Maximal solutions of viscous Hamilton–Jacobi equations with degeneracy

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Qualitative Methods for Hamilton–Jacobi Equations and Applications

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Setting of the problem

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded open set and let us consider the following second order degenerate elliptic equation

$$-\text{tr} \left( A(x) D^2 u \right) + |Du|^p + u = f(x), \quad x \in \Omega,$$

where $A : \overline{\Omega} \mapsto S_N$ is a continuous map from $\overline{\Omega}$ into the space of symmetric $N \times N$ matrices satisfying

$$O \leq A(x) \leq \Lambda \text{Id}_N \quad \forall x \in \overline{\Omega},$$

$\sqrt{A}$ is Lipschitz continuous in $\overline{\Omega}$,

and with

$$p > 1, \quad f \in C(\overline{\Omega}).$$

We are going to focus on viscosity solutions of equation $(E)$ satisfying special boundary conditions.
Where does equation \((E)\) come from?

Equations like \((E)\) arise in degenerate stochastic control problems. Indeed, let us consider the following stochastic differential equation

\[
dX_t = a(X_t) \, dt + \sqrt{A}(X_t) \, dW_t, \quad X_0 = x,
\]

where \(a(X_t)\) is interpreted as a feedback control and \(W_t\) is a standard Brownian motion.

By using the control \(a\), we want to force the solution \(X_t\) to stay in \(\Omega\) with probability 1 for all \(t \geq 0\) and for all initial points \(x \in \Omega\); in other words, we impose a state constraint on the controlled system.
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where \( a(X_t) \) is interpreted as a feedback control and \( W_t \) is a standard Brownian motion. By using the control \( a \), we want to force the solution \( X_t \) to stay in \( \Omega \) with probability 1 for all \( t \geq 0 \) and for all initial points \( x \in \Omega \); in other words, we impose a state constraint on the controlled system.
Note that if $A$ is non degenerate and if $a$ is bounded, then the probability that $X_t$ hits the boundary $\partial \Omega$ is positive for all time $t > 0$. Thus, in this case, the only way we have to keep the solution $X_t$ constrained in the domain is to use an *unbounded* control $a$ which pushes back the state process with an infinite intensity.

We then define in such a way the class $A_x$ of admissable controls for the initial point $x \in \Omega$, and we consider for $a \in A_x$ (provided $A_x$ is non empty) the following *cost functional* associated with the problem

$$J(x, a) = E \int_0^\infty \left[ f(X_t) + \frac{1}{q} |a(X_t)|^q \right] e^{-t} dt.$$
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We then define in such a way the class $A_x$ of admissible controls for the initial point $x \in \Omega$, and we consider for $a \in A_x$ (provided $A_x$ is non empty) the following cost functional associated with the problem

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If \( q = \frac{p}{p-1} \), then equation \( (E) \) (up to a multiplicative constant in front of \( |Du|^p \)) is expected for the value function

\[
u(x) = \inf_{a \in A_x} J(x, a).
\]

Moreover, the state constraint on the process \( X_t \) yields the boundary condition for \( u \)

\[
-\text{tr} (A(x)D^2u) + |Du|^p + u \geq f(x), \quad x \in \partial \Omega.
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In other words, the value function \( u \) turns out to be a solution in \( \Omega \) and a supersolution in \( \overline{\Omega} \).
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In other words, the value function $u$ turns out to be a solution in $\Omega$ and a supersolution in $\overline{\Omega}$. 
Some basic references

For the deterministic case (i.e. \( A(x) \equiv O \))


For the uniformly stochastic case (i.e. \( A(x) \equiv Id_N \))

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Related papers for the stochastic case (with $A$ depending on the control and degenerating on the boundary)


Motivated by the above discussion, we give the following

**Definition**
A maximal solution of equation \((E)\) is a function \(u \in C(\Omega)\) which is a viscosity solution in \(\Omega\) and such that

\[
u_*(x) = \begin{cases} 
  u(x) & \text{if } x \in \Omega, \\
  \liminf_{y \to x} u(y) & \text{if } x \in \partial \Omega,
\end{cases}
\]

is a viscosity supersolution in \(\overline{\Omega}\).
In the language of \textit{generalized viscosity solutions}, a maximal solution is nothing but a generalized viscosity solution of equation (E) equipped with the boundary condition

\[
u = +\infty \quad \text{on} \quad \partial \Omega.
\]

Then, the following result is very natural.

\textbf{Proposition}

\textit{Let} \( u \) \textit{be a maximal solution. If} \( v \in C(\overline{\Omega}) \) \textit{is a viscosity subsolution in} \( \Omega \), \textit{then} \( v(x) \leq u_*(x) \) \textit{for every} \( x \in \overline{\Omega} \).
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For the proof, use the comparison principle by


jointly with a technicality to deal with the power-like nonlinearity of equation (E) as in

Conversely, we have the following

**Proposition**

Let \( u \in C(\Omega) \) be a viscosity subsolution such that \( u \geq v \) for every subsolution \( v \in \text{USC}(\Omega) \). Then, \( u \) is a maximal solution.

The proof can be obtained by arguing as in the first order case (see e.g. Capuzzo Dolcetta & Lions) and by using the “bump lemma“ of the *User’s guide*.

**Goal**: existence, uniqueness and regularity properties for maximal solutions.
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**Goal**: existence, uniqueness and regularity properties for maximal solutions.
Known results for the deterministic case

If \( A \equiv O \), then equation (E) reduces to the Hamilton–Jacobi equation

\[ |D u|^p + u = f \quad \text{in } \Omega. \]

It has been proved that there exists a unique maximal solution \( u \in C(\bar{\Omega}) \), which can be characterized also as the unique generalized supersolution of the associated homogeneous Neumann problem.

In this case, the maximal solution \( u \) is Lipschitz continuous in \( \bar{\Omega} \) and the optimal control is \( a(x) = -|D u(x)|^{p-2} D u(x) \).

Remark

Note that the Lipschitz continuity holds for any bounded from below subsolution.
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Remark

Note that the Lipschitz continuity holds for any bounded from below subsolution.
Known results for the uniformly stochastic case

In this case, equation \( (E) \) has the form

\[-\Delta u + |Du|^p + u = f \quad \text{in } \Omega,
\]

and it has been shown to have a unique maximal solution \( u \in C(\Omega) \) which is \textit{locally} Lipschitz continuous.

Moreover:

- if \( p \leq 2 \), then \( u \) uniformly blows up at the boundary, with a rate of order \( \text{dist}(x, \partial \Omega)^{\frac{p-2}{p-1}} \) if \( p < 2 \), and like \( |\log \text{dist}(x, \partial \Omega)| \) if \( p = 2 \). Then, \( u \) is a so called \textit{large solution}. 
Known results for the uniformly stochastic case

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Moreover:

- if $p \leq 2$, then $u$ uniformly blows up at the boundary, with a rate of order $\text{dist}(x, \partial \Omega)^{\frac{p-2}{p-1}}$ if $p < 2$, and like $|\log \text{dist}(x, \partial \Omega)|$ if $p = 2$. Then, $u$ is a so called large solution.
• if $p > 2$, then $u$ is bounded in $\Omega$ and it can be extended to a globally Hölder continuos function with exponent 
\[ \alpha = \frac{p-2}{p-1}. \]

In any case (with an additional assumption if $p > 2$), the optimal feedback control is 
\[ a(x) = -|Du|^{p-2}Du, \]
which is unbounded on $\partial \Omega$. 
• if $p > 2$, then $u$ is bounded in $\Omega$ and it can be extended to a globally Hölder continuous function with exponent $\alpha = \frac{p-2}{p-1}$.

In any case (with an additional assumption if $p > 2$), the optimal feedback control is $a(x) = -|Du|^{p-2}Du$, which is unbounded on $\partial\Omega$. 
The case of a general $A$, $p > 2$

Proposition

If $u \in C(\Omega)$ is a maximal solution, then there exist constants $m < M$ depending only on $\Omega$, $p > 2$ and $f$ such that

$$m \leq u(x) \leq M \quad \forall x \in \Omega.$$ 

The proof basically uses the same barrier functions depending on the distance from $\partial \Omega$ constructed in Lasry & Lions.
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Our main result is the following regularity theorem.

**Theorem**

*Every viscosity subsolution* \( u \in BUSC(\Omega) \) *of equation* \((E)\) *can be extended up to the boundary to a function satisfying*

\[
u \in C^{0,\alpha}(\overline{\Omega}), \quad \alpha = \frac{p - 2}{p - 1}.
\]

The idea of the proof is to use strongly the *coercitivity* of the first order term, as to partially absobrbe the second order perturbation.

The \(\alpha\)-hölderianity is the sharp regularity for subsolutions, as it is exhibited by the viscosity subsolution \( u(x) = |x|^{\alpha} \) in any ball centered at the origin (if the dimension \( N \) is at least 2).
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As a consequence of the above regularity result, one easily gets the *existence* of a maximal solution by using any approximation argument (on the matrix $A(x)$, or adding to $f$ a forcing datum defined on $\mathbb{R}^N$ and blowing up on the complement of $\Omega$, or.....) Note that any approximating sequence of solutions will be bounded and equicontinuous, and thus uniformly converging to a solution.

The *uniqueness* of the maximal solution follows from the comparison principle proved in Barles & Da Lio for generalized sub- and supersolution, and the proof can be highly simplified by using the continuity of any subsolution.
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The uniqueness of the maximal solution follows from the comparison principle proved in Barles & Da Lio for generalized sub- and supersolution, and the proof can be highly simplified by using the continuity of any subsolution.
As for the regularity of solution is concerned, by adapting the Bernstein technique developed in Lasry & Lions, one can obtain the local Lipschitz continuity of any bounded solution by assuming $f$ to be Lipschitz.

More precisely, for any solution $u \in C(\Omega)$ one gets the bound

$$|Du(x)| \leq \frac{C}{d(x)^{1-\alpha}}, \quad x \in \Omega, \quad d(x) = \text{dist}(x, \partial \Omega).$$

As in the uniformly stochastic case, then one can show that the maximal solution $u$ is the value function of the initial stochastic control problem, and $a(x) = -|Du|^{p-2}Du(x)$ is the optimal control.
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The case of a general $A$, $p \leq 2$

If $p \leq 2$, in general a maximal solution $u$ satisfies Dirichlet boundary conditions of mixed type (bounded and unbounded). In fact, one can easily show in this case that if $u$ is any bounded from below supersolution in $\Omega$, then

$$u_* (x) = +\infty \quad \forall x \in \partial \Omega \text{ such that } A(x) \nu(x) \cdot \nu(x) > 0,$$

where $\nu(x)$ is the outward unit normal vector to $\partial \Omega$ at the point $x$.

On the other hand, in the boundary region where $A$ degenerates along the normal direction, $u$ is expected to be bounded (as in the deterministic case). This makes the uniqueness still open, unless one specifies the rate of blowing up at the boundary.
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with the same exponent $\alpha = \frac{p-2}{p-1}$, which now satisfies $\alpha \leq 0$.

From this, one can derive the existence of a maximal solution, even if, because of the lackness of uniqueness result, it will depend on the method of approximation.
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Open problems and perspectives

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