The left-invariant sub-Riemannian problem on the Engel group is considered. This problem is very important as nilpotent approximation of nonholonomic systems in four-dimensional space with two-dimensional control (see [1, 2]), for instance of a system which describes movement of mobile trailer robot. Parameterization of extremal curves by elliptic Jacob's functions was obtained. Discrete symmetries of Exponential mapping were considered and the corresponding Maxwell sets were constructed. Thus global bound of the cut time (i.e., the time of loss of global optimality) was found which gives necessary optimality conditions for extremal curves. It was shown that the function that gives the upper bound of the cut time provides the lower bound of the first conjugate time.

**Hamiltonian system**

Existence of optimal solutions of problem (1)–(3) is implied by Filippov’s theorem (4). By Cauchy–Schwarz inequality, it follows that sub-Riemannian length minimization problem (3) is equivalent to action minimization problem:

\[
\int_0^t \sqrt{u^2 + v^2} \, dt \to \min.
\]

Pontryagin’s maximum principle [3, 4] was applied to the resulting optimal control problem (1), (2), (4). Abnormal extremals were parameterized. Denote vector fields at the controls in the right-hand side of system (1):

\[
X_t = (0.1, -\frac{1}{3} t^3, 0, 0),
\]

which is a solution of the corresponding bundle T**M** Hamiltonians **h**(λ) = (λ(x, q)), λ ∈ T**M** | H = 1.2.

Normal extremals satisfy the Hamiltonian system

\[
\lambda = h(\lambda), \quad \lambda \in T^* M,
\]

where H = 1/2 |X_1|^2 + 1/2 |X_2|^2.

The normal Hamiltonian system (6) is given, in certain natural coordinates, as follows on a level surface [λ | T* M | H = 1/2,

\[
\dot{\theta} = c, \quad \dot{c} = -\kappa \sin \theta, \quad \dot{\theta} = 0, \quad \kappa = \cos \theta + \sin \theta \dot{q}, \quad \dot{q} = 0.
\]

**Parameterization of normal extremal trajectories**

The family of all normal extremals is parameterized by points of the cylinder parameter of pendulum

\[
\mathcal{C} = \{ \lambda \in T^* M | H(\lambda) = \frac{1}{2} \} = \{ (\theta, c, \alpha) | \theta \in S^1, c, \alpha \in \mathbb{R} \},
\]

and is given by the exponential mapping

\[
\exp : N = \mathbb{C} \times |_{\mathbb{R}} \to M, \quad \exp (\lambda, t) = q_t = (q_t(y, z, v), y, z, v),
\]

where \( q_t = (x, y, z, v) \) is a critical point of the exponential mapping and that is why \( q_t \) is the corresponding critical value.

\[
\exp : T^* N \to T^* M = \text{degenerate}, \quad \exp (\lambda, t), \quad \exp (\lambda, t) = \text{conjugate time along extremal trajectory}
\]

Note that in this case is called a conjugate time used in this work was successfully applied earlier to Euler’s elastic problem [7] and sub-Riemannian problem on the group of rototranslations [8]. There is no doubt that this method is also valid for nilpotent sub-Riemannian problem with the growth vector (2.3.0) [9, 10, 11, 12]. The method can be used for other invariant sub-Riemannian problems on Lie groups of low-dimensional integrable in non-elementary functions. The first natural step in this direction is investigation of invariant sub-Riemannian 3D Lie groups which are classified by A.A. Agrachev and D.Baranov [13].

**System of algebraic equations**

In order to compute the optimal trajectory for a given terminal point \( (x_1, y_1, z_1, v_1) \), the following system of algebraic equations should be solved:

\[
\begin{align*}
0 &= \{ k(\lambda, k, \alpha) - x_1, \\
0 &= \{ y(\lambda, k, \alpha) - y_1, \\
0 &= \{ z(\lambda, k, \alpha) - z_1, \\
0 &= \{ v(\lambda, k, \alpha) - v_1.
\end{align*}
\]

Using one symmetry (dilations) the system (9) was reduced to the system with three algebraic equations in three unknown variables:

\[
\begin{align*}
0 &= \{ \phi(\lambda, k, \alpha) - Y_1, \\
0 &= \{ \phi(\lambda, k, \alpha) - Z_1, \\
0 &= \{ \phi(\lambda, k, \alpha) - V_1.
\end{align*}
\]

**BIBLIOGRAPHY**


