Generalized Contact Structures

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The bundle $TM \oplus T^*M \to M$

- A symmetric bilinear form $\langle X + \alpha, Y + \beta \rangle = \frac{1}{2}(\alpha(Y) + \beta(X))$.
- Courant bracket

$$\llbracket X + \alpha, Y + \beta \rrbracket = [X, Y] + \mathcal{L}_{X}\beta - \mathcal{L}_{Y}\alpha - \frac{1}{2}d(\iota_{X}\beta - \iota_{Y}\alpha).$$

Remarks:

- $\langle -, - \rangle$ is non-degenerate.
- $TM$ and $T^*M$ are maximally isotropic. $\langle X, Y \rangle = 0$, $\langle \alpha, \beta \rangle = 0$ for all $X, Y, \alpha, \beta$.
- Courant bracket does not satisfy Jacobi identity.

Lemma

If $V$ is a subbundle of $(TM \oplus T^*M)_\mathbb{C}$ such that its space of sections is closed: $\llbracket v_0, v_1 \rrbracket \in C^\infty(M, V)$, and if $V$ is isotropic: $\langle v_0, v_1 \rangle = 0$, for any sections $v_0$ and $v_1$ of $V$, with $\rho : V \hookrightarrow (TM \oplus T^*M)_\mathbb{C} \to TM_\mathbb{C}$, then the triple $(V, \llbracket -, - \rrbracket_V, \rho)$ is a Lie algebroid.
Lie bialgebroids, Liu-Weinstein-Xu (90’s)

Definition

$L$ and $K$ form a Lie bialgebroid pair in $TM \oplus T^*M$ if

- $L$ and $K$ are maximally isotropic with respect to $\langle -, - \rangle$,
- $L \oplus K = TM \oplus T^*M$;
- (space of sections of) $L$ and $K$ are closed under $[[-, -]]$.
- When $d_K [[\ell_1, \ell_2]] = [[d_K \ell_1, \ell_2]] + [[\ell_1, d_K \ell_2]]$, where
  
  $$(d_K \ell)(k_1, k_2) := 2 \left( \rho(k_1) \langle \ell, k_2 \rangle - \rho(k_2) \langle \ell, k_1 \rangle - \langle \ell, [[k_1, k_2]] \rangle \right).$$

Treat $L$ as $K^{\ast}$, $d_K : \wedge^m L \rightarrow \wedge^{m+1} L$. 

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Generalized Contact Structures
Suppose that \((L, K)\) is a Lie bialgebroid. \(L \oplus K = (TM \oplus T^*M)_\mathbb{C}\).

Let \(\Gamma \in \mathcal{C}^\infty(M, \wedge^2 L) \subset \mathcal{C}^\infty(M, \text{Hom}(L^*, L)) = \mathcal{C}^\infty(M, \text{Hom}(K, L))\).

Let \(K_\Gamma\) be the graph of \(K\) with respect to \(\Gamma\):

\[
K_\Gamma = \{k + \Gamma(k) : k \in \mathcal{C}^\infty(M, K)\}.
\]

\(L \oplus K_\Gamma \cong L \oplus K \cong L \oplus L^*\). \(K_\Gamma \subset L \oplus K\).

**Theorem (LWX)**

\((L, K_\Gamma)\) is a Lie bialgebroid pair if and only if \(d_{K_\Gamma} + \frac{1}{2} [\Gamma, \Gamma] = 0\).

\([-, -]\) on \(\wedge^\bullet L\), \(d_K\) is C-E differential of \([-, -]\) on \(\wedge^\bullet K\).
A generalized almost complex structure on an even-dimensional manifold $M$ is a bundle automorphism $\mathcal{J} : TM \oplus T^*M \to TM \oplus T^*M$ such that $\mathcal{J}^2 = -\mathrm{I}$ and $\mathcal{J}^* + \mathcal{J} = 0$.

$\mathcal{J} = \begin{pmatrix} \varphi & \pi \\ \theta & -\varphi^* \end{pmatrix}$,

$\varphi$ a $(1,1)$-tensor, $\pi$ a bivector field, $\theta$ a 2-form.

$$(TM \oplus T^*M)_\mathbb{C} = L \oplus \overline{L} = +i \text{ eigenspace} \oplus -i \text{ eigenspace}$$

Equivalent definition: choice of maximally isotropic subspace $L$ in $(TM \oplus T^*M)_\mathbb{C}$ as $(+i)$ eigenspace. The dual space $L^*$ as $(-i)$ eigenbundle.
Integrability

**Definition**

\( \mathcal{J} \) is integrable if \( C^\infty(M, L) \) and/or \( C^\infty(M, \bar{L}) \) are closed with respect to \( [\cdot, \cdot] \).

When \( (M, \mathcal{J}) \) is a generalized complex structure,\n
\[
L \oplus \bar{L} = (TM \oplus T^*M)_\mathbb{C}, \quad \bar{L} \cong L^*.
\]

In particular, the pair \( (L, \bar{L}) \) forms a Lie bialgebroid.
Examples

(1) When \( J : TM \rightarrow TM \) is a (classical) complex structure. On \( TM \oplus T^*M \), define

\[
\mathcal{J} = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix},
\]

\( L = T^{1,0} \oplus T^{*(0,1)} \).

(2) \( \theta \) is a symplectic form. \( \pi \) Poisson (bi)vector field. Define

\[
\mathcal{J} = \begin{pmatrix} 0 & \pi \\ \theta & 0 \end{pmatrix}.
\]

\( L = \text{Span}\{X - i_\iota_X \theta : X \in C^\infty(M, TM)\} \).

Integrability of \( L \) and \( \overline{L} \) is equivalent to \( d\theta = 0 \).
1. Given \((M, \mathcal{J})\) a classical complex structure.
2. Treat it as generalized complex structure.
3. Construct the Lie bialgebroid: \(L \oplus \overline{L}\).
4. Given \(\Gamma_1\) in \(H^2(M, \mathcal{O}) \oplus H^1(M, T^{1,0}) \oplus H^0(M, \wedge^2 T^{(1,0)})\), find \(\Gamma \in C^\infty\left(M, \wedge^2 (T^*^{(0,1)} \oplus T^{(1,0)})\right)\) such that

\[
\Gamma_1 \equiv_1 \Gamma, \quad \overline{\partial}\Gamma + \frac{1}{2} \llbracket \Gamma, \Gamma \rrbracket = 0.
\]
5. Use LWX-theory for \(\Gamma\) to get \(\overline{L}_\Gamma\).
6. Use LWX-theory for \(\overline{\Gamma}\) to get \(L_{\overline{\Gamma}}\).
7. \((L_{\overline{\Gamma}}, \overline{L}_\Gamma)\) is a new generalized complex structure.
8. Sometimes, the deformed object could be a symplectic structure.
### Issue and motivation

#### Theorem (Moser, 65)

*Symplectic structures on compact manifolds are rigid.*

Gautieri 04, Poon 06 (Kodaira-Thurston surfaces)

#### Theorem (Gray, 59)

*Contact structures on compact manifolds are rigid.*

#### Problem

*Is it possible to enlarge the category of geometry so that contact structures could be deformation in a non-trivial and controlled manner?*

#### Remarks:

- Similarity
- Difference
- Classical structures: Jacobi, Dirac, conformal Dirac, Lichnerowicz.
Definition (After Vaisman 07)

A generalized almost contact structure is a collection of tensors: 
\( \mathcal{J} = (\xi, \eta, \pi, \theta, \varphi) \), \( \xi + \eta \in C^\infty(M, TM \oplus T^*M) \)

\[
\Phi = \begin{pmatrix} \varphi & \pi \\ \theta & -\varphi^* \end{pmatrix} : TM \oplus T^*M \to TM \oplus T^*M
\]

such that \( \Phi + \Phi^* = 0, \ \eta(\xi) = 1, \ \Phi(\xi) = 0, \ \Phi(\eta) = 0, \ \Phi \circ \Phi = -\mathbb{I} + \xi \circ \eta. \)

where \( (\xi \circ \eta)(X + \alpha) := \eta(X)\xi + \alpha(\xi)\eta. \)

\( \Phi_{\ker} : \ker \eta \oplus \ker \xi \to \ker \eta \oplus \ker \xi, \ \Phi_{\ker} \circ \Phi_{\ker} = -\mathbb{I}. \)

Remark: Focus on tensorial objects only. No equivalence. Formal.
Subbundles

\[ \Phi^2_{\ker} = -\mathbb{I}. \]

\[ E^{1,0} = \{ e - i\Phi(e) : e \in C^\infty(M, \ker \eta \oplus \ker \xi) \} \]

\[ L := L_\xi \oplus E^{1,0}, \quad \overline{L} = L_\xi \oplus E^{0,1}, \quad L^* = L_\eta \oplus E^{0,1}, \quad \overline{L}^* = L_\eta \oplus E^{1,0}. \]

\[ L \oplus L^* = (TM \oplus T^*M)_\mathbb{C}, \quad \overline{L} \oplus \overline{L}^* = (TM \oplus T^*M)_\mathbb{C}. \]

Fact: \( E^{1,0}, E^{0,1}, L, \overline{L}, L^*, \overline{L}^* \) are isotropic.

But \( \overline{L} \neq L^* \) !!
Integrability, or the lack of it

Definition
Given a $\mathcal{J} = (\xi, \eta, \pi, \theta, \varphi)$-structure, if $C^\infty(M, L)$ is Courant-closed, (but $C^\infty(M, L^*)$ is not necessarily closed,) then $\mathcal{J}$ is a generalized contact structure.

Remember: $\overline{L} \neq L^*$  !!!

Definition
Given a $\mathcal{J} = (\xi, \eta, \pi, \theta, \varphi)$-structure, if both $C^\infty(M, L)$ and $C^\infty(M, L^*)$ are Courant-closed, then $\mathcal{J}$ is a generalized “complex” structure. (Even though $\dim M = 2n + 1$.)

Key: (Not) Lie bialgebroid.

Avoiding terminology: ”generalized normal contact structures”.
Problem

Assume $C^\infty(M, L)$ is closed, determine whether or when $C^\infty(M, L^*)$ is also closed.

LWX’s obstruction for formation of Lie bialgebroids: For any three sections $v_0, v_1, v_2$ of $L^* = L_\eta \oplus E^{0,1}$,

$$Nij(v_0, v_1, v_2) = \frac{1}{3}(\langle[[v_0, v_1]], v_2\rangle + \langle[[v_1, v_2]], v_0\rangle + \langle[[v_2, v_0]], v_1\rangle).$$

$Nij \in C^\infty(M, \wedge^3 L), L = L_\xi \oplus E^{1,0}, \wedge^3 L = \wedge^3 E^{1,0} \oplus L_\xi \otimes \wedge^2 E^{1,0}$.

Proposition

Given $\mathcal{J} = (\xi, \eta, \pi, \theta, \varphi)$ and $C^\infty(M, L)$ closed. Then $L^*$ is closed if and only if $\xi \wedge (\rho^* d\eta)^{2,0} = 0$, where $\rho : E^{1,0} \rightarrow TM_C$. 
Odd dimensional analogue of symplectic structures

Definition (Libermann, 1958)
An almost cosymplectic structure on $M^{2n+1}$ is a reduction from $GL(2n+1, \mathbb{R})$ to $Sp(n, \mathbb{R})$. That is the choice of a 1-form $\eta$ and a 2-form $\theta$ such that $\eta \wedge \theta^n \neq 0$ everywhere.

Definition
$(\eta, \theta)$ is a cosymplectic structure if $d\eta = 0$ and $d\theta = 0$.

Definition
$\eta$ is a contact 1-form on $M^{2n+1}$ if $\eta \wedge (d\eta)^n \neq 0$ everywhere.

A contact 1-form determines an almost cosymplectic structure $(\eta, d\eta)$, but it is NEVER a cosymplectic structure without qualification.
As generalized almost contact structures

- Almost cosymplectic \((\eta, \theta)\): (1-form, 2-form). \(\eta \wedge \theta^n \neq 0\) everywhere. Define \(\flat : TM \to T^*M\) by

\[
\flat(X) = \iota_X \theta - \eta(X)\eta,
\]

then \(\pi(\alpha, \beta) := \theta(\flat^{-1}(\alpha), \flat^{-1}(\beta))\).

\(\flat\) is an isomorphism. There exists a unique \(\xi\) such that \(\eta(\xi) = 1\) and \(\iota_\xi \theta = 0\). Choose \(\varphi = 0\).

\[
\Phi = \begin{pmatrix} 0 & \pi \\ \theta & 0 \end{pmatrix}.
\]

- If \(\eta\) is a contact 1-form, choose \(\theta = d\eta\). Then follow the above construction.
Definition (Sasaki (60))

A $(\xi, \eta, \varphi)$-structure on $M^{2n+1}$ consists of $\varphi$ a (1,1)-tensor, a vector field $\xi$ and a 1-form $\eta$ such that $\varphi^2 = -I + \eta \otimes \xi$, and $\eta(\xi) = 1$. 

Definition

A $(\xi, \eta, \varphi)$-structure on $M$ is “normal” if and only if a naturally defined almost complex structure on $M \times \mathbb{R}^+$ is integrable. Equivalently, $\mathcal{L}_\xi \varphi = 0$, $\mathcal{L}_\xi \eta = 0$, and $\mathcal{N}_\varphi = -\xi \otimes d\eta$, where $\mathcal{N}_\varphi(X, Y) := [\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi([\varphi X, Y] + [X, \varphi Y])$. 

A contact 1-form $\eta$ does not determine a $(\xi, \eta, \varphi)$-structure until a “compatible” metric is chosen. $\varphi$ is metric dependence. Too many choices. $\mathcal{J} = (\xi, \eta, \varphi, \pi = 0, \theta = 0)$. 

Odd dimensional analogue of complex structures
Theorem (Examples of generalized contact structures)

\( C^\infty(M, L) \) is closed for

- Cosymplectic \((\eta, \theta)\). i.e. \( d\eta = 0 \) and \( d\theta = 0 \).
- Contact 1-form \( \eta \). i.e. \( \theta = d\eta \neq 0 \).
- Normal \((\xi, \eta, \varphi)\)-structures. i.e. \( \mathcal{N}_\varphi = -\xi \otimes d\eta \).

Proof: DBH.

Theorem (Examples of generalized complex structures)

Both \( C^\infty(M, L) \) and \( C^\infty(M, L^*) \) are closed for

- Cosymplectic \((\eta, \theta)\). \((G\text{-structure})\)
- Normal \((\xi, \eta, \varphi)\)-structure. \((\text{Sasaki Cone})\)

Proof: For the latter, check the "type" of \( d\eta \).
Focus on contact 1-form $\eta$

Local picture: $(x_j, y_j, z)$ on $\mathbb{R}^{2n+1}$.

$$\eta = dz - \sum_j y_j \, dx_j. \quad \xi = \frac{\partial}{\partial z}, \quad \theta = d\eta = \sum_j dx_j \wedge dy_j.$$

$$X_j := \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j}, \quad Y_j = \frac{\partial}{\partial y_j}, \quad \pi = \sum_j X_j \wedge Y_j.$$

The obstruction $(\rho^* d\eta)^{2,0}$ is equal to

$$\frac{1}{4} \sum_j (dx_j - iY_j) \wedge (dy_j + iX_j).$$

$$(\rho^* d\eta)^{2,0} + (\rho^* d\eta)^{0,2} = \frac{1}{4} (d\eta - \pi).$$

Proposition

The obstruction for $L^*$ being closed is not equal to zero anywhere when $\mathcal{J}$ is due to a contact 1-form.
For generalized complex structures (on odd-dimensional manifolds), use LWX’s Lie bialgebroid theory.
Deformation of classical cosymplectic structures away from classical objects e.g. $H_3$ Heisenberg group or cocompact quotient. Co-symplectic structure.
For generalized contact (not complex)?
Deformation theory due to Lie bialgebroid structure fails.
Alternative:

**Proposition**

Let $M$ be the principal $\text{SO}(2)$-bundle over $N$ with connection $\eta$ and curvature $d\eta = p^*\omega$. Then the family $J_t$ of generalized complex structures on $N$ is lifted to a family $\mathcal{J}_t$ of generalized contact structures on $M$.

Proof. A Boothby+Wang type theorem. In their 1958 paper: "On contact manifolds". (A backbone)
On $N$ the Kodaira surface, there exists

- a complex structure $J = J_0$,
- a symplectic form $\omega = J_1$, with $\omega$ being type $(2,0)+(0,2)$ w.r.t. $J$.
- a family of generalized complex structures $J_t$ containing $J_0$ and $J_1$.

Use Boothby-Wang construction.

Get a family of generalized contact structures $\mathcal{J}_t$. $\mathcal{J}_1$ is contact. $\mathcal{J}_t$ are non-classical objects for $t \neq 1$.

Remark: No more ”Gray’s Theorem“ :-)

Remark: No deformation theory :-(
Further development

Contact 1-form and Reeb field: \( \iota_\xi \eta = 0. \) \( \mathcal{L}_\xi \eta = 0. \)

**Theorem**

\[ \mathcal{J} = (\xi, \eta, \pi, \theta, \varphi) \] a generalized contact structure (not necessarily \( \text{cx} \)), with \( \mathcal{L}_\xi \eta = 0 \), then

- \( \mathcal{L}_\xi \mathcal{J} = 0; \) and
- the pair \( E^{1,0} \) and \( E^{0,1} \) forms a transversal Lie bialgebroid over \( (M, \xi) \).

Reversing Boothby-Wang construction.
Cohomology theory.
Deformation.
Equivalence.
Another story.

Conclusion: contact vs symplectic.
Non-integrability vs integrability. Difference vs similarity.