MFG Models in Economics

Examples from capital dynamic theory

Jean-Michel Lasry, 13.5.2011

Example 1.

Krussel - Smith (and others..) framework (1/11)

"K&S" (PLL, JML)

- Individual agents caracteristics

  θ_i: productivity parameter of agent i

  k_i capital owned by agent i

- Production of agent i

$$Y_i = \theta_i \sqrt{k_i + \tilde{k}_i} \qquad (Y_i = F(\theta_i, k_i, \tilde{k}_i))$$

where $\tilde{k}_i$ is capital rented by agent i (controlled by i)

- State space: $\underline{\theta} \leq \theta_i \leq \bar{\theta}$, $k_i > 0$

$$\Omega = [\underline{\theta}, \bar{\theta}] \times \mathbb{R}_+$$

K&S framework (2/11)

Individual productivity are random

$$d\theta_i = \sigma\, dW_i + \varepsilon\, dB$$

where W_i, B are independent Brownians

dW_i individual risk of agent i

dB common risk shared by all agents

Capital is both needed

- to produce
- to insure against risks

(Incomplete market models)

K&S framework (3/11)

- Market for capital
  - price p_t at time t
  - prices are endogeneous, fixed by equilibrium
- Cost to maintain capital δk_i (δ constant)

Hence Dynamic of agent i capital is:

$$\dot{k}_i = \theta_i \sqrt{k_i + \tilde{k}_i} - p_t \tilde{k}_i - \delta k_i - c_i$$

At each time t, agent i will optimize the choice of $\tilde{k}_i$ borrowed, c_i consumption

K&S framework (4/11)

Agent i optimization problem

- $\text{Max } \mathbb{E} \int_0^{+\infty} e^{-rt}\, U(C_i(t))\, dt$ $\quad$ (U = utility function)

- under constraints:

  - $\tilde{k}_i \leq \lambda k_i$ $\quad$ (λ = leverage parameter)

  - $k_i \geq 0$

- $d\theta_i = \sigma_0\, dW_i + \varepsilon\, dB$

- $\dot{k}_i = \theta_i \sqrt{k_i + \tilde{k}_i} - p_t \tilde{k}_i - \delta k_i - C_i(t)$

K&S framework (5/11)

Given prices (p_t)

agent i faces a (classic) stochastic optimization problem

Closed loop solution

$$c_i^* = f_i(t, \theta_i, k_i)$$
$$\tilde{k}_i^* = g_i(t, \theta_i, k_i)$$

As all agents are identical, agents in the same state (θ, k) at the same time t, will take the same optimal choice.

$c^* = f(t, \theta, k)$ optimal consumption of an agent in state (θ, k)

$\tilde{k}^* = g(t, \theta, k)$ optimal amount of capital borrowed by an agent

Denote $\varphi(t, \theta, k) = \theta\sqrt{k + \tilde{k}^*} - p_t \tilde{k}^* - \delta k - c^*$

the dynamic of individual capital induced by optimal closed loop choices

(K&S) framework (6/11)

Population dynamic

$m(t,\theta,k)$ density of agents, at time t, in state (θ,k)

$$dm = \left[\frac{\partial}{\partial k}\left(\varphi(t,\theta,k)\, m(t,\theta,k)\right) + \frac{1}{2}\left(\sigma^2+\varepsilon^2\right)\frac{\partial^2 m}{\partial \theta^2}\right] dt + \frac{\partial m}{\partial \theta}\, \varepsilon\, dB$$

- optimal individual choices + independent risks $\rightarrow$ deterministic dynamic (drift)
- shared risk dB $\rightarrow$ stochastic population move

K&S framework (7/11)

Optimal individual choice φ $\rightarrow$ (random) population dynamic

population state m_t $\rightarrow$ price p_t

(random) price dynamic $\rightarrow$ optimal individual choice, φ

K&S framework (8/11)

optimal choice of agent : HJB equation on value function

Stationary case on population dynamic

value function $\mathcal{U}(\theta, k, m)$ defined by

$$\mathcal{U}(\theta_0, k_0, m_0) = \max \mathbb{E} \int_0^{+\infty} e^{-rt} u(c_t)\,dt \quad ; \quad \theta(0)=\theta_0,\ k(0)=k_0,\ m(0)=m_0$$

s.t. dynamics equations

Formal derivation of the HJB equation :

$$\mathcal{U}(\theta_0, k_0, m_0) = \max \mathbb{E}\left[u(c)\,dt + e^{-r\,dt}\,\mathcal{U}(\theta_0+d\theta_0, k_0+dk_0, m_0+dm_0)\right]$$

$$= \max_{c,\tilde{k}} \mathbb{E}\left[u(c)\,dt + (1-r\,dt)\left(\mathcal{U}(\theta_0,k_0,m_0) + \frac{\partial \mathcal{U}}{\partial \theta_0}\,d\theta_0 + \frac{\partial \mathcal{U}}{\partial k_0}\,dk_0 + \dots\right)\right]$$

$$0 = -r\,\mathcal{U} + \max_{c,\tilde{k}} \mathbb{E}\left[u(c) + \frac{\partial \mathcal{U}}{\partial \theta}\,d\theta + \frac{\partial \mathcal{U}}{\partial k}\,dk + \dots\right]$$

KnS framework (91 ??)

MFG equation on value $\mathcal{U}$.

$$0 = -r\mathcal{U} + \max_{c,\tilde{k},\text{s.t.}} \mathbb{E}\Big[u(c) + \frac{\partial \mathcal{U}}{\partial \theta}(\sigma dW + \varepsilon dB) + \frac{\partial \mathcal{U}}{\partial k}\left(\theta\sqrt{k+\tilde{k}} - \delta k - p\tilde{k} - c\right)$$
$$+ \frac{\partial^2 \mathcal{U}}{\partial \theta^2}\frac{\sigma^2+\varepsilon^2}{2} + \nabla_m \mathcal{U}\cdot dm + \frac{1}{2}\langle \partial^2 \mathcal{U}\cdot dm, dm\rangle\Big]$$

$$0 = -r\mathcal{U} + \max_{c,\tilde{k},\text{s.t.}} \mathbb{E}\left[u(c) + \frac{\partial \mathcal{U}}{\partial k}\left(\theta\sqrt{k+\tilde{k}} - \delta k - p\tilde{k} - c\right) + \frac{1}{2}(\sigma^2+\varepsilon^2)\frac{\partial^2 \mathcal{U}}{\partial \theta^2} + \frac{1}{2}\langle \partial^2 \mathcal{U}\, dm, dm\rangle\right]$$

$$\max_c \left[u(c) - \frac{\partial \mathcal{U}}{\partial k} c\right] \Rightarrow u'(c^*) = \frac{\partial \mathcal{U}}{\partial k} \qquad \boxed{c^* = (u')^{-1}\left(\frac{\partial \mathcal{U}}{\partial k}\right)}$$

$$\max_{\tilde{k}\le \lambda k} \left[\frac{\partial \mathcal{U}}{\partial k}\left(\theta\sqrt{k+\tilde{k}} - p\tilde{k}\right)\right] \Rightarrow \boxed{\tilde{k}^* = \max\left\{\lambda k, \frac{\theta^2}{4p^2} - k\right\}}$$

K&S framework (10/11)

population state m_t $\rightarrow$ price p_t

Equilibrium on capital borrowing market

$$\int \max\left\{\lambda k, \frac{\theta^2}{4p^2} - k\right\} m_t(\theta, k)\, dk = 0 \qquad \left(\Rightarrow p = \psi(\theta, m)\right)$$

For ex., if $\lambda = +\infty$ (no "friction" on capital borrowing market)

$$\int \left(\frac{\theta^2}{4p^2} - k\right) m(\theta, k)\, dk = 0 \Rightarrow \frac{\theta^2}{4p^2} = \int k\, m(\theta, k)\, dk$$

$$p = \frac{\theta}{2} \frac{1}{\sqrt{m_1}}$$ $\Rightarrow$ p depends only on first moment of the density m

K&S framework (11/ 11)

Back to MFG equation on value $\mathcal{U}$

$$
\begin{cases}
0 = -r\,\mathcal{U} + u(c^*) + \dfrac{\partial \mathcal{U}}{\partial k}\left(\theta\sqrt{k+\tilde{k}^*} - \delta k - p\,\tilde{k}^*\right) + \frac{1}{2}\left(\sigma^2 + \varepsilon^2\right)\dfrac{\partial^2 \mathcal{U}}{\partial \theta^2} + \frac{1}{2}\left(\partial^2 \mathcal{U}\, dm, dm\right) \\[2ex]
\tilde{k}^* = \max\left\{ \lambda k ,\ \dfrac{\theta^2}{4p^2} - k \right\} \\[2ex]
c^* = u'^{-1}\left(\dfrac{\partial \mathcal{U}}{\partial k}\right) \\[2ex]
p = \psi(\theta, m) \qquad \left(\text{for ex. } p = \dfrac{\theta}{2\sqrt{m_1}}\right), \text{ given by } \int \tilde{k}^*\, m(\theta,k)\, dk = 0
\end{cases}
$$

. solve by small noise expansion on ε ... (work in progress)

Example 2 . Time to build models (1/4)

(OG, JML, PLL)

$$\xrightarrow{c} \boxed{m_1} \xrightarrow{\alpha} \boxed{m_2} \xrightarrow{\delta}$$

state 1 — value $u_1(m_1, m_2)$

state 2 — value $u_2(m_1, m_2)$

1 = building stage

2 = production state

α, β, transition probabilities

c entrance flow

c = given exogenous

α = controled ; i.e: each agent controls his own α_i

δ = controled ; i.e: each agent controls his own δ_i

costs : $\frac{1}{2}\alpha_i^2$, $\frac{1}{2}\delta_i^2$

Time to build (2/4)

agents in box 1 optimization problem:

$$\max_{\alpha} \left[\alpha (u_2 - u_1) - \frac{\alpha^2}{2} \right]$$

agents in box 2 optimization problem

$$\max_{\delta} \left[-\delta u_2 - \frac{1}{2\delta^2} \right]$$

Optimal choice of agents given (m_1, m_2) and (u_1, u_2)

$$\alpha^* = u_2 - u_1$$

$$\delta^* = (u_2)^{-1/3}$$

Time to build models (3/4)

- flows

(net flow in box 1) $= 1 - \alpha^* m_1 = 1 - (u_2 - u_1) m_1$

(net flow in box 2) $= \alpha^* m_1 - \delta^* m_2 = (u_2 - u_1) m_1 - (u_2)^{-1/3} m_2$

- earning of agents

$$\int_0^{+\infty} e^{-rt} p_t \, dt$$

where price p_t given by market equilibrium

demand $= D(p)$ (for ex: $D(p) = 1/(\sqrt{p})$)

offer $= m_2$

$\Rightarrow$ price $p = D^{-1}(m_2)$

MFG system

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = (1 - m_1(u_2 - u_1))\dfrac{\partial u_1}{\partial m_1} + ((u_2-u_1)m_1 - (u_2)^{-1/3} m_2)\dfrac{\partial u_1}{\partial m_2} - \dfrac{(u_2-u_1)^2}{2} \\[2ex] \dfrac{\partial u_2}{\partial t} = (1 - m_1(u_2 - u_1))\dfrac{\partial u_2}{\partial m_1} + ((u_2-u_1)m_1 - u_2^{-1/3} m_2)\dfrac{\partial u_2}{\partial m_2} - \frac{1}{2}(u_2)^{2/3} + D^{-1}(m_2) \end{cases}$$

If demand is stochastic, price are stochastic.

For example demand $= D(p) + \varepsilon W$ $\quad$ W Brownian

then price $p = D^{-1}(m_2 - \varepsilon W)$

Value functions depends on (m_1, m_2, W)

$$\frac{\partial u_i}{\partial t} = [\text{.. as previously}] + \frac{1}{2}\frac{\partial^2 u_i}{\partial W^2}$$