STOCHASTIC ORDER RELATIONS AND LATTICES OF PROBABILITY MEASURES∗

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Abstract. We study various partially ordered spaces of probability measures and we determine which of them are lattices. This has important consequences for optimization problems with stochastic dominance constraints. In particular we show that the space of probability measures on \( \mathbb{R} \) is a lattice under most of the known partial orders, whereas the space of probability measures on \( \mathbb{R}^d \) typically is not. Nevertheless, some subsets of this space, defined by imposing strong conditions on the dependence structure of the measures, are lattices.

Key words. Copula, zonoid, lattice, univariate stochastic orders, multivariate stochastic orders, optimization with stochastic dominance constraints.

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1. Introduction. A partially ordered set is called a lattice if every pair of elements has a supremum and an infimum. A great deal of literature has appeared in the recent decades about ordered sets of probability measures (see for instance [34] and [25] for the state of the art), but surprisingly little attention has been given to the lattice structure of these sets. To the best of our knowledge the only exceptions are [19], [17], [7], [11], and [20, 21].

Lattice structures of ordered sets of probability measures have important implications for optimization problems where ordered sets of probability measures occur as constraint sets. As a concrete example of an application where a lattice structure is very helpful we consider optimization problems with stochastic dominance constraints as considered e.g. in [5]. There optimization problems of the form

\[
\max f(X)
\]

subject to \( X \leq Y_i, \quad i = 1, \ldots, n, \) \( X \in C \) (1.2)

are considered, where \( f \) is some real valued functional, \( C \) is a set of random variables, and \( \leq \) is some stochastic order relation. If the stochastic order relation \( \leq \) leads to a lattice, then the problem with multiple stochastic dominance constraints is equivalent to the problem

\[
\max f(X)
\]

subject to \( X \leq Y_1 \wedge \ldots \wedge Y_n, \) \( X \in C \)

with only one constraint, which is much more easy to solve. Similar optimization problems can be found in [8] and [16].

In several other fields of research it is also often useful to resort to classes of distributions rather than one single distribution. This happens for instance in robust
Bayesian statistics, where one considers families of prior distributions, for instance the so called $\varepsilon$-contamination classes (see e.g. [2], [3], and [28]).

Whereas in decision theory à la Savage, the decision maker maximizes her expected utility with respect to her subjective probability measure, more recent developments in the field have led to paradigms of choice that involve a whole class of probability measures rather than a single probability. This allows to incorporate in the model the idea of ambiguity (see e.g. [12] and the subsequent literature).

An interesting area where the concepts of robustness and ambiguity coexist is robust control under economic model uncertainty (see for instance [14] and [13]).

In mathematical finance, classes of equivalent martingale measures occur when dealing with incomplete markets (see e.g. [18]).

In all these situations it may be useful to compute bounds with respect to some order. That is, given a class $C$ of distributions, it may be interesting to find, among all the distributions that are larger than all distributions in $C$, the smallest one, where of course larger and smaller refer to some pre-specified partial order. For the above problem to be well defined it is necessary to have a lattice structure on the space of distributions.

In this paper we will try to study this issue in a more systematic way. It will turn out that the space of probability measures on $\mathbb{R}$ (or some suitable subsets of it) is a lattice when endowed with most of the well known stochastic orders like usual stochastic order, convex order, dispersion order, hazard rate order etc., whereas the space of probability measures on $\mathbb{R}^d$ is in general not a lattice. In order to obtain a lattice structure for sets of probability measures on $\mathbb{R}^d$ we need to put severe restrictions on their dependence structure.

The paper is organized as follows. Section 2 contains some preliminary definition and results, Section 3 is devoted to the space of probability measures on $\mathbb{R}$, and Section 4 deals with the space of probability measures on $\mathbb{R}^d$. Section 5 studies in details the properties of a special order.

2. Preliminaries. In this section we will introduce notation and all the definitions used in the rest of the paper.

2.1. Orders and lattices. We first recall the basic definitions of the theory of lattices.

**Definition 2.1.** Let $(\mathcal{X}, \leq_\ast)$ be an ordered set. For $x, y \in \mathcal{X}$ let $U(x, y) = \{ z \in \mathcal{X} : x \leq_\ast z, y \leq_\ast z \}$. If $U(x, y)$ has a smallest element $\tilde{z}$ such that $\tilde{z} \leq_\ast z$ for all $z \in U(x, y)$ then $\tilde{z}$ is called the supremum of $x$ and $y$, denoted by $\tilde{z} = x \lor_\ast y = \sup \{ x, y \}$. Similarly, if there is a unique largest element $z'$ smaller than $x$ and $y$, then this is called the infimum, denoted by $z' = x \land_\ast y = \inf \{ x, y \}$.

If $x \lor_\ast y$ and $x \land_\ast y$ exist for all $x, y \in \mathcal{X}$, then $(\mathcal{X}, \leq_\ast)$ is called a lattice.

A subset $Z \subset \mathcal{X}$ of a lattice is called a sublattice, if $x, y \in Z$ implies $x \lor_\ast y \in Z$ and $x \land_\ast y \in Z$. Notice that $(Z, \leq_\ast)$ can be a lattice in its own right without being a sublattice.

For properties of lattices the reader is referred to [4] and [1].

The following lattice and some of its sublattices will be used quite frequently in the following. Let $S$ be an arbitrary set and let $\mathcal{X}$ be the set of all functions $f : S \rightarrow \mathbb{R}$, endowed with the pointwise order

$$f \leq g \text{ if } f(s) \leq g(s) \text{ for all } s \in S.$$
This is obviously a lattice with
\[ f \lor g(\cdot) = \max\{f(\cdot), g(\cdot)\} \quad \text{and} \quad f \land g(\cdot) = \min\{f(\cdot), g(\cdot)\}. \tag{2.1} \]

Remark 1. Consider the lattices induced by the two orders \( \preceq_1, \preceq_2 \) on the same space \( X \). Let \( \preceq_1 \) be stronger than \( \preceq_2 \), namely, for all \( x, y \in X \), let \( x \preceq_1 y \) imply \( x \preceq_2 y \). Then \( x \land_1 y \preceq_2 x \land_2 y \), and \( x \lor_2 y \preceq_2 x \lor_1 y \). Therefore comparability of the orders induces comparability of the suprema and infima (with respect to the weaker order).

The following order \( \leq_1 \) will be of interest in the sequel. Let \( S = [a, b] \subset \mathbb{R} \) be a finite interval and let \( \text{BV}(S) \) be the set of functions \( F : S \to \mathbb{R} \), which are of bounded variation. Endow \( \text{BV}(S) \) with the following relation \( \leq_1 \).

For \( F, G \in \text{BV}(S) \), \( F \leq_1 G \) if \( s \mapsto G(s) - F(s) \) is increasing.

Properties of \( \leq_1 \) will be studied in Section 5.

2.2. Probability. Next, we will collect some basic notations from probability that are needed in the sequel.

Definition 2.2. Given a topological space \( \mathcal{Y} \), \( \mathcal{B}(\mathcal{Y}) \) will indicate its Borel \( \sigma \)-field, \( \mathcal{M}(\mathcal{Y}) \) will be the set of \( \sigma \)-additive probability measures on \( (\mathcal{Y}, \mathcal{B}(\mathcal{Y})) \), and for some measure \( \mu \) on \( (\mathcal{Y}, \mathcal{B}(\mathcal{Y})) \), \( \mathcal{M}^\mu(\mathcal{Y}) \subset \mathcal{M}(\mathcal{Y}) \) will be the set of probability measures dominated by the measure \( \mu \).

When \( \mathcal{Y} \) is a linear space, then \( \mathcal{M}^*(\mathcal{Y}) \subset \mathcal{M}(\mathcal{Y}) \) will be the set of probability measures with finite expectation, and \( \mathcal{M}_a(\mathcal{Y}) \subset \mathcal{M}^*(\mathcal{Y}) \) will be the set of probability measures with finite expectation equal to \( a \).

We denote by \( \delta_x \) the degenerate probability measure at \( x \).

Definition 2.3. Given a probability measure \( P \in \mathcal{M}(\mathbb{R}) \), define \( F_P \) the associated distribution function, and \( \bar{F}_P \) the associated survival function, i.e.

\[
F_P(x) = P((-\infty, x]), \quad \bar{F}_P(x) = P((x, \infty)) = 1 - F_P(x).
\]

Define the quantile function
\[
F_P^{-1}(u) = \sup\{x : F_P(x) \leq u\}, \quad 0 < u < 1.
\]

When \( P \in \mathcal{M}^\mu(\mathbb{R}) \), define \( f_P \) the associated density function, and \( r_P \) the associated hazard rate function, i.e.

\[
f_P(x) = \frac{dP}{d\mu}(x), \quad \mu\text{-a.s.,}
\]
\[
r_P(x) = \frac{f_P(x)}{\bar{F}_P(x)}, \quad \mu\text{-a.s.}
\]

Given \( P \in \mathcal{M}^*(\mathbb{R}) \), let \( \bar{F}_P^{(2)} \) be the associated integrated survival function,

\[
\bar{F}_P^{(2)}(x) = \int_x^\infty F_P(t) \, dt.
\]

For \( P \in \mathcal{M}^*(\mathbb{R}_+) \) we call \( m_P \) the associated mean residual life function, i.e.

\[
m_P(x) = \frac{\bar{F}_P^{(2)}(x)}{\bar{F}_P(x)}.
\]

The name is due to the fact that, if the nonnegative random variable \( X \) has law \( P \), then \( m_P(x) = E[X - x|X > x] \).
2.3. Univariate stochastic orders. The following definitions of stochastic orders can be found e.g. in [34] or [25].

Definition 2.4. Given \( P, Q \in \mathcal{M}(\mathbb{R}) \) we define

\[
P \preceq_{\text{st}} Q \quad \text{if} \quad \int \phi \, dP \leq \int \phi \, dQ \quad \text{for all increasing } \phi,
\]

\[
P \preceq_{\text{disp}} Q \quad \text{if} \quad F_{P}^{-1}(t) - F_{P}^{-1}(s) \leq F_{Q}^{-1}(t) - F_{Q}^{-1}(s) \quad \text{for all } 0 < s < t < 1,
\]

\[
P \preceq_{\text{hr}} Q \quad \text{if} \quad t \mapsto \frac{F_{Q}(t)}{F_{P}(t)} \text{ is increasing}.
\]

Given \( P, Q \in \mathcal{M}^{*}(\mathbb{R}) \) we define

\[
P \preceq_{\text{cx}} Q \quad \text{if} \quad \int \phi \, dP \leq \int \phi \, dQ \quad \text{for all convex } \phi,
\]

\[
P \preceq_{\text{icx}} Q \quad \text{if} \quad \int \phi \, dP \leq \int \phi \, dQ \quad \text{for all increasing convex } \phi.
\]

Given \( P, Q \in \mathcal{M}^{*}(\mathbb{R}+) \) we define

\[
P \preceq_{\text{mri}} Q \quad \text{if} \quad m_{P}(t) \leq m_{Q}(t) \quad \text{for all } t \in \mathbb{R}+.
\]

Given \( P, Q \in \mathcal{M}^{\mu}(\mathbb{R}) \) we define

\[
P \preceq_{\text{lr}} Q \quad \text{if} \quad f_{P}(t)f_{Q}(s) \leq f_{P}(s)f_{Q}(t) \quad \text{for all } s \leq t.
\]

Remark 2. In all the integral orders (\( \preceq_{\text{st}}, \preceq_{\text{cx}}, \preceq_{\text{icx}} \)) the defining inequality is assumed to hold whenever the expectations exist.

Remark 3. If \( P, Q \in \mathcal{M}^{\mu}(\mathbb{R}) \), then \( P \preceq_{\text{hr}} Q \) iff \( r_{P}(x) \geq r_{Q}(x) \) for all \( x \in \mathbb{R} \). If furthermore \( f_{P} > 0 \), then \( P \preceq_{\text{hr}} Q \) iff \( f_{Q}/f_{P} \) is increasing.

Remark 4. Given a random variable \( X \) with law \( P \), and a number \( a \in \mathbb{R} \), we denote by \( P_{a} \) the law of \( X + a \). Then we have

\[
P \preceq_{\text{disp}} Q \quad \text{iff} \quad P_{a} \preceq_{\text{disp}} Q \quad \text{for all } a \in \mathbb{R}.
\]

Therefore the relation \( \preceq_{\text{disp}} \) is not antisymmetric, which implies that it is not a partial order on \( \mathcal{M}(\mathbb{R}) \). It is a partial order on the quotient space \( (\mathcal{M}(\mathbb{R})/_{\sim}) \) with respect to the relation

\[
P \sim Q \quad \text{iff} \quad Q = P_{a} \quad \text{for some } a \in \mathbb{R}.
\]

2.4. Multivariate stochastic orders. Definition 2.5. Any function \( C : [0,1]^{d} \to [0,1] \) which is (the restriction of) a \( d \)-variate distribution function with uniform marginals on \([0,1]\) is called a copula.

Lemma 2.6 ([35]). Let \( P \in \mathcal{M}(\mathbb{R}^{d}) \). For \( i \in \{ 1, \ldots, d \} \) let \( P_{i} \in \mathcal{M}(\mathbb{R}) \) be the \( i \)-th unidimensional marginal of \( P \) (i.e. \( P_{i}(A) = P(\mathbb{R} \times \cdots \times \mathbb{R} \times A \times \mathbb{R} \cdots \times \mathbb{R}) \)).

Then there exists a copula \( C_{P} \) such that

\[
P(\times_{i=1}^{d}(-\infty, x_{i}]) = C_{P}(P_{1}((-\infty, x_{1}]), \ldots, P_{d}((-\infty, x_{d}))).
\]

For the properties of copulae the reader is referred to [32], [15], or [26]. For the properties of copulae of probability measures on more general product spaces, see [31].
**Definition 2.7.** Let \( \mathcal{M}^{(C)}(\mathbb{R}^d) \) be the set of probability measures with a common copula \( C \), and let \( \mathcal{M}_a(\mathbb{R}^d) \) be the set of probability measure with expectation equal to \( a \).

**Definition 2.8.** A random variable \( X \) is stochastically increasing in the random vector \( Y \) (denoted by \( X \uparrow_{st} Y \)) if for all \( s \leq t \) we have \( \mathcal{L}(X|Y=s) \leq_{st} \mathcal{L}(X|Y=t) \), where \( \mathcal{L}(X|A) \) is the conditional law of \( X \) given \( A \).

**Definition 2.9.** ([24]) A random vector \( \mathbf{X} = (X_1, \ldots, X_d) \) is said to be conditionally increasing (CI), if

\[
X_i \uparrow_{st} (X_j, \quad j \in J) \quad \text{for all } J \subset \{1, \ldots, d\} \text{ and } i \notin J.
\]

A copula is called conditionally increasing if it is the copula of the distribution of a CI random vector.

**Definition 2.10.** A function \( \phi : \mathbb{R}^d \to \mathbb{R} \) is called supermodular if

\[
\phi(x) + \phi(y) \leq \phi(x \lor y) + \phi(x \land y) \quad \text{for all } x, y,
\]

where \( \mathbb{R}^d \) is endowed with the usual componentwise order and the corresponding lattice structure.

A function \( \phi : \mathbb{R}^d \to \mathbb{R} \) is called directionally convex if for all \( x_i \in \mathbb{R}^d, i = 1, 2, 3, 4 \), such that \( x_1 \leq x_2 \leq x_4, x_1 \leq x_3 \leq x_4 \) and \( x_1 + x_4 = x_2 + x_3 \),

\[
\phi(x_2) + \phi(x_3) \leq \phi(x_1) + \phi(x_4).
\]

Notice that a function is directionally convex, if it is supermodular and convex in each variable, when the others are held fixed.

For the following definitions of stochastic orders the reader is referred again to [34] or [25].

**Definition 2.11.** Given \( P, Q \in \mathcal{M}(\mathbb{R}^d) \) we define

\[
P \leq_{st} Q \quad \text{if} \quad \int \phi \ dP \leq \int \phi \ dQ \quad \text{for all increasing } \phi,
\]

\[
P \leq_{cx} Q \quad \text{if} \quad \int \phi \ dP \leq \int \phi \ dQ \quad \text{for all convex } \phi,
\]

\[
P \leq_{sm} Q \quad \text{if} \quad \int \phi \ dP \leq \int \phi \ dQ \quad \text{for all supermodular } \phi,
\]

\[
P \leq_{dcx} Q \quad \text{if} \quad \int \phi \ dP \leq \int \phi \ dQ \quad \text{for all directionally convex } \phi,
\]

\[
P \leq_{lcx} Q \quad \text{if} \quad \int \phi \ dP \leq \int \phi \ dQ, \quad \text{for all } \phi \text{ such that}
\]

\[
\phi(x) = \psi(\ell(x)) \quad \text{with } \psi : \mathbb{R} \to \mathbb{R} \text{ convex and } \ell : \mathbb{R}^d \to \mathbb{R} \text{ linear},
\]

\[
P \leq_{pdx} Q \quad \text{if} \quad \int \phi \ dP \leq \int \phi \ dQ, \quad \text{for all } \phi \text{ such that}
\]

\[
\phi(x) = \psi(\ell(x)) \quad \text{with } \psi : \mathbb{R} \to \mathbb{R} \text{ convex and } \ell : \mathbb{R}^d \to \mathbb{R} \text{ linear and increasing},
\]

\[
P \leq_{lo} Q \quad \text{if} \quad P \left( x \overset{d}{=} (\infty, \infty) \right) \leq Q \left( x \overset{d}{=} (\infty, \infty) \right) \quad \text{for all } (x_1, \ldots, x_d) \in \mathbb{R}^d,
\]

\[
P \leq_{wo} Q \quad \text{if} \quad P \left( x \overset{d}{=} ((1, \infty) \right) \leq Q \left( x \overset{d}{=} ((1, \infty) \right) \quad \text{for all } (x_1, \ldots, x_d) \in \mathbb{R}^d.
\]
3. Lattices of measures on $\mathbb{R}$. Now we will investigate whether the orders defined in Subsection 2.3 lead to a lattice structure. In case of $\leq_{st}$ this is easy. The well known fact that $P \leq_{st} Q$ if and only if $F_P(x) \leq F_Q(x)$ for all $x \in \mathbb{R}$ immediately implies the following result.

**Theorem 3.1.** The ordered set $(\mathcal{M}(\mathbb{R}), \leq_{st})$ is a lattice with

$$F_{P \wedge_{st} Q} = \min\{F_P, F_Q\} \quad \text{and} \quad F_{P \vee_{st} Q} = \max\{F_P, F_Q\}.$$ 

In the next result we use the notation

$$\text{vex}(f)(x) = \sup\{g(x) : g \text{ is convex and } g(y) \leq f(y) \text{ for all } y \in \mathbb{R}\}.$$

for the convex hull operator, yielding the largest convex function smaller than a given one.

**Theorem 3.2.** The ordered set $(\mathcal{M}^*(\mathbb{R}), \leq_{icx})$ is a lattice with

$$F_{P \wedge_{icx} Q} = \text{vex}(\min\{F_P^{(2)}(x), F_Q^{(2)}(x)\}),$$

$$F_{P \vee_{icx} Q} = \max\{F_P^{(2)}(x), F_Q^{(2)}(x)\}.$$ 

**Proof.** It is well known that increasing convex order can be characterized by pointwise comparison of the integrated survival functions, i.e. $P \leq_{icx} Q$ holds if and only if

$$F_P^{(2)}(x) = \int_x^\infty F_P(t) \, dt \leq \int_x^\infty F_Q(t) \, dt = F_Q^{(2)}(x)$$

for all $x$. Denote by $\mathcal{F}^{(2)}(\mathbb{R})$ the class of all integrated survival functions. $\mathcal{F}^{(2)}(\mathbb{R})$ contains all functions $f$ that are continuous, decreasing, convex, and satisfy $\lim_{x \to -\infty} f(x) - x = a$ for some $a \in \mathbb{R}$, and $\lim_{x \to -\infty} f(x) = 0$, see e.g. Theorem 1.5.10 in [25]. Therefore the pointwise maximum of two such functions $f$ and $g$ again is such a function, and clearly the smallest integrated survival function larger than $f$ and $g$. The pointwise minimum of two such functions is not necessarily convex, but there is always a largest integrated survival function $h \in \mathcal{F}^{(2)}(\mathbb{R})$ smaller than $f$ and $g$, namely $h = \text{vex}(\min\{f, g\})$. \(\Box\)

The ordered set $(\mathcal{M}^*(\mathbb{R}), \leq_{icx})$ is not a lattice, as $P \leq_{icx} Q$ can only hold for distributions with the same mean. Therefore only the set $(\mathcal{M}^*(\mathbb{R}), \leq_{icx})$ containing all distributions with some fixed mean $a \in \mathbb{R}$ can be a lattice. Since in this case

$$\lim_{x \to -\infty} F_P^{(2)}(x) - x = a,$$

the following result can be proved exactly as 3.2.

**Theorem 3.3.** For all $a \in \mathbb{R}$ the ordered set $(\mathcal{M}_a(\mathbb{R}), \leq_{icx})$ is a lattice with

$$F_{P \wedge_{icx} Q} = \text{vex}(\min\{F_P^{(2)}(x), F_Q^{(2)}(x)\}),$$

$$F_{P \vee_{icx} Q} = \max\{F_P^{(2)}(x), F_Q^{(2)}(x)\}.$$ 

The problem examined in Theorem 3.2 and 3.3 has been studied extensively by [20, 21]. The reader is also referred to [19], [7], and [11].

In the next result we investigate the lattice structure of the mean residual life order $\leq_{mrl}$ for distributions on $\mathbb{R}_+$ with a finite mean.
Theorem 3.4. The ordered set \((\mathcal{M}^*(\mathbb{R}^+_r), \leq_{\text{disp}})\) is a lattice with
\[
\overline{F}_{P \land_{\text{disp}} Q}(x) = \exp \left( - \int_0^x \frac{1}{\min\{m_P(t), m_Q(t)\}} \, dt \right) \cdot \frac{\min\{m_P(0), m_Q(0)\}}{\min\{m_P(x), m_Q(x)\}},
\]
\[
\overline{F}_{P \lor_{\text{disp}} Q}(x) = \exp \left( - \int_0^x \frac{1}{\max\{m_P(t), m_Q(t)\}} \, dt \right) \cdot \frac{\max\{m_P(0), m_Q(0)\}}{\max\{m_P(x), m_Q(x)\}}.
\]

Proof. A function \(m : \mathbb{R}^+_r \to \mathbb{R}\) is a mean residual life function of some probability measure \(P\) on \(\mathbb{R}^+_r\), if and only if it has the following properties: It is non-negative, right-continuous, and such that \(t \mapsto m(t) + t\) is increasing, and if there exists \(t_0\) such that \(m(t_0) = 0\), then \(m(t) = 0\) for all \(t > t_0\). If such a \(t_0\) does not exist, then
\[
\int_0^\infty \frac{1}{m(t)} \, dt = \infty
\]
(see [36], [33, 34]). The class of these functions is closed under pointwise minimum and maximum. As the survival function \(\overline{F}_P\) is uniquely determined by the mean residual life function \(m_P\) via
\[
\overline{F}_P(x) = \exp \left( - \int_0^x \frac{1}{m_P(t)} \, dt \right) \cdot \frac{m_P(0)}{m_P(x)},
\]
the given representation for the survival functions of \(P \lor_{\text{disp}} Q\) and \(P \land_{\text{disp}} Q\) follows. \(\Box\)

The following theorems will make use of properties of the order \(\leq_{\text{disp}}\) that are proved in Section 5.

Theorem 3.5. The ordered set \(((\mathcal{M}(\mathbb{R}^+_r), \sim), \leq_{\text{disp}})\) is a lattice with
\[
F_{P \land_{\text{disp}} Q} = \left( F_{P^{-1}} \land F_{Q^{-1}} \right)^{-1},
\]
\[
F_{P \lor_{\text{disp}} Q} = \left( F_{P^{-1}} \lor F_{Q^{-1}} \right)^{-1}.
\]

Proof. Notice that \(P \leq_{\text{disp}} Q\), if and only if \(F_{P^{-1}} \leq_{\text{disp}} F_{Q^{-1}}\). Therefore the assertion follows from Lemma 5.2. \(\Box\)

Remark 5. The set \((\mathcal{M}(\mathbb{R}), \leq_{hr})\) is not a lattice. Notice that \(P \leq_{hr} Q\) holds if and only if \(\log(F_P) \leq_{\text{disp}} \log(F_Q)\). Therefore \((\mathcal{M}(\mathbb{R}^+_r), \leq_{hr})\) would be a lattice, if the set of logarithms of survival functions, endowed with \(\leq_{\text{disp}}\) were a lattice. However, whereas \(\log(F_P) \land \log(F_Q)\) is always a logarithm of a survival function, this in not necessarily the case for \(\log(F_P) \lor \log(F_Q)\). In this case it may happen that the limit for \(x \to \infty\) is finite. If for example \(P\) has as support all even numbers and \(Q\) has as support the odd numbers, then \(\log(F_P) \lor \log(F_Q) \equiv 0\), and this obviously is not a logarithm of a survival function of a distribution on \(\mathbb{R}\).

However, the order relation \(\leq_{hr}\) defines a lattice for distributions on the extended real line \(\mathbb{R} \cup \{+\infty\}\), allowing explicit mass on \(+\infty\). Thus the logarithm of the survival function is allowed to have a finite limit as \(x \to \infty\). The function \(f \equiv 0\), for instance, then is the logarithm of the distribution with \(P(\{+\infty\}) = 1\).

Theorem 3.6. The set \((\mathcal{M}(\mathbb{R} \cup \{+\infty\}), \leq_{hr})\) is a lattice with
\[
\overline{F}_{P \land_{hr} Q} = \exp(\log(F_P) \land \log(F_Q)),
\]
\[
\overline{F}_{P \lor_{hr} Q} = \exp(\log(F_P) \lor \log(F_Q)).
\]
Proof. As $P \leq_{hr} Q$ holds if and only if $\log(E_P) \leq \log(E_Q)$, this is an immediate consequence of Lemma 5.1.

Example 1. We will illustrate how the suprema and infima vary with respect to the various orders by comparing two simple discrete distributions, which have the same mean and variance, and therefore are not comparable with respect to any of the mentioned orders.

Let

$$P = \frac{1}{2} \delta_1 + \frac{1}{2} \delta_3, \quad Q = \frac{1}{8} \delta_0 + \frac{3}{4} \delta_2 + \frac{1}{8} \delta_4.$$ 

Then

$$P \wedge_{st} Q = \frac{1}{8} \delta_0 + \frac{3}{8} \delta_1 + \frac{3}{8} \delta_2 + \frac{1}{8} \delta_3,$$

$$P \vee_{st} Q = \frac{1}{8} \delta_1 + \frac{3}{8} \delta_2 + \frac{3}{8} \delta_3 + \frac{1}{8} \delta_4;$$

$$P \wedge_{cx} Q = P \wedge_{icx} Q = \frac{1}{4} \delta_1 + \frac{1}{2} \delta_2 + \frac{1}{4} \delta_3;$$

$$P \vee_{cx} Q = P \vee_{icx} Q = \frac{1}{8} \delta_0 + \frac{3}{8} \delta_4/3 + \frac{3}{8} \delta_8/3 + \frac{1}{8} \delta_4;$$

$$P \wedge_{disp} Q = \delta_1,$$

$$P \vee_{disp} Q = \frac{1}{8} \delta_1 + \frac{3}{8} \delta_3 + \frac{3}{8} \delta_5 + \frac{1}{8} \delta_7;$$

$$P \wedge_{hr} Q = \frac{1}{8} \delta_0 + \frac{7}{16} \delta_1 + \frac{3}{16} \delta_2 + \frac{1}{16} \delta_3,$$

$$P \vee_{hr} Q = \delta_4;$$

$$P \wedge_{mrl} Q = \frac{2}{9} \delta_1 + \frac{5}{9} \delta_2 + \frac{2}{9} \delta_3;$$

$$P \vee_{mrl} Q = \frac{1}{8} \delta_0 + \frac{10}{32} \delta_1 + \frac{9}{32} \delta_2 + \frac{9}{32} \delta_4.$$ 

Notice that in most cases the support is contained in the union of the supports of $P$ and $Q$, whereas

$$\text{supp}(P \vee_{cx} Q) \nsubseteq \text{supp}(P) \cup \text{supp}(Q) =: S = \{0, 1, 2, 3, 4\}.$$ 

Therefore $(M(S), \leq_{cx})$ is not a sublattice of $(M(\mathbb{R}), \leq_{cx})$. Notice, however, that $(M(S), \leq_{cx})$ is still a lattice in its own right. In the lattice $(M(S), \leq_{cx})$ the supremum of $P$ and $Q$ from the above example is given by

$$P \vee_{cx} Q = \frac{1}{8} \delta_0 + \frac{1}{4} \delta_1 + \frac{1}{4} \delta_2 + \frac{1}{4} \delta_3 + \frac{1}{8} \delta_4.$$ 

Theorem 3.7.

(a) For any $P, Q \in M(\mathbb{R})$ we have

$$\text{supp}(P \vee_{st} Q) \subseteq \text{supp}(P) \cup \text{supp}(Q),$$

$$\text{supp}(P \wedge_{st} Q) \subseteq \text{supp}(P) \cup \text{supp}(Q).$$

(b) For any $P, Q \in M_{a}(\mathbb{R})$ we have

$$\text{supp}(P \vee_{cx} Q) \subseteq \text{conv}(\text{supp}(P) \cup \text{supp}(Q)),$$

$$\text{supp}(P \wedge_{cx} Q) \subseteq \text{supp}(P) \cup \text{supp}(Q).$$
(c) For any \( P, Q \in \mathcal{M}^+(\mathbb{R}_+) \) we have
\[
\text{supp}(P \lor_{\text{st}} Q) \subseteq \text{supp}(P) \cup \text{supp}(Q),
\]
\[
\text{supp}(P \land_{\text{st}} Q) \subseteq \text{supp}(P) \cup \text{supp}(Q).
\]

Proof. We show the case \( \land_{\text{ex}} \). The other cases are similar. Let \( K = F_{P \land_{\text{ex}} Q} \) and fix \( x \notin \text{supp}(P) \cup \text{supp}(Q) \). Then there is a neighborhood \( U_x \) of \( x \) where \( F^{(2)}_P \) and \( F^{(2)}_Q \) are affine. Hence \( \overline{K}^{(2)} = \text{vex}(\min(F^{(2)}_P, F^{(2)}_Q)) \) is also affine on \( U_x \) and thus \( x \notin \text{supp}(P \land_{\text{ex}} Q) \). □

As a consequence of Theorem 3.7 we get the following result. The proof is omitted.

**Theorem 3.8.**

(a) For any measurable subset \( S \subset \mathbb{R} \) the partially ordered set \( (\mathcal{M}(S), \leq_{\text{st}}) \) is a sublattice of \( (\mathcal{M}(\mathbb{R}), \leq_{\text{st}}) \).

(b) For any convex subset \( S \subset \mathbb{R} \) and any \( a \in S \) the partially ordered set \( (\mathcal{M}_a(S), \leq_{\text{ex}}) \) is a sublattice of \( (\mathcal{M}_a(\mathbb{R}), \leq_{\text{ex}}) \).

(c) For any measurable subset \( S \subset \mathbb{R}_+ \) the partially ordered set \( (\mathcal{M}^+(S), \leq_{\text{st}}) \) is a sublattice of \( (\mathcal{M}^+(\mathbb{R}_+), \leq_{\text{st}}) \).

**Remark 6.** It is well known that on \( \mathcal{M}(\mathbb{R}) \) we have
\[
\leq_{\text{lr}} \subset \leq_{\text{hr}} \subset \leq_{\text{st}} \subset \leq_{\text{ex}},
\]
and on \( \mathcal{M}_a(\mathbb{R}) \) we have
\[
\leq_{\text{disp}} \subset \leq_{\text{ex}}.
\]

For some of the orders examined in this section we have a stronger comparability result than the one stated in Remark 1, that is the infima and suprema are comparable with respect to the stronger order, as the following proposition shows.

**Proposition 3.9.**

(a) \( P \lor_{\text{ex}} Q \leq_{\text{st}} P \lor_{\text{st}} Q \).

(b) \( P \land_{\text{st}} Q \leq_{\text{st}} P \land_{\text{ex}} Q \).

(c) \( P \lor_{\text{st}} Q \leq_{\text{lr}} P \lor_{\text{hr}} Q \).

(d) \( P \land_{\text{hr}} Q \leq_{\text{lr}} P \land_{\text{st}} Q \).

Proof.

(a) Define \( H = F_{P \lor_{\text{ex}} Q} \). Then \( \overline{H}^{(2)}(x) = \max(\overline{F}^{(2)}_P(x), \overline{F}^{(2)}_Q(x)) \). As \( \overline{H}(x) = -d^+ \overline{H}^{(2)}(x)/dx \), we have that \( \overline{H}(x) \) equals either \( \overline{F}_P(x) \) or \( \overline{F}_Q(x) \), and therefore
\[
\overline{H}(x) = \max(\overline{F}_P(x), \overline{F}_Q(x)) = \overline{F}_{P \lor_{\text{st}} Q}(x),
\]
which implies the desired result.

(b) Define \( K = F_{P \land_{\text{ex}} Q} \). Then \( \overline{K}^{(2)}(x) = \text{vex}(\min(\overline{F}^{(2)}_P(x), \overline{F}^{(2)}_Q(x))) \). For fixed \( x \) we have that \( \overline{K}(x) \) either equals \( \overline{F}_P(x) \) or \( \overline{F}_Q(x) \), or there exists a largest interval \([a, b]\) containing \( x \) on which \( \overline{K}^{(2)}(x) \) is affine, hence \( \overline{K} \) is constant. Since in the latter case \( \overline{K}^{(2)}(x) \) equals either \( \overline{F}_P^{(2)}(x) \) or \( \overline{F}_Q^{(2)}(x) \) in the point \( a \) and it is smaller than both these functions between \( a \) and \( b \), we then have
\[
\overline{K}(x) = \overline{K}(a) \geq \min(\overline{F}_P(a), \overline{F}_Q(a)) \geq \min(\overline{F}_P(x), \overline{F}_Q(x)) = \overline{F}_{P \land_{\text{st}} Q}(x),
\]
which implies the desired result.
(c) Define $R = P \lor_{st} Q$. Then $\overline{F}_R(x) = \max(\overline{F}_P(x), \overline{F}_Q(x))$. Let $\mu$ be a dominating measure of $P$ and $Q$ and let $r_P$ and $r_Q$ be the corresponding hazard rate functions. Then $r_R(x)$ equals either $r_P(x)$ or $r_Q(x)$ $\mu$-a.s.. Therefore

$$r_R(x) \leq \max\{r_P(x), r_Q(x)\} = r_{P_{\lor_{st}Q}}(x) \mu - a.s.$$ and therefore $R \leq_{hr} P \lor_{hr} Q$. The proof of (d) is similar.

\[ \square \]

**Theorem 3.10.** The following sets of probability measures on $\mathbb{R}$ are lattices, if they are endowed with the order $\leq_{hr}$:

(i) the set of all probability measures with a common finite support,
(ii) the set of all probability measures on a bounded interval $(a,b)$ having a strictly positive Lebesgue density $f$ such that $\log f$ is of locally bounded variation.

**Proof.**

(i) In general $P \leq_{hr} Q$ holds if they both have densities $f_P, f_Q$ with respect to some dominating measure such that $\log f_P \leq_1 \log f_Q$. Therefore, if both probability measures have a common finite support $\{x_1, \ldots, x_n\}$ with $x_1 < \ldots < x_n$ then we get, by (5.2) and (5.3),

$$\frac{(f_{P_{\lor_{st}Q}}(x_{i+1}))}{(f_{P_{\lor_{st}Q}}(x_i))} = \max \left\{ \frac{f_Q(x_{i+1})}{f_Q(x_i)}, \frac{f_P(x_{i+1})}{f_P(x_i)} \right\} \quad (3.1)$$

and

$$\frac{(f_{P \land_{st}Q})(x_{i+1})}{(f_{P \land_{st}Q})(x_i)} = \min \left\{ \frac{f_Q(x_{i+1})}{f_Q(x_i)}, \frac{f_P(x_{i+1})}{f_P(x_i)} \right\} \quad (3.2)$$

(ii) We have

$$\log(f_{P_{\lor_{st}Q}}) = c(\log f_P \lor_1 \log f_Q)$$

where $c$ is such that \(\int_{\mathbb{R}}^b (f_{P_{\lor_{st}Q}})(x) \, dx = 1\). Similarly,

$$\log(f_{P \land_{st}Q}) = c'(\log f_P \land_1 \log f_Q)$$

where $c'$ is such that \(\int_{\mathbb{R}}^b (f_{P \land_{st}Q})(x) \, dx = 1\).

\[ \square \]

**Remark 7.** For a probability measure $P$ with values in $\{0, 1, \ldots, N\}$ the so called equilibrium rate function $e_P$ is defined as

$$e_P(n) := \frac{P(\{n-1\})}{P(\{n\})}, \quad n = 1, \ldots, N.$$ 

It is well known that $e_P$ uniquely determines $P$ and that $P \leq_{hr} Q$ holds if and only if $e_P(n) \geq e_Q(n)$ for all $n = 1, \ldots, N$, see e.g. [34], p. 435ff. Thus the proof of part (i) of Theorem 3.10 is not surprising. It just states that

$$e_{P_{\lor_{st}Q}}(n) = \min\{e_P(n), e_Q(n)\} \quad \text{and} \quad e_{P \land_{st}Q}(n) = \max\{e_P(n), e_Q(n)\}.$$ 

**Example 2.** (a) Let $S = \{1, 2, 3\}$ and let

$$f_P(1) = f_P(3) = \frac{1}{4}, \quad f_P(2) = \frac{1}{2}.$$
and let
\[ f_Q(1) = f_Q(2) = f_Q(3) = \frac{1}{3}. \]

It follows from (3.1) that
\[ f_{P \vee Q}(3) = f_{P \vee Q}(2) = 2f_{P \vee Q}(1). \]

Normalization yields
\[ f_{P \vee Q}(3) = f_{P \vee Q}(2) = \frac{2}{5} \quad \text{and} \quad f_{P \vee Q}(1) = \frac{1}{5}. \]

Similarly, one obtains
\[ f_{P \wedge Q}(3) = f_{P \wedge Q}(2) = \frac{2}{5} \quad \text{and} \quad f_{P \wedge Q}(1) = \frac{1}{5}. \]

(b) Let \( S = [0, 1] \) and consider the following example for Lebesgue-densities:
\[ f_P(x) = \begin{cases} 
4x, & x \leq \frac{1}{2}, \\
4 - 4x, & x > \frac{1}{2}
\end{cases} \quad \text{and} \quad f_Q(x) = 1, \quad 0 \leq x \leq 1. \]

From (5.1) it follows that
\[ \frac{d}{dx} \log f_{P \vee Q}(x) = \begin{cases} 
\frac{1}{x}, & 0 < x \leq \frac{1}{2}, \\
0, & x > \frac{1}{2}
\end{cases} \]

Normalization yields
\[ f_{P \vee Q}(x) = \begin{cases} 
\frac{4x}{5}, & x \leq \frac{1}{2}, \\
\frac{4}{5}, & x > \frac{1}{2}
\end{cases}. \]

Similarly, one obtains
\[ f_{P \wedge Q}(x) = \begin{cases} 
\frac{4}{5}, & x \leq \frac{1}{2}, \\
\frac{8}{5}(1 - x), & x > \frac{1}{2}
\end{cases}. \]

(c) The set of all probability measures with support \( \mathbb{N}_0 \), endowed with the order \( \leq_{lr} \), is not a lattice. To see this, let
\[ f_P(n) = \frac{1}{2(k+1)}, \quad n = 2k, 2k + 1, \quad k = 1, 2, ... \]

and
\[ f_Q(n) = \frac{1}{2(k+1)}, \quad n = 2k - 1, 2k, \quad k = 1, 2, ... \]

A density \( h \) with \( h/f_P \) increasing and \( h/f_Q \) increasing would have to be increasing on the whole of \( \mathbb{N}_0 \) which is impossible. Therefore the set \( \{P, Q\} \) has no upper bound with respect to \( \leq_{lr} \). A very similar argument can be used to show that the set of all probability measures on \( \mathbb{R} \) having Lebesgue densities, endowed with the order \( \leq_{lr} \), is not a lattice. Let
\[ f_P(x) = \frac{1}{2(k+3)}, \quad 2k \leq |x| < 2k + 2, \quad k = 0, 1, 2, ... \]
and
\[ f_Q(x) = \begin{cases} 
1/4, & |x| < 1, \\
(1/2)^{k+3}, & 2k - 1 \leq |x| < 2k + 1, \quad k = 1, 2, \ldots
\end{cases} \]

Then a density \( h \) with \( h/f_P \) increasing and \( h/f_Q \) increasing would have to be increasing on the whole of \( \mathbb{R} \) which is impossible.

4. Lattices of measures on \( \mathbb{R}^d \). In this section we will study the lattice structure of the orders defined in Subsection 2.4.

**Theorem 4.1.** The following ordered sets of probability measures are lattices:

(a) for any copula \( C \), the set \( (\mathcal{M}(C)(\mathbb{R}^d), \leq_{st}) \),

(b) for any conditionally increasing copula \( C \), and for all \( a \in \mathbb{R}^d \), the set \( (\mathcal{M}_a(C)(\mathbb{R}^d), \leq_{dcx}) \),

(c) for any conditionally increasing copula \( C \), and for all \( a \in \mathbb{R}^d \), the set \( (\mathcal{M}_a(C)(\mathbb{R}^d), \leq_{plcx}) \).

The following lemmas will be needed.

**Lemma 4.2** ([29, 30]). Let \( P, Q \in \mathcal{M}(C)(\mathbb{R}^d) \). Then \( P \leq_{st} Q \) iff, for \( i \in \{1, \ldots, d\} \), \( P_i \leq_{st} Q_i \).

**Lemma 4.3** ([24]). Let \( C \) be a conditionally increasing copula, and let \( P, Q \in \mathcal{M}(C)(\mathbb{R}^d) \). Then \( P \leq_{dcx} Q \) iff, for \( i \in \{1, \ldots, d\} \), \( P_i \leq_{cx} Q_i \).

**Proof.** [Proof of Theorem 4.1]

(a) By Lemma 4.2 and Theorem 3.1, we obtain that the ordered set \( (\mathcal{M}(C)(\mathbb{R}^d), \leq_{st}) \) is a product of lattices, hence a lattice.

(b) By Lemma 4.3 and by Theorem 3.3, we obtain that, if \( C \) is CI, then the set \( (\mathcal{M}_a(C)(\mathbb{R}^d), \leq_{dcx}) \) is a product of lattices, hence a lattice,

(c) As \( P \leq_{dcx} Q \) implies \( P \leq_{plcx} Q \) which in turn implies \( P_i \leq_{cx} Q_i \) for all marginals, it follows from Lemma 4.3 that the orderings \( \leq_{dcx} \) and \( \leq_{plcx} \) are equivalent on the set of all probability measures with a fixed conditionally increasing copula \( C \). Thus the result follows immediately from part (b).

Example 3. Theorem 4.1 can be helpful for solving some multivariate versions of optimization problems with stochastic ordering constraints of the type described in (1.1). Let \( X = (X_1, \ldots, X_d) \) describe a portfolio of \( d \) risks, corresponding e.g. to different lines of business. Consider a portfolio optimization problem of the following type:

\[
\text{max } f(X) \\
\text{subject to } X \leq_{plcx} Y^{(i)}, \quad i = 1, \ldots, n, \\
C_X = C^+.
\]

Notice that \( X \leq_{plcx} Y \) is equivalent to

\[
\sum_{j=1}^d w_j X_j \leq_{cx} \sum_{j=1}^d w_j Y_j \quad \text{for all } w_1, \ldots, w_d \geq 0,
\]

which can be interpreted as follows: a portfolio consisting of the risks \( (X_1, \ldots, X_d) \) is less risky than a portfolio consisting of the benchmark risks \( (Y_1, \ldots, Y_d) \), for all possible portfolio weights \( w_1, \ldots, w_d \). The assumption that the copula of \( X \) is given
by the upper Fréchet bound $C^+$ (or in other words that the risks are comonotonic) is a quite common assumption in the calculation of risk measures for portfolios, see e.g. [6]. The two main reasons for this are first that comonotonicity typically yields a worst case bound, and second that many risk measures are easy to evaluate in the case of comonotonicity.

It follows from Theorem 4.1 that the problem (4.1) is equivalent to

$$\max \ f(X)$$
$$\text{subject to } X \leq_{plcx} Y^{(1)} \land_{plcx} \cdots \land_{plcx} Y^{(n)}, \quad C_X = C^+,$$

which in turn is equivalent to

$$\max \ f(X)$$
$$\text{subject to } X_j \leq_{cx} Y_j^{(1)} \land_{cx} \cdots \land_{cx} Y_j^{(n)}, \quad j = 1,\ldots,n, \quad C_X = C^+.$$

[17] were the first to recognize that, for $d > 1$, the stochastic order on $\mathbb{R}^d$ does not generate a lattice.

**Proposition 4.4 ([17]).** The ordered set $(\mathcal{M}(\mathbb{R}^d), \leq_{st})$ is not a lattice.

The counterexample that they use in their proof is the following. Consider for $d = 2$ the probability measures

$$P = \frac{1}{2} \delta_{(0,0)} + \frac{1}{2} \delta_{(1,1)}, \quad Q = \frac{1}{2} \delta_{(0,1)} + \frac{1}{2} \delta_{(1,0)}.$$

Given the upper sets

$$A = \{(x_1, x_2) : x_1 \geq 1, x_2 \geq 1\}, \quad B = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0, \max\{x_1, x_2\} \geq 1\},$$

any upper bound (with respect to $\leq_{st}$) $R$ for $P$ and $Q$ has to satisfy

$$R(A) \geq \frac{1}{2}, \quad R(B) = 1.$$

The measures

$$R_1 = \frac{1}{2} \delta_{(0,1)} + \frac{1}{2} \delta_{(1,1)} \quad \text{and} \quad R_2 = \frac{1}{2} \delta_{(1,0)} + \frac{1}{2} \delta_{(1,1)}$$

are upper bounds for $\{P, Q\}$ with respect to $\leq_{st}$. But $\bar{R} \leq_{st} R_1, \bar{R}(A) \geq 1/2$, and $\bar{R}(B) = 1$ imply $\bar{R} = R_1$, therefore $R_1$ is a smallest upper bound. By symmetry it can be shown that $R_2$ is a smallest upper bound with respect to $\leq_{st}$. Therefore no supremum exists for $\{P, Q\}$.

**Remark 8.** Even if (a) of Theorem 4.1 holds, in general two distributions in the set of all distributions on $\mathbb{R}^d$ do not have a supremum with respect to $\leq_{st}$, even if they have a common copula. Consider for instance $\mathcal{N}(0,1) \times \delta_0$ and $\delta_0 \times \mathcal{N}(0,1)$, where $\mathcal{N}(0,1)$ is the standard normal distribution. Notice that any distribution with marginals $1/2 \cdot (\mathcal{N}^+(0,1) + \delta_0)$ is an upper bound with respect to $\leq_{st}$, where we denote by $\mathcal{N}^+(0,1)$ a standard normal distribution conditioned to be positive.

**Proposition 4.5.** For any $a \in \mathbb{R}^d$ the ordered set $(\mathcal{M}_a(\mathbb{R}^d), \leq_{cx})$ is not a lattice.
Proof. Without loss of generality we will consider the case $a = 0$. Any other case can be obtained by translation. Let

$$P = \frac{1}{2}\delta_{(-1,-1)} + \frac{1}{2}\delta_{(1,1)},$$

$$Q = \frac{1}{2}\delta_{(-1,1)} + \frac{1}{2}\delta_{(1,-1)}.$$ 

The measure

$$R = \frac{1}{4}\delta_{(-2,0)} + \frac{1}{4}\delta_{(0,-2)} + \frac{1}{4}\delta_{(2,0)} + \frac{1}{4}\delta_{(0,2)}$$

(4.4)

dominates both $P$ and $Q$ in $(\mathcal{M}_0(\mathbb{R}^d), \leq_{cx})$. This can be seen by using the idea of fusion studied by [9, 10].

In fact $P$ can be obtained from $R$ by fusing $\frac{1}{4}\delta_{(-2,0)} + \frac{1}{4}\delta_{(0,-2)}$ into $\frac{1}{2}\delta_{(-1,-1)}$, and $\frac{1}{4}\delta_{(2,0)} + \frac{1}{4}\delta_{(0,2)}$ into $\frac{1}{2}\delta_{(1,1)}$.

Similarly $Q$ can be obtained from $R$ by fusing $\frac{1}{4}\delta_{(-2,0)} + \frac{1}{4}\delta_{(0,2)}$ into $\frac{1}{2}\delta_{(-1,1)}$, and $\frac{1}{4}\delta_{(2,0)} + \frac{1}{4}\delta_{(0,-2)}$ into $\frac{1}{2}\delta_{(1,1)}$.

Consider now a measure

$$S = \frac{5}{11}\delta_{\left(\frac{3}{2}, \frac{3}{2}\right)} + \frac{3}{11}\delta_{\left(\frac{3}{2}, -4\right)} + \frac{3}{11}\delta_{(-4, \frac{3}{2})}.$$ 

Since any measure with support in the convex hull of

$$\left(\frac{3}{2}, \frac{3}{2}\right), \left(\frac{3}{2}, -4\right), \left(-4, \frac{3}{2}\right)$$

and expectation $(0,0)$ is convexly dominated by $S$ (see [10]), we have that $S$ is an upper bound for $\{P, Q\}$.

On the other hand $R$ and $S$ are not comparable on $(\mathcal{M}_0(\mathbb{R}^d), \leq_{cx})$, since the convex hulls of their supports are not ordered by inclusion (again see [10]).

If $P \lor_{cx} Q$ existed, then it would have to be dominated by both $R$ and $S$, hence its support would have to be contained in the intersection of the convex hulls of the supports of $R$ and $S$ (indicated in grey in figure 1). Assume that this is possible. Then, in order to dominate $P$ the measure $P \lor_{cx} Q$ would have to deposit mass $1/2$ on the segment $B = \left(\frac{1}{2}, \frac{3}{2}\right), \left(\frac{3}{2}, \frac{1}{2}\right)$. In order to dominate $Q$ it would have to deposit mass $1/2$ on the segment $A = (-2,0), (-\frac{1}{2}, \frac{3}{2})$ and mass $1/2$ on the segment $C = (0,-2), (\frac{3}{2}, -\frac{1}{2})$, see figure 1. Since the three segments are disjoint, this leads to a contradiction.

Hence $\{P, Q\}$ have no supremum in $(\mathcal{M}_0(\mathbb{R}^d), \leq_{cx})$. $\Box$

Proposition 4.6. The ordered set $(\mathcal{M}^*(\mathbb{R}^d), \leq_{cx})$ is not a lattice.

In order to prove the above proposition we need the following definition and result, for which the reader is referred to [23].

Definition 4.7. Given a probability measure $P \in \mathcal{M}^*(\mathbb{R}^d)$, we define $\ell(P)$ its lift-zonoid

$$\ell(P) = \text{conv} \left\{ \left( P(B), \int_B x \, P(dx) \right) : B \in \mathcal{B}(\mathbb{R}^d) \right\}.$$

Lemma 4.8 ([22]). For $a \in \mathbb{R}^d$, and for $P, Q \in \mathcal{M}_a(\mathbb{R}^d)$, the following two conditions are equivalent:
(a) $P \preceq_{\text{lex}} Q$,
(b) $\ell(P) \subseteq \ell(Q)$.

Lemma 4.9. The class of zonoids in $\mathbb{R}^{d+1}_+$ having one common vertex in 0 and another one in $(1, \mu)$, ordered by inclusion, is not a lattice.

Proof. Given two sets $A, B \in \mathbb{R}^{d+1}_+$, let $A \oplus B = \{s + t : s \in A, t \in B\}$ be their Minkowski sum. Consider the following zonotopes in $\mathbb{R}^4$:

$$Z_1 = \overline{0, a_1} \oplus \overline{0, a_2} \oplus \overline{0, a_3},$$
$$Z_2 = \overline{0, b_1} \oplus \overline{0, b_2} \oplus \overline{0, b_3},$$
$$Z_3 = \overline{0, c_1} \oplus \overline{0, c_2} \oplus \overline{0, c_3},$$
$$Z_4 = \overline{0, d_1} \oplus \overline{0, d_2} \oplus \overline{0, d_3},$$

where

$$a_1 = \begin{pmatrix} 1 & 2 & 2 & 5 \\ 3 & 9 & 9 & 9 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 & 2 & 5 & 2 \\ 3 & 9 & 9 & 9 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 1 & 5 & 2 & 2 \\ 3 & 9 & 9 & 9 \end{pmatrix},$$
$$b_1 = \begin{pmatrix} 1 & 4 & 4 & 1 \\ 3 & 9 & 9 & 9 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 1 & 4 & 1 & 4 \\ 3 & 9 & 9 & 9 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 1 & 1 & 4 & 4 \\ 3 & 9 & 9 & 9 \end{pmatrix},$$
$$c_1 = \begin{pmatrix} 1 & 3 & 4 & 2 \\ 3 & 9 & 9 & 9 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 9 & 9 & 9 \end{pmatrix}, \quad c_3 = \begin{pmatrix} 1 & 4 & 2 & 3 \\ 3 & 9 & 9 & 9 \end{pmatrix},$$
$$d_1 = \begin{pmatrix} 1 & 3 & 2 & 4 \\ 3 & 9 & 9 & 9 \end{pmatrix}, \quad d_2 = \begin{pmatrix} 1 & 4 & 3 & 2 \\ 3 & 9 & 9 & 9 \end{pmatrix}, \quad d_3 = \begin{pmatrix} 1 & 2 & 4 & 3 \\ 3 & 9 & 9 & 9 \end{pmatrix}.$$
It is not difficult to verify that the zonotopes \( Z_1, Z_2, Z_3, Z_4 \) have one vertex in \( 0 \) and the other in \( 1 := (1, 1, 1) \). Let \( S_2 \) be the simplex

\[
S_2 = \{(x_1, x_2, x_3) : x_1, x_2, x_3 \geq 0, \sum_{j=1}^{3} x_j = 1\}.
\]

Then each of the above zonoids is generated by segments of the type \( 0, (\frac{1}{3}, x_1, x_2, x_3) \), with \( (x_1, x_2, x_3) \in S_2 \).

It is enough to look at the simplex \( S_2 \) to notice that both \( Z_3 \) and \( Z_4 \) are included in \( Z_1 \cap Z_2 \), so they are lower bounds for \( \{Z_1, Z_2\} \). Therefore an infimum of \( \{Z_1, Z_2\} \) would have to contain \( Z_3 \) and \( Z_4 \). We observe that each of the six segments that generate \( Z_3 \) and \( Z_4 \) is extreme for the convex hull of \( Z_3 \) and \( Z_4 \), which coincides with the intersection of \( Z_1 \) and \( Z_2 \), see figure 2. Therefore any zonoid that includes both \( Z_3 \) and \( Z_4 \) would have to contain all six generating segments among its generators, but then it would not have a vertex in \( 1 \). This proves that the set \( \{Z_1, Z_2\} \) has no infimum. \( \square \)

**Proof.** [Proof of Proposition 4.6] It is enough to combine Lemma 4.8 and Lemma 4.9. \( \square \)

**Proposition 4.10.** The ordered set \( (\mathcal{M}(\mathbb{R}^d), \leq_{sm}) \) is not a lattice.

**Proof.** Let

\[
P = \frac{1}{3} (\delta_{(2,1)} + \delta_{(1,3)} + \delta_{(3,2)}),
\]

\[
Q = \frac{1}{3} (\delta_{(1,2)} + \delta_{(3,1)} + \delta_{(2,3)}).
\]
The measures
\[
R = \frac{1}{3}(\delta_{(1,1)} + \delta_{(3,2)} + \delta_{(2,3)}), \\
S = \frac{1}{3}(\delta_{(1,2)} + \delta_{(2,1)} + \delta_{(3,3)})
\]
are upper bounds for \(\{P, Q\}\). To prove this notice that for instance
\[
\int f \, dR - \int f \, dP = \frac{1}{3}(f(x \lor y) + f(x \land y) - f(x) - f(y)),
\]
for \(x = (2, 1), y = (1, 3)\). Similar results hold for the other cases. The definition of supermodularity implies that \(R, S\) are upper bounds for \(\{P, Q\}\).

The distribution function \(F_{P \vee_{sm} Q}\) would have to satisfy
\[
F_P, F_Q \leq F_{P \vee_{sm} Q} \leq F_R, F_S,
\]
which implies
\[
F_{P \vee_{sm} Q}(1, 1) = 0, \quad F_{P \vee_{sm} Q}(1, 2) = \frac{1}{3}, \quad F_{P \vee_{sm} Q}(2, 1) = \frac{1}{3}, \quad F_{P \vee_{sm} Q}(2, 2) = \frac{1}{3}.
\]
This is not possible. 

The argument in the above proof can be used to prove also the following proposition.

**Proposition 4.11.** The ordered sets \((\mathcal{M}(\mathbb{R}^d), \leq_{\text{dcx}}), (\mathcal{M}(\mathbb{R}^d), \leq_{\text{in}})\) and \((\mathcal{M}(\mathbb{R}^d), \leq_{\text{sm}})\) are not lattices.

A different argument showing that \((\mathcal{M}(\mathbb{R}^d), \leq_{\text{in}})\) is not a lattice can be derived by Example 2.1 in [27].

5. Properties of the order \(\leq_{\uparrow}\). The relation \(\leq_{\uparrow}\) on \(\text{BV}(S)\) induces a partial order \(\leq_{\uparrow}\) on \(\text{BV}/\sim(S)\), the set of all equivalence classes \(F/\sim\) defined by the equivalence relation \(F \sim G\) if \(G - F\) is constant.

Then \((\text{BV}/\sim(S), \leq_{\uparrow})\) is a lattice (see e.g. section 8.6 in [1]). It is easy to see that this can be extended to arbitrary measurable subsets \(S \subset \mathbb{R}\), denoting by \(\text{BV}_{\text{loc}}(S)\) the set of functions \(F : S \rightarrow \mathbb{R}\) that are of bounded variation on closed bounded subsets.

The subset \(\text{BV}_{\text{loc}}^{+}(S), \leq_{\uparrow}\) of all right-continuous functions of local bounded variation forms a sublattice, which is strongly related with the set of all signed measures of local bounded variation endowed with the natural partial order. Indeed, let \(S(S)\) be the set of all signed measures on \((S, \mathcal{B}(S))\) of local bounded variation. Define on it the order relation \(\preceq\) as follows:

\[
\mu \preceq \nu \text{ if } \mu(B) \leq \nu(B) \text{ for all } B \in \mathcal{B}(S).
\]

Then \((S(S), \preceq)\) is a lattice, where \(\mu \vee \nu\) and \(\mu \wedge \nu\) are given as follows. Let \(\rho\) be a dominating measure of \(\mu\) and \(\nu\) (e.g. take \(\rho = |\mu| + |\nu|\)), and denote by
\[
f_\mu = \frac{d\mu}{d\rho} \text{ and } f_\nu = \frac{d\nu}{d\rho}.
\]
the corresponding Radon-Nikodym derivatives. Then \( \mu \lor \nu \) and \( \mu \land \nu \) are the signed measures with the Radon-Nikodym derivatives
\[
\frac{d(\mu \lor \nu)}{d\rho} = \max\{f_\mu, f_\nu\} \quad \text{and} \quad \frac{d(\mu \land \nu)}{d\rho} = \min\{f_\mu, f_\nu\}.
\]
The mapping from \( (\mathcal{BV}_{loc}/\sim}(S), \leq_\uparrow) \) to \((S(S), \preceq)\), assigning to (the equivalence class of) a distribution function \( F \) the corresponding signed measure \( \mu_F \) with
\[
\mu_F((a, b]) = F(b) - F(a) \quad \text{for all } a, b \in S, \ a < b,
\]
is a lattice isomorphism, see [1], Theorem 9.61. There are two important special cases, where this lattice isomorphism can be used to derive explicit formulas for \( F \lor_\uparrow G \) and \( F \land_\uparrow G \). If \( S \) is an open set, and \( F \) and \( G \) are differentiable, then
\[
(F \lor_\uparrow G)'(s) = \max\{F'(s), G'(s)\} \quad \text{and} \quad (F \land_\uparrow G)'(s) = \min\{F'(s), G'(s)\}, \ s \in S.
\]
If \( S = \mathbb{N}_0 \), then
\[
(F \lor_\uparrow G)(s+1) - (F \lor_\uparrow G)(s) = \max\{F(s+1) - F(s), G(s+1) - G(s)\}, \ s \in \mathbb{N}_0, \ (5.1)
\]
and
\[
(F \land_\uparrow G)(s+1) - (F \land_\uparrow G)(s) = \min\{F(s+1) - F(s), G(s+1) - G(s)\}, \ s \in \mathbb{N}_0. \ (5.3)
\]
The following special cases of spaces of functions endowed with the order \( \leq_\uparrow \) are needed in Section 3. Let \( \mathcal{F}^{\log}(\mathbb{R}) \) be the set of all functions \( f : \mathbb{R} \to \mathbb{R} \cup \{-\infty\} \), which are decreasing and right-continuous with \( \lim_{x \to -\infty} f(x) = 0 \), in other words \( \mathcal{F}^{\log}(\mathbb{R}) \) is the set of logarithms of survival function of distributions with support \( \mathbb{R} \cup \{+\infty\} \), where we define \( \log(0) = -\infty \), see Remark 5.

**Lemma 5.1.** The partially ordered set \( (\mathcal{F}^{\log}(\mathbb{R}), \leq_\uparrow) \) is a lattice.

**Proof.** For \( f, g \in \mathcal{F}^{\log}(\mathbb{R}) \) define
\[
S_{f,g} = \{ x \in \mathbb{R} : f(x) > -\infty, g(x) > -\infty \}
\]
\[
= : (-\infty, \alpha_{f,g}).
\]
Then \( (\mathcal{BV}_{loc}/\sim}(S_{f,g}), \leq_\uparrow) \) is a lattice as described above, and therefore \( f \land_\uparrow g(x) \) and \( f \lor_\uparrow g(x) \) are well-defined for \( x \in S_{f,g} \). This can be extended to the whole of \( \mathbb{R} \) by defining
\[
f \land_\uparrow g(x) := -\infty, \quad \text{for } x \geq \alpha_{f,g}
\]
and
\[
f \lor_\uparrow g(x) := \begin{cases} \lim_{t \uparrow} f \lor_\uparrow g(t) + g(x) - g(\alpha_{f,g}), & \text{for } x \geq \alpha_{f,g}, \ g(x) > -\infty, \\ \lim_{t \uparrow} f \lor_\uparrow g(t) + f(x) - f(\alpha_{f,g}), & \text{for } x \geq \alpha_{f,g}, \ f(x) > -\infty, \\ -\infty, & \text{for } f(x) = g(x) = -\infty. \end{cases}
\]
As monotonicity and right-continuity are preserved under the lattice operations, it is straightforward to see that \( (\mathcal{F}^{\log}(\mathbb{R}), \leq_\uparrow) \) becomes a lattice under these operations. \( \Box \)

Now let \( \mathcal{Q} \) be the set of all quantile functions, i.e. the set of all right-continuous increasing functions \( f : (0, 1) \to \mathbb{R} \). The proof of the following result is immediate.

**Lemma 5.2.** The partially ordered set \( (\mathcal{Q}, \leq_\uparrow) \) is a lattice.
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