A PDE approach to the numerical computation of the effective Hamiltonian and of the Aubry set

Marco Rorro
rorro@caspur.it

CASPUR
Numerical Methods for Viscosity Solutions and Applications
Rome, September, 6-8, 2004
1. The Cell Problem
2. A Homogenization Problem
3. Numerical Approximation of the Cell Problem
4. Numerical Approximation of the Effective Hamiltonian
5. Numerical Approximation of the Aubry Set
6. Two Dimensional Experiments
The cell problem

We consider the family of periodic Hamilton-Jacobi equations

\[
\frac{1}{2} |Du(x) + p|^2 = f(x) + c
\]  

(1)

with \( x \in \mathbb{T}^2 \equiv \mathbb{R}^2 / \mathbb{Z}^2 \) for the parameters \( p \in \mathbb{R}^2 \) and \( c \in \mathbb{R} \).

- Existence: for a unique value \( c = \bar{H}(p) \) equation (1) has a viscosity solution, see [5].

- Uniqueness: in general such solution is not unique on the torus \( \mathbb{T}^2 \).
The critical value such that the cell problem (1) has solution as function of $p$, $\bar{H}(p)$, is known as the effective Hamiltonian.

The set $\mathcal{A}$ which characterizes the uniqueness of the solution of equation (1) is known as the Aubry set.

The function $\bar{H}(p)$ and the set $\mathcal{A}(p)$ are the objects of our approximation.
The effective Hamiltonian appears for the first time in the study of the limit for $\epsilon \to 0$ of strong oscillating systems of the kind

$$\begin{cases}
    u_t^\epsilon + H(\frac{x}{\epsilon}, Du^\epsilon) = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\
    u^\epsilon(x, 0) = u_0 & \text{on } \mathbb{R}^n.
\end{cases} \quad (2)$$

with $H$ periodic in $x$. In fact the solution of (2) converges to solution of

$$\begin{cases}
    u_t + \bar{H}(Du) = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\
    u(x, 0) = u_0 & \text{on } \mathbb{R}^n.
\end{cases} \quad (3)$$
We use a first-order semi-lagrangian scheme \cite{2} on a structured rectangular grid applied to the cell problem (1) rewritten in the form

\[ |Du(x) + p| = \sqrt{2(f(x) + c)} \]  \hspace{1cm} (4)

The fully discrete scheme is \( u^{n+1} = F^-(u^n) \) where

\[ F^-(u^n)_{ij} = \min_{a \in \partial B(0,1) \cup \{0\}} \{ u^n(x_{ij} + a \Delta t) + a p \Delta t \} + \Delta t \sqrt{2(f(x_{ij}) + c)} \]  \hspace{1cm} (5)

To get a minimal subsolution we replace in (5) the \( \min \) operator with the \( \max \) operator and ... we call such map \( F^+ \).

\( F^\pm \) are monotone non expansive maps.
Numerical Discretization

$F_-$ computes the solution of the equivalent evolutive problem:

\[
\begin{cases}
  u_t + |Du(x) + p| = \sqrt{2(f(x) + c)} & \text{in } \mathbb{T}^n \times (0, \infty) \\
  u(x, 0) = u_0(x) & \text{on } \mathbb{T}^n.
\end{cases}
\]  \tag{6}

which, under suitable hypotheses and for $c = \bar{H}(p)$ tends to the solution of the steady cell problem,

- for $c < \bar{H}(p)$ the solution tends to $-\infty$
- for $c > \bar{H}(p)$ the solution tends to $+\infty$

and in both cases it behaves like $u(x) + vt$, \cite{[1]}.
Effective Hamiltonian approximation

We get a numerical approximation of the speed $v$ by

$$v^n = \frac{1}{N} \sum_{i,j} \frac{u_{ij}^{n+1} - u_{ij}^n}{\Delta t}$$  \hspace{1cm} (7)

($N$ the number of the nodes), until the variation $v^{n+1} - v^n$ is sufficiently small and we use it to approximate the effective Hamiltonian by:

$$c_{n+1} = c_n + \frac{1}{2} v^2 - v \min_x \left\{ \sqrt{2(f + c_n)} \right\}$$  \hspace{1cm} (8)

with $c_0 = -\min f \leq \bar{H}(p)$ or some better approximation obtained from the (already computed) values of $\bar{H}(p)$. 

The following properties are verified on $\mathcal{A}$:

- if a minimal subsolution $u_1$ is greater than a maximal solution $u_2$ somewhere, then $u_1 \geq u_2$ on $\mathcal{A}$ [4].

- all the subsolutions are differentiable on the Aubry set and their differential coincides on $\mathcal{A}$ [4].

Both properties need another subsolution to be applied. Starting the scheme from a different $u_0$ we can obtain a different solution, while with the $F^+$ map we get a minimal subsolution. In both cases we get an external approximation of $\mathcal{A}$. 
The following optimal paths converge to a connected component of $\mathcal{A}$, \cite{3},

\begin{equation}
\begin{cases}
y' = a^*(x) \\ y(0) = x
\end{cases}
\end{equation}

where $a^*(x)$ is the control where the right-hand side of our scheme

\[
F^-(u^n)_{ij} = \min_{a \in \partial B(0,1) \cup \{0\}} \left\{ u^n(x_{ij} + a\Delta t) + ap\Delta t \right\} + \Delta t \sqrt{2(f(x_{ij}) + c)}
\]

attains its minimum. This gives an approximation from inside.
\[ f(x, y) \equiv \cos(2\pi x) + \cos(2\pi y) \text{ on } \mathbb{T}^2 = [0, 1]^2 \]

**Figure 1:** \( f(x, y) \)

**Figure 2:** \( \bar{H}(p) \) (computed)

<table>
<thead>
<tr>
<th>( \Delta x )</th>
<th>0.1</th>
<th>0.05</th>
<th>0.025</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_\infty(\Delta x) )</td>
<td>0.0229</td>
<td>0.0062</td>
<td>9.2614e − 4</td>
</tr>
</tbody>
</table>
For $p = (0, 0)$ $\mathcal{A} \equiv \{(0.5, 0.5)\}$.

Figure 3: $u_1, u_2$

Figure 4: $|D(u_1 - u_2)|$

In Figure 3, we have the minimal subsolution $u_1$ (brown) and the maximal solution $u_2$ (blue).
Figure 5: $u_1, u_2$

Figure 6: $|D(u_1 - u_2)|$

A computed optimal path starting from $x_0 = (1, 1)$.
For $p = (2, 0)$, $\mathcal{A} \equiv \{(x, y) : y = 0.5\}$.

In Figure 8, we have the minimal subsolution $u_1$ (brown) and the maximal solution $u_2$ (blue).
A computed optimal path starting from $x_0 = (1, 1)$. The path reaches $\mathcal{A}$ and the circles on it for ever.
For $p = (2, 2)$ $\mathcal{A}$ is the whole torus.

In Figure 11, we have the minimal subsolution $u_1$ (brown) and the maximal solution $u_2$ (blue).
A computed invariant optimal path.
\[ f(x, y) \equiv \cos(2\pi x) \cos(2\pi y) \text{ on } \mathbb{T}^2 = [0, 1]^2 \]

Figure 14: \( f(x, y) \)

Figure 15: \( \tilde{H}(p) \) (computed)

Figure 15 shows the approximate \( \tilde{H} \) which looks very much like the result obtained in [6, 7].
For $p = (0, 0)$ $\mathcal{A} \equiv \{(0.5, 0) \equiv (0.5, 1), (0.5, 0) \equiv (1, 0.5)\}$.

In Figure 16, we have the minimal subsolution $u_1$ (brown) and the maximal solution $u_2$ (blue).
Some computed optimal paths starting from different points.

Figure 18: $u_1, u_2$

Figure 19: $|D(u_1 - u_2)|$
For $p = (2, 2)$ the computed $\mathcal{A}$ is $\{y = x - 0.5 \cup y = x + 0.5\}$.

In Figure 20, we have the minimal subsolution $u_1$ (brown) and the maximal solution $u_2$ (blue).
Figure 20: $u_1, u_2$

Figure 21: $|D(u_1 - u_2)|$

A computed optimal path starting from $x_0 = (1, 1)$.  

Test 2
For $p = (8, 0)$ the computed $\mathcal{A}$ is $\{y = 0.25 \cup y = 0.75\}$.

In Figure 22, we have the minimal subsolution $u_1$ (brown) and the maximal solution $u_2$ (blue).
Figure 24: $u_1, u_2$

Figure 25: $|D(u_1 - u_2)|$

Two invariant optimal paths.
In Figure 26, we have two different solutions obtained starting the scheme from different $u_0$. Figure 27 shows their intersection and the two previous optimal paths.
Conclusions

The experiments just seen and many others made (also for discontinue function $f$) show that our scheme is able to compute efficiently the effective Hamiltonian in the cases studied. At the same time the solution of the cell problem gives relevant information about the Aubry set and allows its computation.
References


7. J. Qian, Two approximations for effective Hamiltonians arising from homogenization of Hamilton-Jacobi equations, UCLA, Department of Mathematics, preprint, 2003.