Stochastic orders: a brief introduction and Bruno's contributions.

Franco Pellerey

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In this presentation:

- A short introduction to the subject
- A description of some of his contributions
- An example of recent applications of his studies

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$$Y = \begin{cases} 2 & \text{with probability } 1/2 \\ 3 & \text{with probability } 1/2 \end{cases} \quad \rightarrow E[Y] = 2.5$$

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Here we can not immediately assert which one is the best choice.

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In fact, a long list of different stochastic orders have been defined, and applied, in a variety of fields (economics and finance, engineering, medicine, etc.)

They are based on comparisons between:

• The *location*, or *magnitude*, of the random variables to be compared (mainly considered in reliability and survival analysis)

Ex: X is better than Y if $P[X > t] \geq P[Y > t]$ for all $t \in \mathbb{R}$ (*usual stochastic order*)

Stochastic orders: what are they?

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- Their variability

Ex: X is more dispersed than Y if $X = \phi(Y)$ for some dispersive function ϕ, i.e., such that $\phi(x_2) - x_2 \geq \phi(x_1) - x_1$ whenever $x_1 \leq x_2$ (*dispersive order*).

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comparisons of stochastic processes.

Stochastic comparisons of stochastic processes

- B.B. and Marco Scarsini. **Convex orderings for stochastic processes.** *Comment. Math. Univ. Carolin.* 32 (1991), no. 1, 115–118.

- B.B. and Marco Scarsini. **On the comparison of pure jump processes.** *RAIRO Rech. Opér.* 26 (1992), no. 1, 41–55.

- B.B., Erhan Cinlar and Marco Scarsini. **Stochastic comparisons of Itô processes.** *Stochastic Process. Appl.* 45 (1993), no. 1, 1–11.

- B.B. and Marco Scarsini. **Positive dependence orderings and stopping times.** *Ann. Inst. Statist. Math.* 46 (1994), no. 2, 333–342.

Multivariate stochastic models

- B.B. and Fabio Spizzichino. **Stochastic comparisons for residual lifetimes and Bayesian notions of multivariate ageing.** *Adv. in Appl. Probab.* 31 (1999), no. 4, 1078–1094.

- B.B. and Fabio Spizzichino. **Dependence and multivariate aging: the role of level sets of the survival function.** System and Bayesian reliability, 229–242, *Ser. Qual. Reliab. Eng. Stat.*, 5, World Sci. Publ., 2001.

- B.B., Subhash Kochar and Fabio Spizzichino. **Some bivariate notions of IFR and DMRL and related properties.** *J. Appl. Probab.* 39 (2002), no. 3, 533–544.

- B.B. and Fabio Spizzichino. **Bivariate survival models with Clayton aging functions.** *Insurance Math. Econom.* 37 (2005), no. 1, 6–12.

- B.B. and Fabio Spizzichino. **Relations among univariate aging, bivariate aging and dependence for exchangeable lifetimes.** *J. Multivariate Anal.* 93 (2005), no. 2, 313–339.

Lifetimes and stochastic orders

Let X and Y represent the lifetimes of two items, or individuals.

Recall: X is said to be smaller in the *usual stochastic order* than Y if and only if $P[X > t] \leq P[Y > t] \quad \text{for all } t \in \mathbb{R}^+$, or, equivalently, if and only if $E[\phi(X)] \leq E[\phi(Y)]$ for every increasing function ϕ.

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Apart the usual stochastic order, other comparisons are commonly considered in survival analysis. In particular, different orders have been defined and studied to compare residual lifetimes, i.e., to compare the variables $[X - t|X > t]$ and $[Y - t|Y > t]$, for $t \in \mathbb{R}^+$.

Among them, the *hazard rate order*, the *mean residual life order*, the *likelihood ratio order*, etc.

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By means of these orders, it is possible to provide a list of alternative characterizations of aging, i.e., of the effect of the age in the residual lifetime of the items.

Ex: X has *Increasing Hazard Rate (IHR)* if $[X - t_1|X > t_1] \geq_{st} [X - t_2|X > t_2]$ for all $0 \leq t_1 \leq t_2$, i.e., if the residual lifetime of X stochastically decreases with the age.

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A clear understanding of the relationships between univariate aging, multivariate aging and dependence

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• The random vector (X, Y) satisfies the *Bivariate Increasing Hazard Rate* (BIHR) property if, and only if, for all $0 \leq t_1 \leq t_2$,

$$[(X - t_1, Y - t_1) | \{X > t_1, Y > t_1\}] \geq_{st} [(X - t_2, Y - t_2) | \{X > t_2, Y > t_2\}].$$

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• Let us say now that the lifetimes X and Y have *positive dependence* if large (respectively, small) values of X tend to go together (in some stochastic sense) with large (respectively, small) values of Y. Similarly define the *negative dependence*: if large (respectively, small) values of X tend to go together (in some stochastic sense) with small (respectively, large) values of Y. [Formal definitions can be given by means of the *copula* of the vector (X, Y).]

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multivariate negative aging;

positive dependence + multivariate positive aging

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univariate positive aging;

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etc..

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multivariate positive aging + univariate positive aging

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positive dependence

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- Defined and studied for the first time by Shantikumar and Yao (1991), Bivariate characterization of some stochastic order relations. *Adv. in Appl. Prob.*, 23, 642–659, the joint stochastic orders have been introduced in order to compare, marginally, univariate lifetimes, but taking into account their possible mutual dependence.

Recently....

Ex: Consider a pair (X, Y) of non-independent lifetimes. It is possible to compare the corresponding residual lifetimes considering the inequalities $[X - t | X > t] \leq_{st} [Y - t | Y > t]$ for all $t \in \mathbb{R}^+$ (*hazard rate order*, $\leq_{hr}$). This is a comparison based on the marginal distributions of X and Y.

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Otherwise, it is possible to compare their residual lifetimes taking into account survivals of both, i.e., consider the following inequalities:

$$[X - t|\{X > t, Y > t\}] \leq_{st} [Y - t|\{X > t, Y > t\}] \quad \text{for all } t \in \mathbb{R}^+.$$

This is a joint stochastic order based on the joint distribution of (X, Y) (*joint weak hazard rate order*, $\leq_{hr:wj}$).

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It can be shown that the stochastic orders above are not equivalent, i.e.,

$$X \leq_{hr} Y \not\Leftrightarrow X \leq_{wj:hr} Y.$$

Actually, the equivalence holds if, and only if, X and Y are independent.

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Ex: Let (X, Y) be such that $X \leq_{hr:wj} Y$. It can be shown that in this case

$$E[g(X, Y)] \geq E[g(Y, X)]$$

for all function $g : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\Delta g(x, y) = g(x, y) - g(y, x)$$

is supermodular in the set $\mathcal{U} = \{(x, y) \in \mathbb{R}^2, x \geq y\}$.

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Consider now a pair (X, Y) of non-independent financial returns, and consider the two portfolios

$$Z_1 = \alpha Y + (1-\alpha)X \text{ and } Z_2 = (1-\alpha)Y + \alpha X, \quad \alpha \in [0,1].$$

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Because of the previous property, for $\alpha \geq 1/2$ one has

$$X \leq_{hr:wj} Y \implies E[u(Z_1)] \geq E[u(Z_2)]$$

for every logarithmic utility function u (like, e.g., $u(t) = \ln(1+t)$).

[The proof easily follows letting $g(x,y) = u(\alpha y + (1-\alpha)x)$.]

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Another interesting property of the joint weak hazard rate order is that it is closed with respect to mixtures (while the standard hazard rate order is not)

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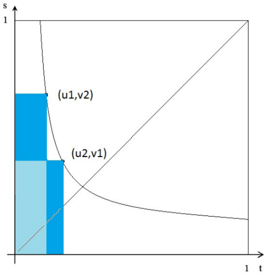
Definition

A bivariate copula $C : [0,1]^2 \longrightarrow [0,1]$ is called *supermigrative* (*submigrative*) if it is symmetric, i.e. $C(u,v) = C(v,u)$ for every $(u,v) \in [0,1]^2$, and it satisfies

$$C(u, \gamma v) \geq (\leq) \; C(\gamma u, v)$$

for all $u \leq v$ and $\gamma \in (0,1)$.

Recently....



As shown by Bruno and Fabio, examples of copulas satisfying the supermigrative property are the Archimedean ones having log-convex inverse generators. Thus, e.g., the Gumbel-Hougaard copula, and the Frank and Clayton copulas with positive value of the parameter θ. The Farlie-Gumbel-Morgenstern copula, with posite value of the parameter, satisfies this property as well.

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Theorem

Let the survival copula $\overline{C}_{(X,Y)}$ *of* (X, Y) *be supermigrative. Then* $X \leq_{hr} Y \implies X \leq_{hr:wj} Y$*. Viceversa, if the survival copula* $\overline{C}_{(X,Y)}$ *of* (X, Y) *is submigrative, then* $X \leq_{hr:wj} Y \implies X \leq_{hr} Y$*.*

A final remark

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Definition

Let $\mathcal{D}$ be the set of functions $\mathcal{D} = \{g \mid g : \mathbb{R}^2 \to \mathbb{R}\}$. We denote:

- $\mathcal{G}_{hr} = \{g \in \mathcal{D} \mid g(x, y) - g(y, x) \text{ is non-decreasing in } x \; \forall y \leq x\}$;
- $\mathcal{G}_{lr} = \{g \in \mathcal{D} \mid g(x, y) \geq g(y, x) \text{ for all } y \leq x\}$.

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Definition

Given two random variables X and Y as above, X is said to be greater than Y

- in the *joint hazard rate order* (denoted $X \geq_{hr:j} Y$) if $E[g(X,Y)] \geq E[g(Y,X)]$ for all $g \in \mathcal{G}_{hr}$;
- in the *joint likelihood ratio order* (denoted $X \geq_{lr:j} Y$) if $E[g(X,Y)] \geq E[g(Y,X)]$ for all $g \in \mathcal{G}_{lr}$.

A final remark

Recall the definition of standard *joint likelihood ratio*:

$$X \geq_{lr} Y \iff [X - t|\ t \leq X \leq t + s] \geq_{st} [Y - t|\ t \leq Y \leq t + s]$$

for all $t \in \mathbb{R}, s \in \mathbb{R}^+$.

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Theorem

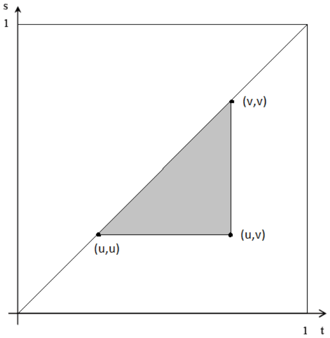
Let (X, Y) be a random vector with an absolutely continuous survival copula $\widehat{C}_{(X,Y)}$. Assume that

- *$\widehat{C}_{(X,Y)}$ is exchangeable (symmetric);*
- *The density $\widehat{c}_{(X,Y)}(u, v)$ is non-increasing in u and non-decreasing in v for all $u \geq v$.*

Then $X \geq_{lr}\ [\geq_{hr}]\ Y$ implies $X \geq_{lr:j}\ [\geq_{hr:j}]\ Y$.

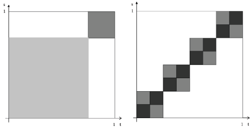
A final remark

The second assumption appearing in this statement has an immediate interpretation: it essentially means that for every point (u, v) below the diagonal and for any point $(\hat{u}, \hat{v})$ contained in the closed triangle with vertices (u, v), (u, u) and (v, v) it holds $\widehat{c}_{(X,Y)}(\hat{u}, \hat{v}) \geq \widehat{c}_{(X,Y)}(u, v)$ (and similarly for points above the diagonal).



A final remark

Roughly speaking, it is satisfied by copulas having probability mass mainly concentrated around the diagonal, thus describing *positive dependence*, as the copulas having density uniformly distributed on regions like the ones shown in the following figure.



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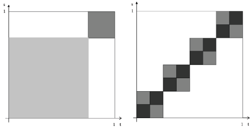
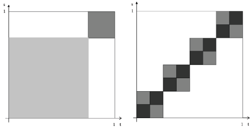


Such an assumption can be further weakened in order to let the relation $X \geq_{lr} Y \;\Rightarrow\; X \geq_{lr:j} Y$ be satisfied also for survival copulas having a singularity on the diagonal $v = u$, like Cuadras-Augé copulas or Frechét copulas.

A final remark

Roughly speaking, it is satisfied by copulas having probability mass mainly concentrated around the diagonal, thus describing *positive dependence*, as the copulas having density uniformly distributed on regions like the ones shown in the following figure.



Such an assumption can be further weakened in order to let the relation $X \geq_{lr} Y \;\Rightarrow\; X \geq_{lr:j} Y$ be satisfied also for survival copulas having a singularity on the diagonal $v = u$, like Cuadras-Augé copulas or Frechét copulas.

A larger class of copulas satisfying this property has been defined and studied in Durante (2006), A new class of symmetric bivariate copulas, *J of Nonparametrical Statistics*, **18**, 499–510.