), \dots, (\zeta_n, b_n)\}$, with $n_i b_i = a_i,\ \zeta_i^{n_i} = 1$.

Let m be the least common multiple of the n_i and set $U := T_{1/m}$. U is a torus that coers T and on which the elements b_i are characters.

Let $\mu = \prod_i (x^{b_i})^{k_i}$ with $0 < k_i \in \mathbb{N}$ and the function su U:

$$\gamma := \frac{1}{\prod_{i=1}^{k}(1 - \zeta_i x^{b_i})},\ i = 1, \dots, n.$$

Lemma 35.8. *(1) The coefficiente of μ in the series expansion of γ is :*

$$(-1)^r \epsilon_{\underline{b}} \sum_P res_{\underline{b},P}(\mu^{-1}\gamma\omega), \tag{70}$$

where P lies in the finite set E of points of U defined by equations $1 - \zeta_i x^{b_i} = 0,\ i = 1, \dots, n$.

In particular if $k < r$ both terms are zero.

The coefficient of μ in the series expansion of γ is:

(2) If $b := \sum_i k_i b_i$ is regular and $\mathfrak{c}$ is the unique cell containing b,

$$(-1)^r \epsilon_{\underline{b}} \sum_P res_{\underline{b},P}(\mu^{-1}\gamma\omega) = JK(\mathfrak{c}, \mu^{-1}\gamma).$$

Where the sum is over the set of P in E. If $b := \sum_i k_i b_i$ is regular and $\mathfrak{c}$ is the unique cell containing b.....

Proof 1) Let $\psi_i = x^{a_i}$ be a basis of characters for a torus S covering T.

From Lemma 35.7 the right hand side of the formula (70) is independent of the torus on which the characters ψ_i are defined.

By definition the same is true for the coefficient of μ.

We can thus reduce to the torus S for which $\underline{b}$ is a basis of the character group.

In this caso, there is a unique point P associated to $\underline{b}$.
Moreover $\omega = \epsilon_S d\log\psi_1 \wedge \cdots \wedge d\log\psi_r$.

We can now separate variables and redurce to the one-dimensional case. This is an elementary instance of the knapsack problem.
(see §1.1 o [30]).

2) To start with, let us apply this theory to the case in which Δ consists of the single basis $\underline{b}$ and the cell is $C(\underline{b})$ (a degenerate situation).

In U and with respect to coordinates y_i with $x_i = y_i^m$ the points of this arrangement are those for which $y^{ma_i} = 1,\ \forall i$.

We see then that :

$$JK(C(\underline{b}), \mu^{-1}\gamma) = (-1)^r \epsilon_{\underline{b}} \sum_P res_{\underline{b},P}(\mu^{-1}\gamma\omega).$$

To reduce to the degenerate case it suffices to show that, if $(\underline{b}', Q)$ non is una delle coppie $(\underline{b}, P)$, $\psi_i(P) = 1$, allora $res_{\underline{b}',Q}(\gamma\omega) = 0$.

If $Q \neq P$ at least one of the factors $1 - a_i\psi_i$ is holomorphic in Q and the form, whose residue we need to compute, is exact.

Suppose now that $P = Q$; we will again use the degenerate case.

Consider, in logarithmic co-ordinates around P, the form $\mu^{-1}\gamma\omega$.

It is evident that such a form has poles along hyperplanes $(a_i, \theta) = 0$ corresponding to characters ψ_i.

In this arrangement of hyperplanes we have only one cohomology class in the corresponding open set $A_{\underline{b}}$:

that of the form $d\log(a_1, \theta) \wedge \cdots \wedge d\log(a_n, \theta)$,

Therefore in such an arrangement, by definition, the cohomology class of the form $\mu^{-1}\gamma\omega$ coincides with:

$$res_{\underline{b},P}(\mu^{-1}\gamma\omega) d\log(a_1, \theta) \wedge \cdots \wedge d\log(a_n, \theta).$$

The open set complementary to the whole arrangement contains the open $A_{\underline{b}}$ and therefore again on this set the classof the form $\mu^{-1}\gamma\omega$ coincides with $res_{\underline{b},P}(\mu^{-1}\gamma\omega) d\log(a_1, \theta) \wedge \cdots \wedge d\log(a_n, \theta)$.

This is equivalent to saying that the other residued are all zero. □

The main theorem

Theorem 35.9. *Let*

$$f := \frac{1}{\prod_{(a,\chi)\in\Delta}(1 - a\chi)^{h_{a,\chi}}}, \ h_{a,\chi} \in \mathbb{N}$$

If β is in the closure of a cell $\mathfrak{c}$:

$$c_\beta = res(\beta^{-1} f\omega) = JK(\mathfrak{c}, \beta^{-1} f) \tag{71}$$

Proof Using the Lemma 35.7, we can go to a finite cover U of T.

For a suitable cover we can find a character ξ such that $\beta\xi$ lies in $\mathfrak{c}$ and the function ξf has a expansion as in proposition 35.1.

In view of this, we only need to prove the identity for each term of this expansion. Write $\beta^{-1} f = (\beta\xi)^{-1}\xi f$.

Conclusion Since each term of the expansion of ξf satisfies the hypothesis of the preceding lemma with $\mu = \beta\xi$ the result follows. □

Remark: From the observations after the definition 32 it follows that, in the preceding, c_β is (as a function of β) a periodic polynomial.

This polynomial has a remarkable property of continuity.

In fact, if β is on the boundary of two different cells, there are two possible periodic polynomials on these cells that coincide on the itersection of their closures, in particular on β.

Lezione.13 Partial fractions on a torus

36. PARTIAL FRACTIONS

We have already noted one obstacle.

Given k characters $a_1, \dots, a_k$ which are linearly independent, these generate a subgroup Λ that does not coincide with $\overline{\Lambda}$.

After a co-ordinate change if necessary, the subgroup Λ can be generated by characters corresponding to monomials $x_i^{d_i},\ i = 1, \dots, k$.

Lemma 36.1. *We want to show that $A_T/(1-x_1^{d_1}, \dots, 1-x_k^{d_k})$ decomposes as a direct sum of $d_1 d_2 \dots d_k$ rings and determine the unit elements of these rings (the primitive idempotents).*

Primitive Idempotents
Let us first treat the case: $\mathbb{C}[x, x^{-1}]/(1-x^d) = \mathbb{C}[x]/(1-x^d)$.

Let $\zeta := e^{\frac{2\pi i}{d}}$. The homomorphism $\pi : p(x) \mapsto (p(1), p(\zeta), \dots, p(\zeta^h), \dots, p(\zeta^{d-1}))$ yields an isomorphism between $\mathbb{C}[x]/(1-x^d)$ and $\mathbb{C}^d$.
Consider the d elements $e_h(x) := \frac{1}{d}\sum_{j=0}^{d-1} \zeta^{-hj} x^j,\ 0 \le h < d$. We have:

$$e_h(\zeta^k) = \frac{1}{d}\sum_{j=0}^{d-1} \zeta^{(-h+k)j} = \begin{cases} 0 & \text{se} \quad h \neq k \\ 1 & \text{se} \quad h \neq k \end{cases}$$

It follows that $e_h(x)$ is a representative of the $h+1-$eme primitive idempotent e_h of $\mathbb{C}[x]/(1-x^d) = \mathbb{C}^d$.
From this we see easily that:

Lemma 36.2. *the products:*

$$e_{h_1,\dots,h_k}(x_1, \dots, x_k) := \prod_{i=1}^{k} e_{h_i}(x_i),\ 0 \le h_i < d_i$$

are represented by $d_1 d_2 \dots d_k$ primitive idempotent of

$$A_T/(1-x_1^{d_1}, \dots, 1-x_k^{d_k}) = A_{T_{n-k}}^{d_1 d_2 \dots d_k}.$$

Explicitly this isomorphism can be obtained by the chaneg of coordinates $p(x_1, \dots, x_n) \mapsto p(\zeta_1^{h_1}, \zeta_2^{h_2}, \dots, \zeta_k^{h_k}, x_{k+1}, \dots, x_n)$.
NOTE The representatives that we have costructed depend on the system of coordinates chosen to normalize the presentation of Λ.
Note: In the quotients $\mathbb{C}[x]/(1-x^d)$ we have $xe_h = \zeta^h e_h$ while forr the representatives:

$$xe_h(x) = \frac{1}{d}\sum_{j=0}^{d-1} \zeta^{-hj} x^{j+1} = \frac{1}{d}\sum_{k=1}^{d-1} \zeta^{-h(k-1)} x^k + x^d = x^d - 1 + \zeta^h e_h(x).$$

$$x^{-1} e_h(x) = x^{-1}(x^d - 1) + \zeta^{-h} e_h(x).$$

$$\sum_{h=0}^{d-1} e_h(x) = \frac{1}{d}\sum_{j=0}^{d-1}[\sum_{h=0}^{d-1}\zeta^{-hj}]x^j = 1. \tag{72}$$

From which, in general for an element $e_{h_1,\dots,h_k}(x_1,\dots,x_k)$ we have:

$$x_i e_{h_1,\dots,h_k}(x_1,\dots,x_k) = \zeta_i^{h_i} e_{h_1,\dots,h_k}(x_1,\dots,x_k) + (x_i^{d_i}-1)\prod_{j\neq i} e_{h_j}(x_j).$$

A simple variation of this analysis will come in useful. Take k nonzero integers α_i and consider $A_T/(\alpha_1 - x_1^{d_1},\dots,\alpha_d - x_k^{d_k})$.

For each i we have d_i possibile integers $\beta_{i,j},\ j = 1,\dots,d_i$ with $\beta_{i,j}^{d_i} = \alpha_i$.

It is easy to se that:

$$A_T/(\alpha_1 - x_1^{d_1},\dots,\alpha_d - x_k^{d_k}) = \oplus_{j_1,j_2,\dots,j_k} A_T/(\beta_{1,j_1} - x_1,\dots,\beta_{k,j_k} - x_k). \tag{73}$$

For a proof, choose for each i a particular β_i with $\beta_i^{d_i} = \alpha_i$ and make a change of coordinate $x_i := \beta_i y_i$ which reduces to the preceding case.

It is useful to think geometrically.

- Given k linearly independent characters $a_1,\dots,a_k$ that generate a subgroup Λ, the subvariety W where these take the value 1 depends only on Λ.
- W consists of d disjoint components $W_1,\dots,W_d$, where d is the index of Λ in $\overline{\Lambda}$.
- The elements $e_i(x)$ defined earlier take the value 1 on the component W_i and 0 on the other components.

Of course, this does not uniquely determine the functions $e_i(x)$ but only up to functions that vanish on $W = \cup_{i=1}^d W_i$.

We can now prove the first reduction lemma, which shall be useful when we turn to the formulation of the expansion in partial fractions.

This lemma is the periodic analogue of lemma (13.1):

Lemma 36.3. *Given a list of elements $a_i \in \mathbb{Z}^n$ (with possibile repetitions) $\Psi = \{a_1,\dots,a_m\}$, the product*

$$\prod_{i=1}^m \frac{1}{(1-x^{a_i})}$$

can be written as a linear combination of elements

$$\frac{x^\gamma}{(1-x^{a_{i_1}})^{h_1}\cdots(1-x^{a_{i_r}})^{h_r}}$$

where :

- $(a_{i_k}) \in \Psi$ *for each* $k = 1,\dots,r$
- $\{a_{i_1},\dots,a_{i_r}\}$ *are linearly independent.*

Proof of the Lemma

Proof by a simple induction, one can assume that $\Psi = \{a_0, \dots, a_r\}$ with $a_0, \dots a_r$ are linearly dependent, $a_1, \dots a_r$ linearly independent, and that $\{a_0, \dots, a_r\}$ generate (after a change of co-ordinates) $\mathbb{Z}^r$.

The difficulty is that instead $\{a_1, \dots, a_r\}$ could generate a subgroup of some finite index, say m.

This is because the relation of dependence will be of type $ma_0 = \sum_{i=1}^r c_i a_i, \ m, c_i \in \mathbb{Z}$.

Let us choose (as in the preceding paragraph) representatives $e_i(x) \in A_T$ of the primitive idempotents e_i of $A_T/(1 - x^{a_1}, \dots, 1 - x^{a_r})$.

In this ring we have $x^{a_0} e_i = \beta_i e_i$ with $\beta_i \in \mathbb{C}^*$.By the definition of the elements $e_i(x)$ (cf. 72), we have $1 = \sum_i e_i(x)$, whence:

$$\frac{1}{(1 - x^{a_0}) \cdots (1 - x^{a_r})} = \sum_i \frac{e_i(x)}{(1 - x^{a_0}) \cdots (1 - x^{a_r})}.$$

Let us, therefore, a term

$$\frac{e_i(x)}{(1 - x^{a_0}) \cdots (1 - x^{a_r})}.$$

We Have that $(1 - x^{a_0}) e_i(x) = \sum_{j=1}^r c_j (1 - x^{a_j}) + (1 - \beta_i) e_i(x), \ c_j \in A_T$, distinguishing two cases.

First case $\beta_i = 1$

If $\beta_i = 1$, we have $(1 - x^{a_0}) e_i(x) = \sum_{j=1}^r c_j (1 - x^{a_j})$ and substituting:

$$\frac{e_i(x)}{(1 - x^{a_0}) \cdots (1 - x^{a_r})} = \frac{\sum_{j=1}^r c_j (1 - x^{a_j})}{(1 - x^{a_0})^2 (1 - a_1 a_1) \cdots (1 - x^{a_r})}$$

we get a sum of terms in which the denominator lacks a factor $1 - x^{a_j}$.

Second case $\beta_i \neq 1$

If $\beta_i \neq 1$, we have $e_i(x) = (1 - \beta_i)^{-1} [(1 - x^{a_0}) e_i(x) - \sum_{j=1}^r c_j (1 - x^{a_j})]$

$$\frac{e_i(x)}{(1 - x^{a_0}) \cdots (1 - x^{a_r})} = (1 - \beta_i)^{-1} \frac{(1 - x^{a_0}) e_i(x) - \sum_{j=1}^r c_j (1 - x^{a_j})}{(1 - x^{a_0}) \cdots (1 - x^{a_r})},$$

and the result follows by induction. □

This is not yet the final form in which we wish to develop in partial fractions. We next need a reduction of the terms x^γ of the numerator.

Suppose given n independent elements $a_1, \dots, a_n$ that generate in $\mathbb{Z}^n$ un sublattice of index m. Choose m representatives of this sublattice $\xi_1, \dots, \xi_m$.

Take a fraction of the type

$$\frac{x^\gamma}{\prod_{i=1}^n (1 - x^{a_i})^{h_i}}.$$

Write $\gamma = \xi_i + \sum_{j=1}^n c_j a_j, \ c_j \in \mathbb{Z}$. We wish to expand

$$\frac{x^{\sum_{j=1}^{n} c_j a_j}}{\prod_{i=1}^{n}(1-x^{a_i})^{h_i}}.$$

To do this, note that for a variable t we have:

$$\frac{t}{1-t}=\frac{1}{1-t}-1,\ \frac{t^{-1}}{1-t}=\frac{1}{1-t}+t^{-1}.$$

It follows by induction (applied to $t=x^{a_i}$) that:

Lemma 36.4. *$\frac{x^{\sum_{j=1}^{n} c_j a_j}}{\prod_{i=1}^{n}(1-x^{a_i})^{h_i}}$ can be expanded as a linear combination with constant coefficients of terms of the type $\frac{1}{\prod_{i=1}^{n}(1-x^{a_i})^{k_i}}$, $k_i \leq h_i$ and then of type $\frac{x^\beta}{\prod_{i=1}^{n}(1-x^{a_i})^{k_i}}$ in which for at least one $k_i=0$.*

This Lemma does not yet give us a normal form of partial fractions and we deal with this below.

To complete the costruction of an expression in partial fractions for each function of $R_\Delta := A_T[(\prod_{i=1}^{m}(1-x^{a_i})^{-1}]$ we make of another notion.

Take a set of linearly independent elements $S := b_1 := a_{i_1}, \dots, b_d := a_{i_d}$, extracted from the list $\Delta := \{a_1, \dots, a_m)\}$.

Fix also a component W of the variety defined by equations $1-x^{b_i}=1,\ i=1,\dots,d$.

Definition 20. *We shall say that S is a non-broken circuit on W if S is a non-broken circuit relative to the subset:*

$$\Delta_W := \{a_i \in \Delta \,|\, x^{a_i}=1, \quad su \quad W\}.$$

Given such an S, we can choose coordinates for the characters and the coordinates $z_1,\dots,z_n$ for the torus such that:

- the lattice generated by $S := b_1,\dots,b_d$ has finite index m_S in the lattice in which the last $n-d$ coordinates are zero,
- More precisely, the monomials x^{b_k}, with respect to the new coordinates are of the form z^{c_i} in the first k coordinates z_i.

Let ius call the coordinates $z_{d+1},\dots,z_n$ un set of of complementary coordinates.

The building blockes of the decomposition

We have seen that the ring $\mathbb{C}[z_1^{\pm 1},\dots,z_d^{\pm 1}]/(1-z^{c_1},\dots,1-z^{c_d}) = \oplus_{i=1}^{m_S}\mathbb{C}e_i$ with e_i primitive idempotents and we have found a method for constructing the representatives $e_i(z)$.

- Let $A_S=\mathbb{C}[z_{d+1}^{\pm 1},\dots,z_n^{\pm 1}]$ be the ring of Laurent polynomials in the last coordinate.

- Given S and one of the idempotents e_i let W be the irreducible component of $x^{b_i} = 1$ that corresponds to e_i.
- We will use the notation $e_W = e_i(x)$.

Definition 21. *Define*

$$R_{S,W} := \{f \in R_\Delta \,|\, f = \frac{ge_W}{\prod_{j=1}^d (1-x^{b_j})^{m_j}} \,|\, g \in A_S,\ m_j > 0,\ \forall j.\}$$

These spaces are the building blocke of the desired expansion

Theorem 36.5. *(1) The space $R_{S,i}$ has as basis the monomials*

$$\frac{z_{d+1}^{h_{d+1}} \dots z_n^{h_n} e_i(z)}{\prod_{j=1}^d (1-x^{b_j})^{m_j}} \,|\, h_i \in \mathbb{Z},\ m_j > 0,\ \forall i, j.$$

(2) $R_\Delta = \oplus_S R_{S,W}$, varying W and S among the non-broken circuits relative to W.

1) Proof Let us prove part of the theorem, and leave the rset to be completed in the Lezione 9, sezione 38.

Put $u_i := x^{b_i}$. The elements $u_1, \dots, u_d, z_{d+1}, \dots, z_n$ are clearly algebraically independent. That is, they generate a polynomial ring.

Thus the elements $1-u_1, \dots, 1-u_d, z_{d+1}, \dots, z_n$ also generate a polynomial ring and R_S is contained i the Laurent polynomial ring in the same variables.

These polynomials have as basis the monomials in the variables with integer exponents.

The proposed monomials are part of this basis, and therefore linearly independent.

2) First, from the lemma (36.3) it follows that each function in R_Δ can be written as a linear combination of expressions

$$f = \frac{g}{\prod_{j=1}^d (1-x^{b_j})^{m_j}} \,|\, g \in A_T,\ m_j > 0,\ \forall j$$

and $S := b_1 := a_{i_1}, \dots, b_d := a_{i_d}$ linearly independent.

For a given term $\frac{g}{\prod_{j=1}^d (1-x^{b_j})^{m_j}}$ let us make the earlier change of veriables $z_1, \dots, z_n$ and expanding, reduce to the case $g = \prod_{i=1}^n z_i^{h_i}$ is a monomial.

Appeal next to the Lemma (36) to obtain a sum of terms of the type $f = \frac{g\xi}{\prod_{j=1}^d (1-x^{b_j})^{m_j}} \,|\, g \in A_T,\ m_j > 0,\ \forall j$ with $S := b_1, \dots, b_d$ linearly independent that generate a lattice Λ, ξ a representative of $\overline{\Lambda}/\Lambda$ and g a monomial in the complementary variables.

Utilize next the fact that the vector space has as basis the elements $e_W$as W runs over the components of $x^{b_i} = 1,\ i = 1, \dots, d$ and rewrite the sum as terms of type $f = \frac{ge_W}{\prod_{j=1}^d (1-x^{b_j})^{m_j}} \,|\, g \in A_T,\ m_j > 0,\ \forall j$.

Let now $S := b_1, \dots, b_d$ be a broken circuit relative to W component of the variety V of equations $x^{b_i} = 1$. There exists e with $1 \le e \le d$ and a b that precedes $b_e, \dots, b_d$ is linearly dipendente of these and takes the value 1 on W.

Observe that e^b is constant on the connected components of the variety V_e defined by equations $x^{b_i} = 1,\ i = e, \dots, d$, and in particular takes the value 1 on the unque component Z that contains W.

Consider now a representative e_Z (relative to $b_e, \dots, b_d$)which takes the value 1 su Z and 0 on the other components of the variety V_e.

It follows that the element $e_W e_Z$ coincides with e_W as a function on the variety V. More precisely, $e_W = e_W e_Z + \sum_i c_i(1 - x^{b_i}),\ c_i \in A_T$.

Therefore the element $\frac{e_W}{\prod_{j=1}^d (1-x^{b_j})} = \frac{e_W e_Z}{\prod_{j=1}^d (1-x^{b_j})}$ modulo terms in which at least one of the factors $(1 - x^{b_j})$ is cancelled by denominators. We can thus make a substitution by induction and assume that $e_W = e_W e_Z$.

We save that $(1-x^b)e_Z$ vanishes identically on the variety V_e and therefore can be written as:

$$(1 - x^b)e_Z = \sum_{i=e}^{d} g_i(1 - x^{b_i}),\ g_i \in A_T$$

It follows that:

$$\frac{e_W}{\prod_{j=1}^d (1 - x^{b_j})} = \frac{e_W(1 - x^b)e_Z}{(1 - x^b)\prod_{j=1}^d (1 - x^{b_j})} = \frac{e_W(\sum_{i=e}^d g_i(1 - x^{b_i}))}{(1 - x^b)\prod_{j=1}^d (1 - x^{b_j})}$$

is a sum of terms in which in denominator a factor $(1-x^{b_i})$ has been replaced by a factor $(1 - x^b)$.

Now, lexicographically ordering the linearly independent seqence $b_1, \dots, b_d$ extracted from Δ, we have shown that a fraction in whose denominator appears a split sequence spezzata can be replaced by a sum in which appaear, in the denominators, strictly smaller (lexicographically) sequences.

Iterating this procedure we are left with terms in whose denominators only non-broken sequences appear.

Therefore the proposed monomials linearly generate linearly the space.

The fact that these constitute a basis, or equivalently that the sum is direct, has been proved in 9.

Remark 36.6. *In a later section we will apply a change of basis of R_Δ, starting from the basis that we have just found.*

The chjange will betriangular relative to apartial order that we now introduce.

Given two elements $M_1 := \frac{z_{d+1}^{h_{d_1+1}} \dots z_n^{h_n} e_i(z)}{\prod_{j=1}^{d_1}(1-x^{b_j})^{m_j}}$, $M_2 := \frac{w_{d+1}^{h_{d_2+1}} \dots w_n^{h_n} e_i(w)}{\prod_{j=1}^{d_2}(1-x^{c_j})^{n_j}}$ let us say that:

$M_1 < M_2$ if $d_1 < d_2$ or if $d_1 = d_2$ and $\sum_j m_j < \sum_i n_i$.

The triangular changes of basis are those that replace a monomial M with a nonzero multiple of M plus a sum of monomials which are strictly smaller with respect to this order.

The earlier expansion in partial fractions yields an algorithm to compute the coefficients of the expansion of a function in R_Δ:

$$F := \frac{f}{\prod_{i=1}^m (1-x^{a_i})^{m_i}} = \sum_{b \in C(A)} S_F(b) e^b.$$

In fact, expanding in partial fractions we get terms of type $\frac{x^b}{\prod_{i=1}^d (1-^{b_i})^{m_i}}$ with the b_i linearly independent, for which we have:

$$\prod_{i=1}^n \frac{x^b}{(1-x^{b_i})^{m_i}} = \sum_{h_1,h_2,\dots,h_n} \prod_{i=1}^n \binom{m_i - 1 + h_i}{h_i} x^{\sum_{i=0}^n h_i b_i + b} \tag{74}$$

The functions that we get are of the following type. Take a polynomial and restrict it to the points in the lattice which lie in one of the cones, possibly translated by a lattice point.

For the generating function ($f = 1$) the theorem is more precise. We have seen that $S_A(b)$ is un quasi polynomial on the closure of each cell. This is a discrete version of continuity.

37. Partial fractions (concluded)

For our purpose it is convenient to reformulate the Theorem on expansion in partial fractions 36.5. We will introduce another family of spaces that decompose R_Δ, each with a distinguished basis and (the new condition) each stable with respect to all the derivations.

We will use the notation of Theorem 14.3. Consider a list $S := \{b_1, \dots, b_d\}$ extracted from Δ, an irreducible component W of the variety defined by equations $x^{b_i} = 1$ and let e_i l'be the idempotent that corrisponds to W. We use again the notation $e_W = e_i(x)$. Let $z_1, \dots, z_d, z_{d+1}, \dots, z_n$ be a system of coordinates adapted to S.

Although the following considerations are valid in eneral, we restrict ourself to the case when S is a a non-broken circuit on W.

Define the following ring of differential operators:

$$D_S := \mathbb{C}[\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_d}, z_{d+1}^{\pm 1}, \dots, z_n^{\pm 1}].$$

Note - given that the variables are disticnt this algebra of operators is *commutativa.*

Also D_S is a module over the algebra of all differential operators with constant coefficients.

Define

$$Z_{S,W} := D_S \frac{e_W}{\prod_{j=1}^d (1-x^{b_j})}.$$

Theorem 37.1. *(1) $Z_{S,W}$ is a free D_S-module with basis the element* $\frac{e_W}{\prod_{j=1}^d (1-x^{b_j})}$.

(2) We have a direct sum decomposition:

$$R_\Delta = \oplus_{W,S} Z_{S,W}$$

as W runs over the components of the arrangementand of S among corresponding non-brokencircuits.

Proof 1) The key consists in knowing the operation of a monomial in operators $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_d}$ on the element $\frac{e_W}{\prod_{j=1}^d (1-x^{b_j})}$.
Differentiating once yields:

$$\frac{\partial}{\partial z_i} \frac{e_W}{\prod_{j=1}^d (1-x^{b_j})} =$$

$$\frac{\frac{\partial}{\partial z_i} e_W}{\prod_{j=1}^d (1-x^{b_j})} + \sum_{j=1}^d \frac{\partial x^{b_j}}{\partial z_i} \frac{e_W}{\prod_{h=1}^{i-1}(1-x^{b_h})(1-x^{b_j})^2 \prod_{h=j+1}^d (1-x^{b_h})}.$$

We compute

$$\sum_{j=1}^d \frac{\partial x^{b_j}}{\partial z_i} \frac{e_W}{\prod_{h=1}^{i-1}(1-x^{b_h})(1-x^{b_j})^2 \prod_{h=j+1}^d (1-x^{b_h})}$$

Espanding this derivative, we get a dominant term which is a nonzero multiple of $\frac{e_W}{\prod_{h=1}^{i-1}(1-x^{b_h})(1-x^{b_j})^2 \prod_{h=j+1}^d (1-x^{b_h})}$ plus other terms that would be smaller with respect to a suitable ordering of a basis of R_Δ.

This proves the first part.
2) As for the second, we use the theory of $D-$ modules in a later Lezione.

The total residue and the top degree cohomology

To find the total residue and the cohomology in the top degree observe that:

Proposition 37.2. *D_S is the dicet sum of subspace generated by applying the derivate and the one-dimesional subspace generated by $\prod_{i=d+1}^n z_i^{-1}$.*

As corollary we deduce by induction that:

Corollary 37.3. *A complement to the subspace generated by the application of derivates to elements of R_Δ has as basis the family of elements:*

$$\omega_{S,W} := \prod_{i=d+1}^{n} z_i^{-1} \frac{e_W}{\prod_{j=1}^{d}(1-x^{b_j})}.$$

Therefore the classes of the forms $\omega_{S,W}\omega_T$ give a basis of the cohomology.

Lezione 14. $D-$modules II

38. Modules

To investigate the linear independence of the expansion in partial fractions in the toric case, we will develop an toric analogue of the $D-$modules of 9.

The algebra of the differential operators that we will use is l'algebra of the operators with coefficients which are the regular functions on the torus – in co-ordinates, the Laurent polynomials $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.

Let us call this algebra *the periodic Weyl algebra* keeping in mind the change of co-ordinates $x_i = e^{\theta_i}$), and denote it by $\tilde{W}(n)$.

$\tilde{W}(n)$ has a basis consisting of the elements:

$$x_1^{h_1} \dots x_n^{h_n} \frac{\partial^{k_1}}{\partial x_1} \dots \frac{\partial^{k_n}}{\partial x_n},\ h_i \in \mathbb{Z},\ k_i \in \mathbb{N}.$$

Remark 38.1. *Non is necessario mettere in testa tutte levariables and poi tutte the derivate, in seguito sarà utile mettere in altri ordinivariables and derivate anche alternandovariables with derivate. Dalle regole of commutazione tutte queste are basi ottenute with cambiamenti of base of tipo triangolare.*

Consider d monomials x^{a_i} where $S := \{a_1, \dots, a_d\} \subset \mathbb{Z}^n$ is a set of d linearly independent vectors with integer coordinates.

Consider next the linear differential operators with constant coefficients that kill these monomials:

$$D_S := \{\sum_{j=1}^{n} \alpha_j \frac{\partial}{\partial x_j} \mid \sum_{j=1}^{n} \alpha_j \frac{\partial x^{a_i}}{\partial x_j} = 0,\ i = 1, \dots, d\} \tag{75}$$

Lemma 38.2. *(1) D_S is a vector space of dimension $n-d$.*

In an adapted system of coordinates z_i for which the a_i have the the lastn $-\, d$ coordinates equal to 0, D_S has as basis the operators $\frac{\partial}{\partial z_i},\ i = d+1, \dots, n$.

Proof Evidently $2 \implies 1$; so we turn to 2.
In an adapted system, the monomials associates to S are of the form

$$\prod_{i=1}^{d} z_i^{a_{ij}},\ j = 1, \dots, d$$

where the matrix a_{ij} has determinant nonzero.
We have

$$\sum_{j=1}^{n} \alpha_j \frac{\partial \prod_{i=1}^{d} z_i^{a_{ij}}}{\partial z_j} = \sum_{j=1}^{d} \alpha_j a_{i,j} \prod_{i=1}^{d} z_i^{a_{ij}} z_j^{-1}.$$

Therefore, if $D := \sum_{j=1}^{n} \alpha_j \frac{\partial}{\partial z_j} \in D_S$ we must have

$$\sum_{j=1}^{d} \alpha_j a_{i,j} z_j^{-1} = 0,\ \forall i.$$

Since the matrix $a_{i,j}$ is invertible this implies that $D = \sum_{j=d+1}^{n} \alpha_j \frac{\partial}{\partial z_j}$.

- (2) Taking $S = \{a_1, \dots, a_d\}$ as before let Λ or, to be precise, Λ_S, the lattice generated.
- Take d numbers $\alpha_i \in \mathbb{C}$ that we think of as giving a multiplicative character $\phi : \Lambda \to \mathbb{C}^*,\ \phi(a_i) = \alpha_i$.
- We next define a $D-$module $N_{S,\phi}$ associated to S and a ϕ. By definition this is the

cyclic $\tilde{W}(n)$-module generated by an element u_S subject to relations:

$$x^a u_S = \phi(a_i) u_S,\ \forall a \in S,\ D u_S = 0,\ \forall D \in D_S. \tag{76}$$

Lemma 38.3. *(1) $N_{S,\phi}$ depend only on the lattice Λ and da ϕ.*
Denote this by $N_{\Lambda,\phi}$

(2) $N_{\Lambda,\phi}$ is irreducible if $\Lambda = \overline{\Lambda}$.

(3) In general $N_{\Lambda,\phi} = \oplus_\psi N_{\overline{\Lambda},\psi}$, where ψ runs over the $m = [\overline{\Lambda} : \Lambda]$ multiplicative characters $\psi : \overline{\Lambda} \to \mathbb{C}^*$ that extend ϕ.

(4) $N_{\overline{\Lambda},\psi}$ is irreducible with characteristic variety the zero set of equations $x^a = \psi(a),\ a \in \overline{\Lambda}$.

Proof 1. Clearly, if $x^a u_S = \phi(a)u_S,\ x^b u_S = \phi(b)u_S$ we have $x^{a+b}u_S = \phi(a+b)u_S$. Since, in addition, D_S also depends only on Λ this first claim is clearly true.

2. First, suppose that $\Lambda = \overline{\Lambda}$. We can then choose coordinates z_i such that $N_{\Lambda,\phi} = \tilde{W}(n)/I$ where I is the left ideal generated by the elements $z_i - \alpha_i,\ i = 1, \dots, d;\ \dfrac{\partial}{\partial z_j},\ j = d+1, \dots, n$.

To start with, let us check that modulo this left ideal the elements of the form

$$x_{d+1}^{h_1} \dots x_n^{h_n} \frac{\partial^{k_1}}{\partial x_1} \dots \frac{\partial^{k_d}}{\partial x_d},\ h_i \in \mathbb{Z},\ k_i \in \mathbb{N} \tag{77}$$

linearly generate the module.

Denote by u_S the class of 1.

We take as basis of $\tilde{W}(n)$ the monomials:

$$x_{d+1}^{h_1} \dots x_n^{h_n} \frac{\partial^{k_1}}{\partial x_1} \dots \frac{\partial^{k_d}}{\partial x_d} x_1^{h_1} \dots x_d^{h_n} \frac{\partial^{k_{d+1}}}{\partial x_{d+1}} \dots \frac{\partial^{k_n}}{\partial x_n},\ h_i \in \mathbb{Z},\ k_i \in \mathbb{N}.$$

In this basis:

$$x_{d+1}^{h_1} \dots x_n^{h_n} \frac{\partial^{k_1}}{\partial x_1} \dots \frac{\partial^{k_d}}{\partial x_d} x_1^{h_1} \dots x_d^{h_n} \frac{\partial^{k_{d+1}}}{\partial x_{d+1}} \dots \frac{\partial^{k_n}}{\partial x_n} u_s = 0$$

if for some $i > d$ we have $k_i > 0$. If, on the other hand, all these k_i are zero we have:

$$x_{d+1}^{h_1} \dots x_n^{h_n} \frac{\partial^{k_1}}{\partial x_1} \dots \frac{\partial^{k_d}}{\partial x_d} x_1^{h_1} \dots x_d^{h_n} u_s =$$

$$x_{d+1}^{h_1} \dots x_n^{h_n} \frac{\partial^{k_1}}{\partial x_1} \dots \frac{\partial^{k_d}}{\partial x_d} \alpha_1^{h_1} \dots \alpha_d^{h_n} u_s$$

At this point to see see that the preceding elements constitute a basis and that the module is irreducible taka an element p, which is a linear combination (with coefficients not all zero) of elements (77). It remains to show that the submodule P generated by p is all of N_S.

The proof of this fact is entirely along the lines of **??** .

Moltiplying p by a monomial $\prod_{i=d+1}^n x_i^N$ we can suppose that the exponents h_i are non-negative.

Order the elements lexicographically with respect to the esponents and argue by induction. Let the the variabile x_k $(k > d)$ appears in p with esponent $h > 0$ the maximum possible. Moltiply p by $\dfrac{\partial}{\partial x_k}$ to obtain an element $q \in P$ in the module generated by p.

We know that that $\frac{\partial}{\partial x_k} x_k^h = h x_k^{h-1} + x_k^h \frac{\partial}{\partial x_k}$; now move $\frac{\partial}{\partial x_k}$ to right of x_k to get an element in the ideal I and thus 0 in the module.

Thus q is now expressed as a new linear combination in which the exponent of x_k is smaller.

By induction all the variables x_k can be removed; then moltiplying by $x_i,\ i \leq d$ also the derivates, obtaining $u_S \in P$ and thus $P = N_{\Lambda,\phi}$.

3. In adapted coordinates Λ is generated by vectors $k_i e_i,\ i = 1, \dots, d$ where the e_i are elementss of the canonical basis and the k_i are positive integers.

If $\phi(k_i e_i) = \alpha_i$ we can extend ϕ to a multiplicative character of $\overline{\Lambda}$ by setting $\psi(e_i)^{k_i} = \alpha_i$. This can be done exactly $\prod_{i=1}^d k_i$ ways. For each such, we have amodule $N_{\overline{\Lambda},\psi}$; and we then seek an iso morphism :

$$\pi : \oplus_\psi N_{\overline{\Lambda},\psi} \to N_{\Lambda,\phi}$$

.

Consider first the generator $u_{\overline{\Lambda},\psi}$ of $N_{\overline{\Lambda},\psi}$. This evidently verifies the equations satisfied by $u_{\Lambda,\phi}$ and thus we have a morphism $\pi_\psi : N_{\Lambda,\phi} \to N_{\overline{\Lambda},\psi}$ per each ψ.

From the formula (73) it follows that a choice can be made of that the representatives of the idempotents that appear in the corresponding deomposition obtaining elements $e_\psi(z)$ for which $e_\psi(z) u_{\Lambda,\phi}$ satisfy the equations of $u_{\overline{\Lambda},\psi}$. We thus have morphisms $i_\psi : N_{\overline{\Lambda},\psi} \to N_{\Lambda,\phi}$ from which

$$i : \oplus_\psi N_{\overline{\Lambda},\psi} \to N_{\Lambda,\phi}$$

. We leave it to the reader to verify that i, π are isomorphisms inverse to one another.

The proof of the claim regarding the characteristic variety follows the lines of the proof in the case of hyperplanes. The equations $(z_i - \beta_i)u = \frac{\partial}{\partial z_j} u = 0,\ i = 1, \dots, d;\ j = d+1, \dots, n$ can be transformed to $z_i - \beta_i = \xi_j = 0,\ i = 1, \dots, d;\ j = d+1, \dots, n$ that define the conormale bundle to the variety defined by equations $z_i - \beta_i = 0,\ i = 1, \dots, d$. This also we leave to the reader.

Lezione 15. Linear independence, $D-$modules

Using the structure of $D-$module on R_Δ, the study of the canonical basis is reduced to proving that the vectors $v_{S,W}$ with S a non-broken circuit are linearly independent.

Consider a basis $b_1, \dots, b_k$ of Σ and costruct adapted coordinates$z_1, \dots, z_n$ such that $y^{b_1}, \dots, y^{b_k}$ are monomials in the first k coordinates z_i.

Take a point with coordinates $z_i = a_i$ in the component W and local coordinates $z_i = a_i e^{\theta_i}$ such that for each vector b one has $z^b = a^b e^{(b,\theta)}$ and locally $b \in \Delta_W$ if and only if $a^b = 1$ and $(b, \theta) = 0$. Then locally W is given with respect to the coordinates θ by the linear equations $(b, \theta) = 0,\ b \in \Delta_W$.

Consider the space transcversal to W with parameters $z_i = a_i e^{\theta_i},\ i \leq k,\ z_i = a_i,\ i > k$. This reduces computations with with k-forms to the case of hyperplanes.

Lezione 16. Cohomology

We encountered, in the lezione on differential forms, de Rham cohomology and and the notion of *differential graded algebra.*

In general, cohomology has an aspect which is formal and algebraic.

Formally a *cochain complex* is a sequence of abelian groups C^i and operators, called differentials, $d^i : C^i \to C^{i+1}$ with the property $d^{i+1} \circ d^i = 0, \forall i$. In our case the C_i are vector spaces and the d^i linear maps.

Often we omit the index i in the differentials and simply use the notation d.

We set:

- $Z^i(C) := \{a \in C^i \,|\, d(a) = 0$ i *cocycles*
- $B^i(C) := \{d(a) \,|\, a \in C^{i-1})$ i *coboundaries.*

  By hypothesis $B^i(C) \subset Z^i(C)$ and we set:

- $H^i(C) = Z^i(C)/B^i(C)$ the *coomology.*

Particularly important is the case in which there is a *multiplication*, more precisely, $\oplus_i C^i$ is a graded algebra.

Definition 22. *One speaks of an differential graded algebra when the following compatibilty holds between multiplication and differentiation:*

$$d(ab) = d(a)b + (-1)^i a\, d(b),\ a \in C^i$$

In this case we see immediately that the cocycles are a subring and that coboundaries an ideal in the ring of cocycles.

It follows that the cohomology is a *graded algebra.*

Lezione 16. Cohomology

We wish now to show how we can compute the cohomology (with complex co-efficients) of the complement of hyperplanes in a vector space or a toric arrangement in a torus, using the decomposition of functions which we have obtained.

In both cases we start with a set Δ of vectors.

In the first case $\Delta := \{\alpha_1, \dots, \alpha_N\}$ ate complex vectors, thought of as linear forms $\alpha_i(y)$ to which correspond the hyperplanes $\alpha_i(y) = 0$.

In the second case $\Delta := \{a_1, \dots, a_N\}$ are integer vectors, to which correspond codimension one subgroups (not necessarily connected) $x^a = 1$.

The complement of an arrangement We will use the following notation:

For the hyperplanes:

$$\mathcal{A}_\Delta := \{p \in V \,|\, \alpha_i(p) \neq 0,\ \forall i \in \Delta\}$$

Per the torus:

$$\mathcal{T}_\Delta := \{p \in T \,|\, x^{a_i}(p) \neq 1,\ \forall i \in \Delta\}$$

Functions on the complement of an arrangement

Indichiamo con: A_V ($= \mathbb{C}[y_1, \dots, y_n]$ in coordinate), l'anello delle funzioni (polinomi) su V.

A_T ($= \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ in coordinates), the ring of functions (Laurent [polynomials) on T.

For the hyperplanes the *functions on* $\mathcal{A}_\Delta$:

$$R_\Delta := A_V[d^{-1}],\ d = \prod_{\alpha_i \in \Delta} \alpha_i(y)$$

For the torus the *functions on* $\mathcal{T}_\Delta$:

$$R(T)_\Delta := A_T[d^{-1}],\ d = \prod_{a_i \in \Delta} (1 - x^{a_i}).$$

In both cases we start with a pair $(W, \mathcal{S})$ where:

- W is a component of the arrangiament.
  - In the case of hyperplanes W is a subspace which is the intersection of the given hyperplanes.
  - In the case of the torus W is a connected component of an intersectionof varieties of the type $x^a = 1$ with a among the characters.
- S is a basis non-broken on W.
  - In the case of hyperplanes this means that $\mathcal{S}$ is una basis which is non-broken for the set of characters (forms?) $\alpha_i \in \Delta$ vanishing on W.
  - In the case of the torus this means that $\mathcal{S}$ is a basis non-broken for the set of $a_i \in \Delta$ wuch that $x^{a_i} = 1$ su W.

The decomposition Fixing a pair (W, S) with $|S| = k$ and W of codimension k, we have made

a choice of coordinates $z_1, \dots, z_n$ such that:

- $z_i = \alpha_i(y),\ i = 1, \dots, k$ for the hyperlanes.
- $x^{a_i},\ i = 1, \dots, k$ is a monomial in $z_i,\ i = 1, \dots, k$ for the torus.

Correspondingly, we have a

commutative ring of differential operators:

- $S_\Delta := \mathbb{C}[\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_k}, z_{k+1}, \dots, z_n]$ for the hyperplanes.
- $S(T)_\Delta := \mathbb{C}[\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_k}, z_{k+1}^{\pm 1}, \dots, z_n^{\pm 1}]$ for the torus.

Finally, we have chosen

a particular element.

- $a_{W,\mathcal{S}} := \frac{1}{\prod_{i=1}^k z_i}$ for the hyperplanes.
- $b_{W,\mathcal{S}} := \frac{e_{W,\mathcal{S}}}{\prod_{i=1}^k (1 - x^{a_i})}$ for the torus.

where $e_{W,\mathcal{S}} \in \mathbb{C}[z_1^{\pm 1}, \dots, z_k^{\pm 1}]$ is a function which takes the value 1 on W and 0 on the other connected components of the variety with equations $x^{a_i} = 1$.

The fundamental Theorem which we have proved is:

Theorem 38.4. *(1)* $R_\Delta = \oplus_{W,\mathcal{S}} S_\Delta a_{W,\mathcal{S}}$

(2) $S_\Delta a_{W,\mathcal{S}}$ *is invariant under differential operators with constant coefficients.*

For the torus:

(3) $R(T)_\Delta = \oplus_{W,\mathcal{S}} S(T)_\Delta b_{W,\mathcal{S}}$

(4) $S(T)_\Delta b_{W,\mathcal{S}}$ *is stable under differential operators with constant coefficients.*

From this we can draw a consequence for the cohomology.

Consider the two exterior algebras of differential forms:

$$E_V := \mathbb{C}[dx_1, \dots, dx_n], \quad E_T := \mathbb{C}[d\log(x_1), \dots, d\log(x_n)]$$

Let us describe the: differential forms

- $\Omega_{\mathcal{A}_V}$ the algebra of algebraic differential forms on $\mathcal{A}_V$.
- $\Omega_{\mathcal{A}_T}$ the algebra of algebraic differential forms on $\mathcal{A}_T$

Naturally these are *differential algebras* with respect to the usual differentiation d.

We have:

Lemma 38.5.

$$\Omega_{\mathcal{A}_V} = R_\Delta \otimes E_V$$

$$\Omega_{\mathcal{A}_T} = R(T)_\Delta \otimes E_V = R(T)_\Delta \otimes E_T.$$

From all these facts we deduce that:

Theorem 38.6. *The complex of differential forms decomposes into the direct sum of complexes:*

$$\Omega_{\mathcal{A}_V} = \oplus_{W,\mathcal{S}} S_\Delta a_{W,\mathcal{S}} \otimes E_V$$

$$\Omega_{\mathcal{A}_T} = \oplus_{W,\mathcal{S}} S(T)_\Delta b_{W,\mathcal{S}} \otimes E_V.$$

Therefore to compute the cohomology (at last as vector spaces) it thus suffices to compute it for the blocks

$$S_\Delta a_{W,\mathcal{S}} \otimes E_V, \qquad S(T)_\Delta b_{W,\mathcal{S}} \otimes E_V.$$

The cohomology of the blocks We shall now use a standard fact *The Künneth Formula* to see that the cohomology of the bloccks is rather simple; in effect,:

Theorem 38.7. *The cohomology of $S_\Delta a_{W,\mathcal{S}} \otimes E_V$ is of dimension 1 generated by the classes of*

$$a_{W,\mathcal{S}} dz_1 \wedge \cdots \wedge dz_k = d\log(z_1) \wedge \cdots \wedge d\log(z_k)$$

That of $S(T)_\Delta b_{W,\mathcal{S}} \otimes E_V$ is the free module
on the algebra generated by the form $d\log(z_i),\ i = k+1, \dots n$
of the class of the form $b_{W,\mathcal{S}} d\log(z_1) \wedge \cdots \wedge d\log(z_k)$.

To prove this theorem, let us begin from the simplest case, to which we shall return.

Consider the complex of algebraic forms on $\mathbb{C}^*$, the punctured plane.

The cohomology of the blocks

This is the simplest arrangement of hyperplanes.

The co-ordinate ring is the ring $\mathbb{C}[x, x^{-1}]$ of Laurent polynomials which we decompose into two pieces, the polynomials and the *polar part*:

$$\mathbb{C}[x, x^{-1}] = \mathbb{C}[x] \oplus x^{-1}\mathbb{C}[x^{-1}]$$

These pieces are both modules over the ring $\mathbb{C}[\frac{d}{dx}]$ of diffrential operators with constant coefficients.

But thes are two very different modules (in some ways *dual to each other*).

- The operator $\frac{d}{dx} : \mathbb{C}[x] \to \mathbb{C}[x]$ is surjective with kernel the one-dimensional space of constant polynomials.
- The operator $\frac{d}{dx} : x^{-1}\mathbb{C}[x^{-1}] \to x^{-1}\mathbb{C}[x^{-1}]$ is injective with image $x^{-2}\mathbb{C}[x^{-1}]$.
- In fact $x^{-1}\mathbb{C}[x^{-1}]$ is a free module with range 1 generated by x^{-1} over the ring $\mathbb{C}[\frac{d}{dx}]$.

The coomology of the blocks

- Thus the complex of algebraic forms decomposes into the sum of the two complexes:
- $$C(x) : \mathbb{C}[x] \xrightarrow{\ d\ } \mathbb{C}[x]dx,$$ $$\hat{C}(x) : x^{-1}\mathbb{C}[x^{-1}] \xrightarrow{\ d\ } x^{-1}\mathbb{C}[x^{-1}]dx$$
- From the above discussion we get:

$$H^0(C(x)) = \mathbb{C}1, \quad H^1(C(x)) = 0,$$
$$H^0(\hat{C}(x)) = 0, \quad H^1(\hat{C}(x)) = \mathbb{C}d\log(x).$$

The Künneth Formula

The Künneth formula describes the cohomology of a complex obtained, starting from two complexes C, D by an operation of composition called *tensor product*.

We will describe this in the simplest case, when the two complexes are formed of vector spaces.

Here are the definitions:

- The complex $C \otimes D$ in degree i is by definition $\oplus_{h+k=i} C^h \otimes D^k$, and for differential we take:
- $d(a \otimes b) := d(a) \otimes b + (-1)^h a \otimes d(b), \quad a \in C^h$
- We have clearly

$$Z^i(C) \otimes Z^j(D) \subset Z^{i+j}(C \otimes D),$$

$$B^i(C) \otimes Z^j(D) + Z^i(C) \otimes B^j(D) \subset B^{i+j}(C \otimes D)$$

- This induces a map:

$$H^i(C) \otimes H^j(D) \to H^{i+j}(C \otimes D).$$

The Künneth formula The Künneth formula affirms (only in the case of complexes of vector spaces) the isomorphism:

$$\oplus_{i+j=n} H^i(C) \otimes H^j(D) \to H^n(C \otimes D).$$

This formula can clearly be iterated.

Let us now take n variables $x_1, \dots, x_n$ and the space A_k of polynomials in which the variables $x_1, \dots, x_k$ appear with strictly negative exponents while the remaining variables appear with non-negative exponents.

It is then clear that the algebraic forms $\Omega(A_k)$ with coefficients in A_k form a subcomplex isomorphic to the tensor product of the complexes:

$$\Omega(A_k) := C(x_1) \otimes \cdots \otimes C(x_k) \otimes \hat{C}(x_{k+1}) \otimes \cdots \otimes \hat{C}(x_n)$$

The Künneth Formula From the Künneth Formula it follows that:

$$H^i(\Omega(A_k)) = 0, \ \forall i \neq k, \quad H^k(\Omega(A_k)) = \mathbb{C} d\log(x_1) \wedge \cdots \wedge d\log(x_k).$$

In the case of the torus on the other hand we haev that the the complex $S(T)_\Delta b_{W,\mathcal{S}} b_{W,\mathcal{S}} \otimes E_T = \mathbb{C}[\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_k}, z_{k+1}^{\pm 1}, \dots, z_n^{\pm 1}] b_{W,\mathcal{S}} \otimes E_T$ is the tensor product:

$$\mathbb{C}[\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_k}] b_{W,\mathcal{S}} \otimes E_{T_1} \otimes \mathbb{C}[z_{k+1}^{\pm 1}, \dots, z_n^{\pm 1}] \otimes E_{T_2}$$

Where E_{V_1} risp. E_{V_2} are the exterior algebras generated by elements $dz_1, \dots, dz_k$ resp., $dz_{k+1}, \dots, dz_n$. Ora,

- the complex $\mathbb{C}[\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_k}] b_{W,\mathcal{S}} \otimes E_{N_1}$ is isomorphic to the product of the complexes $\hat{C}(z_i), \ i = 1, \dots, k$
- thus its cohomology is 1-dimensional, in degree k and generated by the class of $b_{W,\mathcal{S}} dz_1 \wedge \cdots \wedge dz_k$
- while the complex $\mathbb{C}[z_{k+1}^{\pm 1}, \dots, z_n^{\pm 1}] \otimes E_{T_2}$ is the complex of algebraic forms on the torus with coordinates $z_{k+1}, \dots, z_n$

- and therefore it cohomology is the algebra E_{T_2}, exterior algebra in the elements $d\log(z_{k+1}), \dots, d\log(z_n)$.

□

The multiplicative structure (hyperplanes)

We have seen that cohomology is an algebra, and so it is interesting to discuss the multiplicative structure.

We begin with the case of the hyperplanes $\Delta = \{\alpha_1, \dots, \alpha_N\}$, considering the cohomology of the complement $\mathcal{A}_\Delta$.

it is useful to change the normalization and define the form $\omega_i := \frac{1}{2\pi i} d\log(\alpha_i)$.

From the theorem of the preceding section we know that the classes of such forms generate the cohomology.

In fact something much more precise holds:

From the formula **??** it follows that, given elements $\alpha_{i_1}, \dots, \alpha_{i_k}$ which are linearly independent we have :

Fundamental Relations

$$\sum_{h=1}^{k} (-1)^h \omega_{i_1} \wedge \cdots \wedge \omega_{i_{h-1}} \wedge \omega_{i_{h+1}} \wedge \cdots \wedge \omega_{i_k} = 0$$

The multiplicative structure (hyperplanes)

Consider formally an esterior algebra $\Lambda(\psi_1, \dots, \psi_N)$, generated by the symbols $\psi_i,\ i = 1, \dots, N$ and in it the ideal I_Δ generated by elements

$$R(i_1, \dots, i_k) := \sum_{h=1}^{k} (-1)^h \psi_{i_1} \wedge \cdots \wedge \psi_{i_{h-1}} \wedge \psi_{i_{h+1}} \wedge \cdots \wedge \psi_{i_k}.$$

Denote by Λ_Δ the algebra $\Lambda(\psi_1, \dots, \psi_N)/I_\Delta$.

This is known as the *Orlik-Solomon* algebra after the authors who discovered and studied it.

Let now $\mathcal{E}_\Delta$ the subalgebra of the algebra of differential forms, generated by elements ω_i.

This is a subalgebra of the algebra of cocycled, and finally H_Δ is algebra of the cohomology of $\mathcal{A}_\Delta$.

From all the above analysis we obtain an isomorphism

$$\rho : \Lambda_\Delta = \Lambda(\psi_1, \dots, \psi_N)/I_\Delta \to \mathcal{E}_\Delta, \quad \rho(\psi_i) := \omega_i$$

$$\pi : \mathcal{E}_\Delta \to H_\Delta.$$

π is associated to a form in $\mathcal{E}_\Delta$, its cohomology class of cohomology.

he multiplicative structure (hyperplanes)

We have the The fondamental Theorem of Orlik-Solomon:

- Both ρ and π are isomorphisms.
- The algebra Λ_Δ has a basis consisting of classes of products $\psi_{i_1} \wedge \cdots \wedge \psi_{i_k}$ corresponding to non-broken circuits $\alpha_{i_1}, \dots, \alpha_{i_k}$.

The multiplicative structure (hyperplanes)

Now the proof of the theorem is easy. We already know that H_Δ is generated by the classes of the elements ω_i and that the classes of the products $\omega_{i_1}\wedge\cdots\wedge\omega_{i_k}$ corresponding to the non-broken circuits $\alpha_{i_1},\dots,\alpha_{i_k}$ constitute a basis.

Thus it suffices to show that the classes modulo I_Δ of the products $\psi_{i_1}\wedge\cdots\wedge\psi_{i_k}$ corresponding to the non-broken circuits $\alpha_{i_1},\dots,\alpha_{i_k}$ linearly generate Λ_Δ.

For this there exists an algorithm: if $\psi_{i_1}\wedge\cdots\wedge\psi_{i_k}$ corresponds to a broken circuit $\alpha_{i_1},\dots,\alpha_{i_k}$ then there exists a $e\leq k$ and an index $i<i_e$ with $\alpha_i,\alpha_{i_e},\dots,\alpha_{i_k}$ linearly dependent.

To such dependence is associated the relation $R(i,i_e,\dots,i_k)=-[\psi_i\wedge R(i_e,\dots,i_k)+\psi_{i_e}\wedge\cdots\wedge\psi_{i_k}]$

The multiplicative structure (hyperplanes)

Thus in Λ_Δ to the product $\psi_{i_1}\wedge\cdots\wedge\psi_{i_k}$ we can substitute:

$$-\psi_{i_1}\wedge\cdots\wedge\psi_{i_{e-1}}\wedge\psi_i\wedge R(i_e,\dots,i_k)$$

which is a sum of lexicographically smaller monomials.

Iterating the algorithm we can express each monomial in terms of monomials associated to non-broken circuits. □

The multiplicative structure (tori)

In the case of the toric arrangement the situation is much more complicated, and in in general the multiplicative structure is not explicit.

There is, however, an important case in which the theory is similar to the case of hyperplanes, the *unimodular case*.

This is when there is is a unique point (vertex?) of the arrangement, to be precise, when each basis of the vector space of characters extracted from Δ is also a basis of the lattice of characters.

We have encountered an interesting class of unimodular examples arising from graphs and networks.

In this case the above theory implies that the classes of the elements $\psi_i := d\log(x_i)$, $\omega_{a_i} := d\log(1-x^{a_i})$ with $a_i\in\Delta$ generate the cohomology.

The multiplicative structure (tori)

Given independent elements $\beta_1,\dots,\beta_k\in\Delta$ and an element $\beta\in\Delta$ which is dependent on these:

Lemma 38.8. *there exists a relation of the type* $\beta=\sum_{i=1}^k\epsilon_i\beta_i,\quad \epsilon_i=\pm1.$

Proof Complete $\beta_1,\dots,\beta_k$ into a basis with elements $\beta_{k+1},\dots,\beta_n\in\Delta$. If by contradiction a coefficient, say, the first is modulo $m>1$? then we would have the base $\beta,\beta_2,\dots,\beta_k,\beta_{k+1},\dots,\beta_n\in\Delta$ extracted from Δ that generates a sublattice of index m. □

We also have a basis of the cohomology arising from non-broken circuits.

Therefore we can attempt to determine the relations and prove a Theorem analogous to that of Orlik-Solomon.

The multiplicative structure (tori)

We thus need to find the relations.

Let us begin with some symbolic identities:

Set, for $i = 1, \dots, n$, $\omega_i := d\log(1 - x_i)$, $\psi_i := d\log x_i$. Also put

$$\theta = d\log(1 - \prod_{i=1}^n x_i) = \frac{\prod_{i=1}^n x_i}{(1 - \prod_{i=1}^n x_i)} \sum_{j=1}^n \psi_j.$$

Let us use the relation 78:

$$1 - \prod_{i=1}^n x_i = \sum_{I \subsetneq \{1,2,\dots,n\}} \prod_{i \in I} x_i \prod_{j \notin I} (1 - x_j). \tag{78}$$

Split the sum in 3 terms: $I = \{1, \dots, n-1\}$, $I \subsetneq \{1, \dots, n-1\}$ ed $n \in I$. We et

$$\prod_{i=1}^{n-1} x_i(1 - x_n) + (1 - \prod_{i=1}^{n-1} x_i)(1 - x_n) + x_n(1 - \prod_{i=1}^{n-1} x_i) = 1 - \prod_{i=1}^n x_i.$$

Da cui

$$\sum_{I \subsetneq \{1,2,\dots,n\}} \frac{1}{(1 - \prod_{i=1}^n x_i)} \prod_{i \in I} \frac{x_i}{(1 - x_i)} = \frac{1}{\prod_{i=1}^n (1 - x_i)}.$$

This can be rewritten in terms of differential forms after multiplying by $dx_1 \wedge \cdots \wedge dx_n$:

Given a proper subset $I = \{i_1 < \cdots < i_t\}$ in $\{1, \dots, n\}$, let $J = \{j_1 < \cdots < i_{n-t}\}$ be its complement. We have

$$\frac{1}{(1 - \prod_{i=1}^n x_i)} \prod_{i \in I} \frac{x_i}{(1 - x_i)} dx_1 \wedge \cdots \wedge dx_n =$$

$$(-1)^{s_I} \frac{\prod_{i=1}^n x_i}{(1 - \prod_{i=1}^n x_i)} \omega_{i_1} \wedge \omega_{i_2} \wedge \cdots \wedge \omega_{i_k} \wedge \psi_{j_1} \wedge \cdots \wedge \psi_{j_{n-k}} =$$

$$(-1)^{s_I} \omega_{i_1} \wedge \omega_{i_2} \wedge \cdots \wedge \omega_{i_k} \wedge \psi_{j_1} \wedge \cdots \wedge \psi_{j_{n-k-1}} \wedge \theta$$

where s_I is the parity of the permutation $(i_1, \dots, i_t, j_1, \dots, i_{n-t})$.

Define the n-form:

$$\Phi_I = (-1)^{s_I} \omega_{i_1} \wedge \cdots \wedge \omega_{i_t} \wedge \psi_{j_1} \wedge \cdots \wedge \psi_{j_{n-t-1}} \wedge \theta$$

We have then the identity

$$\sum_{I \subsetneq \{1,2,\dots,n\}} \Phi_I = \omega_1 \wedge \cdots \wedge \omega_n. \tag{79}$$

For $s = 0, \dots n$, put

$$\theta^{(s)} = d\log(1 - \prod_{i=1}^{s} x_i^{-1} \prod_{j=s+1}^{n} x_j) =$$

$$\frac{\prod_{i=1}^{n} x_i}{(1 - \prod_{i=1}^{s} x_i^{-1} \prod_{j=s+1}^{n} x_j)}[-\sum_{j=1}^{s} \psi_j + \sum_{j=s+1}^{n} \psi_j]$$

To find a relation involving $\theta^{(s)}$ it is enough to make the substitution of variables $x_i \mapsto x_i^{-1}, 1 \le i \le s$.

In general observe that

$$d\log(1 - x^{-1}) = d\log\frac{x-1}{x} = -d\log(x) - d\log(1-x).$$

Thus, the substitution of x_i with x_i^{-1} for $i = 1, \dots, s$, corresponds, in the formula (79) to the substitution of ω_i with $-\omega_i - \psi_i$. This gives us a new identity that we label (79'). □

For consistency, given an element γ in the lattice of characters, set$\omega_\gamma := d\log(1 - x^\gamma)$, $\psi_\gamma := d\log(x^\gamma)$.

Now let us consider the exterior algebra $\Lambda(\omega_{a_1}, \dots, \omega_{a_N}, \psi_a)$, formally generated by the symbols $\omega_{a_i},\ i = 1, \dots, N,\ \psi_a,\ a \in \Lambda$.

For the ψ_a suppose that $a \to \psi_a$ is an linear isomorphism of Λ to additive group generated by these elements.

That is we have the identities $\psi_{ma+nb} = m\psi_a + n\psi_b,\ a, b \in \Lambda,\ m, n \in \mathbb{Z}$

Let us take in this algebra the ideal I_Δ generated by the following elements:

Given k independent elements β_1, β_k from Δ, let $\beta = \sum_{i=1}^{k} \beta_i \in \Delta$.

Complete to a basis of the lattice, adding elements $\beta_{k+1}, \dots, \beta_n$.

Substituting in the expression $R := \sum_{I \subsetneq \{1,2,\dots,n\}} \Phi_I - \omega_1 \wedge \dots \wedge \omega_n$ ad x_i the element x^{β_i} has the effect that ω_i substitutes for ω_{β_i} and θ for ω_β.

The relation R becomes $R(\beta, \beta_1, \beta_k) \in \Lambda(\omega_1, \dots, \omega_N, \psi_1, \dots, \psi_n)$.

More generally if $\beta = \sum_{i=1}^{k} \epsilon_i \beta_i,\ \epsilon_i = \pm 1, \beta \in \Delta$ we can again make a substitution, utilising the rule $\omega_{-a} = -\psi_a - \omega_a$, obtaining an alement which we again denote $R(\beta, \beta_1, \beta_k)$.

As in the case of hyperplanes we can consider: the ideal I_Δ generated by:

- The elements $\psi_{ma+nb} - m\psi_a - n\psi_b$.
- the elements $\omega_{a_{i_1}} \wedge \dots \wedge \omega_{a_{i_k}} \wedge \psi_a$ when a is in the lattice generated by the a_i.
- gli elements $R(\beta, \beta_1, \beta_k)$ per tutte the relazioni of dipendenza $\beta = \sum_{i=1}^{k} \beta_i$ fra elements of Δ, with i β_i linearly independent.

Denote by Λ_Δ the algebra $\Lambda(\psi_1, \dots, \psi_N)/I_\Delta$.

Let now $\mathcal{E}_\Delta$ be the subalgebra of the algebra of differential forms generated by the elements ω_i, ψ_a.

This is a subalgebra of the algebra of cocycles, and finally, H_Δ is the cohomology algebra of $\mathcal{T}_\Delta$.

From all the above analyis we have homomorphisms

$$\rho : \Lambda_\Delta = \Lambda(\omega_{a_1}, \dots, \omega_{a_N}, \psi_a)/I_\Delta \to \mathcal{E}_\Delta, \quad \rho(\psi_a) := d\log(x^a),$$
$$\rho(\omega_a) := d\log(1 - x^a).$$
$$\pi : \mathcal{E}_\Delta \to H_\Delta.$$

π associates to a form in $\mathcal{E}_\Delta$ its class of cohomology.

We have the Fundamental Theorem of presentation

- Both ρ and π are isomorphisms.
- The algebra Λ_Δ haa a basis consisting of classes of products $\omega_{a_{i_1}} \wedge \cdots \wedge \omega_{a_{i_k}} \wedge \psi_{a_{j_1}} \wedge \cdots \wedge \psi_{a_{j_t}}$ where $a_{i_1}, \dots, a_{i_k}$ is a non-broken circuit and $a_{j_1}, \dots, a_{j_t}$ are elements of Δ, part of $n-k$ elements chosen to complete $a_{i_1}, \dots, a_{i_k}$ to a basis.

The multiplicative structure-hyperplanes

At this point the proof of this Teorema is easy.

We already know that H_Δ is generatated by the classes of the elements ω_{a_i}, ψ_a and that the classes of the elements $\omega_{a_{i_1}} \wedge \cdots \wedge \omega_{a_{i_k}} \wedge \psi_{a_{j_1}} \wedge \cdots \wedge \psi_{a_{j_t}}$, presented in the theorem, form a basis.

Thus it is sufficient to show that the classes modulo I_Δ of such elements generate linearly Λ_Δ.

Given any element $\omega_{a_{i_1}} \wedge \cdots \wedge \omega_{a_{i_k}} \wedge \psi_{b_{j_1}} \wedge \cdots \wedge \psi_{b_{j_t}}$ we have two reduction algorithms.

First of all, such a product is 0 modulo I_Δ is 0 if the elements b_{j_h} are not linearly independent modulo the lattice generated of the elements a_{i_s}.

Also, if $\omega_{a_{i_1}} \wedge \cdots \wedge \omega_{a_{i_k}}$ corresponds to a broken circuit $a_{i_1}, \dots, a_{i_k}$ then there exists an $e \leq k$ and an index $i < i_e$ with $a_i, a_{i_e}, \dots, a_{i_k}$ linearly dependent.

This dependence is expressed by a relation $R(a_i, a_{i_e}, \dots, a_{i_k})$.

Such a relation lets us express the product $\omega_{a_{i_1}} \wedge \cdots \wedge \omega_{a_{i_k}}$ as a sum of other terms which either contain fewer factors ω_a or are lexicographically smaller.

Iterating the algorithm one can express each monomial in terms of monomials associated to non-broken circuits. □

Part 10. **Four appendixs**

39. Appendix 1, Cyclotomic Polynomials

In our formulae, we encountered series with integer coefficients expressed by means of calculations with roots of unity.

Roots of unity are special examples of algebraic numbers.

Algebraic Numbers

- A number α is said to be algebraic if it satisfies an equation $f(\alpha) = 0$ where $f(x)$ is a polynomial with rational coefficients,
- To handle rational espressions in an algebraic number α we need to know the minimal degree monic polynomial f satisfied by α. (This polynomial is irreducible and unique).
- If $f(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$ then every rational expression in α has a unique normal form as a polynomial in α of degee $< n$ with rational coefficients.

In more abstract terms, the numbers $1, \alpha, \dots, \alpha^{n-1}$ form a basis for the field $\mathbb{Q}(\alpha)$ generated by α.

Calculational rules

Here are two rules of calculation. The first, which is evident, consists of the substitution $\alpha^n = -(a_1\alpha^{n-1} + \cdots + a_{n-1}\alpha + a_n)$.

The second permits us to compute, given an element $g(\alpha) \neq 0$ with $g(x)$ a polynomial with ratonal coefficients, the polynomial expression for $g(\alpha)^{-1}$.

For this there is a standard algorithm based on euclidean division: Find polynomials $a(x), b(x)$ with $a(x)g(x) + b(x)f(x) = 1$. Substituting for x the number α and using the fact that $f(\alpha) = 0$ we get $g(\alpha)^{-1} = a(\alpha)$.

With this technique, it is necessary to calculate the minimal polynomial satisfied by a $m-$th root of unity $\zeta = e^{2\pi i/m}$ (or euivalently, as we shall see, by a root of the form $e^{2k\pi i/m}$ with $k < m$ and coprime with m.

Such a number is said to be *a primitive root of unity*.

The primitive roots of unity of order m are therefore as many as the natural numbers smaller than m and coprime to m.

The number $\phi(m)$ of such integers is called *Euler function* ϕ and can be easily computed using the following facts:

$\phi(ab) = \phi(a)\phi(b)$ if a, b are coprime.

$\phi(p^k) = (p-1)p^{k-1}$ if p isprime.

The minimal polynomial satisfied by an $m-$th root of unity is called *mth cyclotomic polynomial* and denotated by $\phi_m(x)$.

The polynomial $\phi_m(x)$ can be calculated recursively from the formula:

$$x^m - 1 = \prod_{d \,|\, m} \phi_d(x).$$

(The espression $d \,|\, m$ signifies that d divides m.)

Example:

If $m = p$ is prime $\phi_p(x) = x^{p-1} + x^{p-2} + \ldots + x + 1$.

$\phi_4(x) = x^2 + 1, \quad \phi_6(x) = x^3 + 1.$

40. Appendix 2, Modules

We have studied modules over the Weyl algebra.

In very simple terms, an algebra R can be described intrinsically or as an algebra of operators on a vector space. A module is essentially an *incarnation* of R as an algebra of operators.

in more formal terms, a module is a homomorphism $p : R \to End(V)$ from R to the algebra of endomorphisms of a vector space V.

Equivalently, a module is a *multiplication*:

$$p : R \times V \to V, \quad \text{written} \quad (r, v) \mapsto rv,$$

satisfying the following axioms:

p is bilinear. $1v = v$ for each $v \in V$,and $(rs)v = r(sv)$ for each $r, s \in R$ and $v \in V$.

The most natural module is the ring R itself, in which the multiplication p coincides with the usual multiplication in thye ring. This module is also called the *regular representation*.

Various operations are possible on modules. In particular, the direct sum of two modules M, N is the space of pairs (m, n), $m \in M$, $n \in N$ with $r(m, n) := (rm, rn)$.

A submodule $N \subset M$ of a module M is a (vector) subspace satisfying $rn \in N$, $\forall r \in R$, $\forall n \in N$, or, in other words, N *is stable with respect to the operators induced from* R.

An important costruction is *the quotient module* M/N. This refers to a new vector space obtained from M putting the vectors of N equal to 0; or in other words letting $m_1 = m_2$ if $m_1 - m_2 \in N$. If N is a submodule the action of R on M induces in anatural way an action on M/N.

Sarting from a ring R we can construct iother modules by taking quotients and direct sums.

The *free module* is a direct sum $R^{\oplus I}$ (not necessarily finite) of copies of R. Formally, given a set I the module $R^{\oplus I}$ is the set of functions $f : I \to R$ with the property $f(i) = 0$ except for finitely many elements i (a condition which is vacuous if I is finite).

Given a module M and a set m_i, $i \in I$ of elements $m_i \in M$ indexed by I ew have a map

$$\pi : R^{\oplus I} \to M, \quad \pi(f) = \sum_{i \in I} f(i) m_i.$$

We say that the elements m_i *generate* M if π is surjective.

In this case M is isomorphic to $R^{\oplus I}/K$ where $K = \{f \in R^{\oplus I} \,|\, \pi(f) = 0\}$ and the *kernel* di π is sometimes called the *module of relations among the lements* m_i.

A particularly important case arises when M is generated by a a single element. m. In this case the module is called *cyclic*. The module of relations of a cyclic module is a left ideal I of R, $I = \{r \in R \,|\, rm = 0\}$ alo known as the *annihillator* of m. We have $M = R/I$.

We have seen some important special types of modules First of all, the *irreducible* ones, modules M which do not contain a submodule N other than 0 or M. We see immediately that an irreducible module is necessarily cyclic and that the aanihilator of any nonzero element is a maximal left ideal.

Semisimple or *completely reducible* modules are direct sums of irreducible modules.

We have also stated some basic theorems on modules, in particular, the structure theorem for semisimple modules and the theorem of Jordan–Hölder.

41. Appendix 3, Lattices

We discuss a simple part of the theory of finitely generated abelian groups.

By a lattice we shall mean a subgroup $\mathbb{Z}^n$ for given n,

It is asy to show that such a lattice is finitely generated. We can exhibit a finite set of m generators by means of a $n \times m$ matrix A with integer entries.

Changing a basis of $\mathbb{Z}^n$ corresponds to multiplying A on the left by an $ntimesn$ invertible matrix B integer entries obtaining a new matrix BA, while a way of changing the generators corresponds to multiplying A on the right by an $m \times m$ integer matrix C, obtaining the matrix AC.

Multiplication on the left by invertible integral matrices can be described in more algorithmic tyerms as a sequence of:

elementary rwo operations

- Exchange two rows.
- Change the sign of a row.
- Add to one row another row multiplied by an integer.

Similarly, multiplying on the right by invertible integer matrices is described in terms of asequences of *elementary column operations.*

the basic algorithm consists of:

(1) permute rows and columns such that $a_{1,1} \neq 0$ is the minimum (in absolute value) nonzero entry in the matrix.
(2) Change the sign if necesary such that $a_{1,1} > 0$.
(3) Fo each $i = 2, \dots, n$, if $a_{i,1} \neq 0$ do Euclidean division $a_{i,1} = a_{1.1}q + r,\ |r| < a_{1,1}$.
(4) Add to the $i-$th row the first multiplied by $-q$.
(5) Repeat the steps 3 and 4, this time on the columns.
(6) Return to step 1 and iterate..

We obtain a new matrix A' in which we have $a_{i,1} = 0 = a_{1,i},\ \forall i > 0$.

At this point, run the same algorithm on the matrix obtained by omitting the first row and column of A

Finally we obtain a matrix which has all entries 0 apart from the *diagonal* ones $a_{i,i}$. Letting $d_i := a_{i,i}$, we have moreover that (for some k) the first k of the d_i are positive and he remaining zero.

With additional elementary operations we can also normalise the d_i such that $d_1 \,|\, d_2 \,|\, d_3 \,|\, \dots \,|\, d_k$.

It is not difficult show that the d_i are invariants of A; more precisely, $\prod_{i=1}^{h} d_i$ is the maximum common divisor of the (nonzero) determinants of the $h \times H$ minors of A.

42. Appendix 4, Laplace transform

In the vector space $\mathbb{R}^n$ we defined the Laplace transform by means of the formula:

$$L(f)(y) := \int_{\mathbb{R}^n} e^{-(y,x)} f(x) dx.$$

Up to a normalisation, this transform is related to the Fourier transform.

$$\hat{f}(y) = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} e^{i(y,x)} f(x) dx.$$

The main difference, especially for our computations, lies in the fact hat in the Fourier transform the kernel $e^{i(y,x)}$ has modulus 1 while in the Laplace transform the kernel goes exponentially to zero at ∞.

The principal property of the Fourier transform is that it preserves L^2 norms; in other words:

Plancherel Theorem:

$$\int |\hat{f}(y)|^2 dy = \int |f(x)|^2 dx.$$

In the cases of interest to us, we can relate the Laplace and Fourier transforms as follows. If the function f has support in a pointed cone and has polynomial growth, we can define the Laplace transform not only for y in the interior of the dual cone, but also for complex $z = y + iu$. Now the resulting function is holomorphic in z in the region with the real part belonging to the dual cone. For y fixed, we have:

$$L(f)(y - iu) := \int e^{-(y-iu,x)} f(x) dx = \hat{g}(u), \quad g(x) := e^{-(y,x)} f(x).$$

In particular, from the Plancherel theorem it follows that if the Laplace transform of f is zero, so is f. It is this fact that permits us to talk of the inverse of the Laplace transform.

References

[1] M. Ahmed, J. De Loera, R. Hemmecke, *Polyhedral cones of magic cubes and squares.* Discrete and computational geometry, 25–41, Algorithms Combin., 25, Springer, Berlin, 2003.

[2] T.V. Alekseevskaya, I.M. Gelfand, I. M., A.V. Zelevinsky, A. V. *Distribution of real hyperplanes and the partition function connected with it.* Dokl. Akad. Nauk SSSR 297 (1987), no. 6, 1289–1293.

[3] W. Baldoni-Silva, J.A. De Loera, M. Vergne, *Counting Integer flows in Networks* , preprint math.CO/0303228

[4] W. Baldoni-Silva, M. Vergne, *Residues formulae for volumes and Ehrhart polynomials of convex polytopes* , preprint math.CO/0103097

[5] E.T. Bell, *Interpolated denumerants and Lambert series* , Amer. J. Math. 65 (1943), 382–386.

[6] M. Brion, M. Vergne, *Residue formulae, vector partition functions and lattice points in rational polytopes.* J.A.M.S. (v. 10) 4 (1997), 797–833.

[7] M. Brion, M. Vergne, *Arrangements of hyperplanes. I. Rational functions and Jeffrey-Kirwan residue.* Ann. Sci. cole Norm. Sup. (4) 32 (1999), no. 5, 715–741.

[8] S. C. Coutinho, *A primer of algebraic D-modules.* London Mathematical Society Student Texts, 33. Cambridge University Press, Cambridge, 1995.

[9] W. Dahmen, C.Micchelli, *The number of solutions to linear Diophantine equations and multivariate splines.* Trans. Amer. Math. Soc. 308 (1988), no. 2, 509–532.

[10] C. DeConcini, C. Procesi, *Wonderful Models of subspace arrangements.* Selecta Math. (N.S.) 1 (1995), no. 3, 459–494.

[11] C. DeConcini, C. Procesi, *Nested sets and Jeffrey Kirwan residues.* in: Geometric methods in Algebra and Number Theory, Bogomolov, F., Tschinkel. Y., Eds., vol. 235, 139–150.

[12] C. DeConcini, C. Procesi, *On the geometry of graph arrangements.* math.CO/0412130

[13] C. DeConcini, C. Procesi, *Toric arrangements.* math.AG/0505351

[14] A. Dimca, G.I. Lehrer, *Purity and equivariant weight polynomials.* Algebraic groups and Lie groups, 161–181, Austral. Math. Soc. Lect. Ser., 9, Cambridge Univ. Press, Cambridge, 1997

[15] J.M. Douglass, *Toral arrangements and hyperplane arrangements.* Rocky Mountain J. Math. 28 (1998), no. 3, 939–956.

[16] E. M. Feichtner, B. Sturmfels *Matroid polytopes, nested sets and Bergman fans,* CO/0411260 .

[17] A. Grothendieck, *On the de Rham cohomology of algebraic varieties.* Inst. Hautes Études Sci. Publ. Math. No. 29 (1966), 95–103.

[18] R. Hartshorne, *Local cohomology.* A seminar given by A. Grothendieck, Harvard University, Fall, 1961. Lecture Notes in Mathematics, No. 41 Springer-Verlag, Berlin-New York 1967 vi+106 pp.

[19] A. Henderson, *The symmetric group representation on cohomology of the regular elements of a maximal torus of the special linear group* , preprint http://www.maths.usyd.edu.au/u/anthonyh/

[20] Lehrer, G. I. *The cohomology of the regular semisimple variety.* J. Algebra 199 (1998), no. 2, 666–689.

[21] Lehrer, G. I. *A toral configuration space and regular semisimple conjugacy classes.* Math. Proc. Cambridge Philos. Soc. 118 (1995), no. 1, 105–113.

[22] E. Looijenga, *Cohomology of* $\mathcal{M}_3$ *and* $\mathcal{M}_3^1$. Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991), Contemp. Math., 150, Amer. Math. Soc., 1993, 205–228.

[23] C. Macmeikan, *Modules of derivations for toral arrangements.* Indag. Math. (N.S.) 15 (2004), no. 2, 257–267.

[24] C. Macmeikan, *The Poincaré polynomial of an mp arrangement.* Proc. Amer. Math. Soc. 132 (2004), no. 6, 1575–1580.

[25] R. Mac Pherson, C. Procesi, *Making conical compactifications into wonderful ones.* Selecta Math. 4, (1998),n.1,125-139. Dekker 1998.

[26] P. Orlik, H. Terao, *Arrangements of hyperplanes* . Grundlehren der Mathematischen Wissenschaften 300. Springer-Verlag, Berlin, 1992.

[27] A. Postnikov, *Permutaedra, associaedra and beyond* , preprint 2004.

[28] A. Szenes, *Iterated residues and multiple Bernoulli polynomials.* Int. Math. Res. Not. (1998), No.18, 937-956.

[29] A. Szenes, *Residue theorem for rational trigonometric sums and Verlinde's formula.* Duke Math. J. 118 (2003), no. 2, 189–227.

[30] A. Szenes, M. Vergne, *Residue formulae for vector partitions and Euler-Maclaurin sums* preprint, CO/0202253.

[31] A. Szenes, M. Vergne, *Toric reduction and a conjecture of Batyrev and Materov* preprint, AT/0306311.