The origin of the development of semi–copulas and their applications

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Outline

1. Introduction
2. Determination of the multivariate ageing function
3. Multivariate ageing

From univariate to multivariate ageing

Univariate notions of aging (e.g., IFR, DFR, NBU) constitute a well–established core of reliability theory: the definitions given in the, by–now classical, literature are very clear and provide the basis for many useful results, which apply when dealing with the analysis of a single unit or of several units with stochastically independent lifetimes.

On the contrary, multivariate definitions of aging, i.e. definitions valid when dealing with several dependent lifetimes, are rather controversial. In fact, starting from univariate notions, several types of multivariate extensions can be defined; furthermore, the non-trivial interactions between aging and dependence contribute to make the analysis quite challenging.

(Bassan and Spizzichino, JMVA, 2005)

From univariate to multivariate ageing

A first natural assumption for dealing with this problem is to consider dependent lifetimes that are judged to be similar, viz., one is interested on a vector $\mathbf{X} = (X_1, , \dots , X_d)$ of exchangeable positive and continuous rv's (also called lifetimes).

Here, we suppose that:

- $\mathbf{X}$ has survival function (=sf) $\overline{F} : \mathbb{R}^d \to \mathbb{I}$,
- for every i, X_i has sf $\overline{G} : \mathbb{R} \to \mathbb{I}$, which is continuous and strictly decreasing with $\overline{G}(0) = 1$;
- $K = K_{\overline{F}}$ is the unique copula associated with $\mathbf{X}$.

Level curves and multivariate ageing

Following the ideas by Barlow and Mendel (JASA, 1992) and Barlow and Spizzichino (JCAM, 1993), the level curves $\mathcal{L}_{\overline{F}}$ of $\overline{F}$ are a primary object of interest in (Bayesian) notion of ageing.

This approach led to the the introduction of concepts like semi–copula (and semi sf's) and multivariate ageing function, introduced by Bassan and Spizzichino (JMVA, 2005).

Remark: $\mathcal{L}_{\overline{F}} = \{A_\alpha\}_{\alpha\in\mathbb{I}}$, where $A_\alpha = \left\{\mathbf{x} \in \mathbb{R}^d \mid \overline{F}(\mathbf{x}) = \alpha\right\}$.

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Level curves for describing ageing

In order to study the ageing properties of $\overline{F}$ by means of level curves, it is convenient to consider an auxiliary function $\overline{L}$, having the same level curves as $\overline{F}$ but, possibly, different marginals.

$\overline{L}$ belongs to the family of continuous semi-sf's $\overline{F} : \mathbb{R}^d \to \mathbb{I}$ such that

(a) $\overline{F}$ is strictly decreasing in each argument;

(b) $\overline{F}(0, \dots, 0) = 1$;

(c) $\overline{F}(x_1, \dots, x_d) \to 0$ when $\max\{x_1, \dots, x_d\} \to +\infty$.

However, in general, $\overline{F}$ does not satisfy the d–increasing property (and, so, it may not be a probability sf).

Actually, a semi–sf can be associated to a capacity.

Level curves for describing ageing

Now, all the semi–sf's sharing the same family of level curves of $\overline{F}$ may be obtained by means of a distortion of $\overline{F}$.

Theorem

Let $\overline{F}_1$ and $\overline{F}_2$ be semi sf's on $\mathbb{R}^d$ with families of level curves given, respectively, by $\mathcal{L}_{\overline{F}_1}$ and $\mathcal{L}_{\overline{F}_2}$. Then the following statements are equivalent:

(a) $\mathcal{L}_{\overline{F}_1} = \mathcal{L}_{\overline{F}_2}$;

(b) *there exists an increasing bijection $h : \mathbb{I} \to \mathbb{I}$ such that* $\overline{F}_2 = h \circ \overline{F}_1$.

(Durante and Spizzichino, 2010)

Describing ageing by means of level curves

Corollary

Let $\overline{F} : \mathbb{R}^d \to \mathbb{I}$ be a joint sf and let $h : \mathbb{I} \to \mathbb{I}$ be an increasing bijection. Let $\mathbb{P}_{\overline{F}}$ be the probability measure induced by $\overline{F}$. Then $\overline{L} := h \circ \overline{F}$ is the joint semi–sf associated to the distorted probability $h \circ \mathbb{P}_{\overline{F}}$, i.e.

$$\overline{L}(x_1, \dots, x_d) = h \circ \mathbb{P}_{\overline{F}} \left([x_1, +\infty] \times \cdots \times [x_d, +\infty] \right).$$

(Durante and Spizzichino, 2010)

The properties of the level curves of $\overline{F}$, and consequently the ageing property of $\overline{F}$, may also be investigated in the more general framework of distorted probabilities and associated joint survival semi–sf's.

Describing ageing by means of level curves–2

As a specific choice for a distortion of $\overline{F}$ that is specially adapt to describe the ageing property of $\overline{F}$ in terms of level curves, one may select

$$\overline{L}(x_1, \dots, x_d) = \exp\left(-\overline{G}^{-1}\left(\overline{F}(x_1, \dots, x_d)\right)\right)$$

Note that:

- if two joint sf's $\overline{F}_1$ and $\overline{F}_2$ have the same level curves, then they share the same $\overline{L}$;
- $\overline{L}(x, 0, \dots, 0) = e^{-x}$ for every $x \in \mathbb{R}$;
- $\overline{L} = \overline{F}$, when the survival marginal $\overline{G}$ of $\overline{F}$ is exponential.

Intuitively, $\overline{L}$ is the semi–sf that allows to express the behaviour of the level curves of a joint sf by reducing it to a function having marginals indifferent to univariate ageing.

Semicopulæ

A semi–copula is a function $S : \mathbb{I}^d \to \mathbb{I}$ such that

- S is increasing in each place,
- $S(1, \dots, 1, x_i, 1 \dots, 1) = x_i$.

Semi–copulæ generalise several other concepts already used in the literature, like copulas, quasi–copulas and triangular norms.

Sklar's Theorem for semi–sf's

Theorem

Let $\overline{F} : \mathbb{R}^d \to \mathbb{I}$ *be a semi–sf with univariate marginals* $\overline{F}_1, \dots, \overline{F}_d$. *Suppose that, for every* $i \in \{1, \dots, d\}$, $\overline{F}_i$ *is strictly decreasing on* $[0, +\infty[$ *and continuous. Then there exists a unique semi–copula* $S = S_{\overline{F}} : \mathbb{I}^d \to \mathbb{I}$ *such that*

$$\forall\, \mathbf{x} \in \mathbb{R}^d \qquad \overline{F}(x_1, \dots, x_d) = S\left(\overline{F}_1(x_1), \dots, \overline{F}_d(x_d)\right) .$$

(Durante and Spizzichino, 2010)

Definition of multivariate ageing function

Definition (Bassan-Spizzichino, 2001)

Given a sf $\overline{F}$, the multivariate ageing function $B = B_{\overline{F}}$ is the unique semi–copula associated with

$$\overline{L}(\mathbf{x}) = \exp\left(-\overline{G}^{-1}\left(\overline{F}(\mathbf{x})\right)\right)$$

It is explicitly given by

$$B(\mathbf{u}) = \exp\left(-\overline{G}^{-1}\left(\overline{F}(-\log u_1, \dots, -\log u_d)\right)\right)$$

Two comments

Two semi–copulæ are associated with each joint sf $\overline{F}$:

- K, interpreting the dependence properties of $\overline{F}$,
- B, interpreting the ageing properties of $\overline{F}$.

$$B(\mathbf{u}) := \exp\left(-\overline{G}^{-1}\left(K_{\overline{F}}(\overline{G}(-\log u_1), \dots, \overline{G}(-\log u_d))\right)\right)$$

is also a suitable transformation of the copula $K_{\overline{F}}$.

The example of Archimedean copulas

Let $\overline{F}$ be a bivariate sf whose copula K is a (strict) Archimedean copula additively generated by a convex and decreasing function $\varphi : \mathbb{I} \to [0, +\infty]$ with $\varphi(1) = 0$ and $\varphi(0) = +\infty$, viz.,

$$K(u, v) = \varphi^{-1} (\varphi(u) + \varphi(v))$$

Then the related bivariate ageing function B is given by

$$B(u, v) = t^{-1} (t(u) + t(v)) \ ,$$

where $t = \varphi \circ \overline{G} \circ (-\log)$. B is a copula iff t is convex.

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The procedure

As argued for the first time by Bassan and Spizzichino (2001), one can define multivariate ageing notions in terms of the level curves of $\overline{F}$, and, hence, of the semi–copula $B_{\overline{F}}$, by means of the following scheme.

- Consider a univariate ageing notion P.
- Take the joint survival $\overline{F} = \overline{F}_{\Pi,\mathcal{G}}$ of d iid lifetimes and prove results of this type: each lifetime has the property P iff $B_{\Pi,\overline{G}}$ satisfies the property $\widetilde{P}$;
- Define a multivariate ageing property as follows:
  an exchangeable sf $\overline{F}$ is multivariate–P if $B_{\overline{F}}$ has the property $\widetilde{P}$.

Special classes of multivariate ageing functions

Let B be a multivariate ageing function. We say that:

(A1) $B \in \mathcal{A}_1^+$ iff, for every $\mathbf{u} \in \mathbb{I}^d$

$$B(\mathbf{u}) \geq \Pi_d(\mathbf{u}) \,.$$

(A2) $B \in \mathcal{A}_2^+$ iff for all $i, j \in \{1, \ldots, d\}$, $i \neq j$, and for every $\mathbf{u} \in \mathbb{I}^d$,

$$B(u_1, \ldots, u_i, \ldots, u_j, \ldots, u_d) \geq B(u_1, \ldots, u_i u_j, \ldots, 1, \ldots, u_d) \,.$$

(A3) $B \in \mathcal{A}_3^+$ iff for all $i, j \in \{1, \ldots, d\}$, $i \neq j$, for all $u_i, u_j \in \mathbb{I}$, $u_i \geq u_j$, and for every $s \in \,]0, 1[$,

$$B(u_1, \ldots, u_i s, \ldots, u_j, \ldots, u_d) \geq B(u_1, \ldots, u_i, \ldots, u_j s, \ldots, u_d) \,.$$

Obviously. . .

$$\mathcal{A}_3^+ \subset \mathcal{A}_2^+ \subset \mathcal{A}_1^+$$

Multivariate ageing for iid lifetimes–1

Theorem

The following statements are equivalent for the joint sf $\overline{F}$ of a vector of independent lifetimes with univariate marginal $\overline{G}$:

(a) $\overline{G}$ *is New Better than Used* (=NBU), *i.e.,* $\overline{G}(x+y) \leq \overline{G}(x)\,\overline{G}(y)$;

(b) $B_{\Pi,\overline{G}} \in \mathcal{A}_1^+$,

(c) $B_{\Pi,\overline{G}} \in \mathcal{A}_2^+$.

(Durante, Foschi and Spizzichino, 2010)

Multivariate ageing for iid lifetimes–2

Theorem

The following statements are equivalent for the joint sf $\overline{F}$ of a vector of independent lifetimes with univariate marginal $\overline{G}$:

(a) $\overline{G}$ *is Increasing Failure Rate (=IFR), i.e., $\overline{G}$ is log–concave;*

(b) $B_{\Pi,\overline{G}} \in \mathcal{A}_3^+$.

(Durante, Foschi and Spizzichino, 2010)

Definitions of multivariate ageing

Let $\mathbf{X} = (X_1, \ldots, X_d)$ be an exchangeable random vector $(d \geq 2)$ of continuous lifetimes with joint survival function $\overline{F} : \mathbb{R}^d \to \mathbb{I}$ and univariate survival marginals equal to $\overline{G}$. We suppose that $\overline{G}$ is strictly decreasing on $\mathbb{R}_+$ with $\overline{G}(0) = 1$ and $\overline{G}(+\infty) = 0$.

One says that:

- $\overline{F}$ is B–multivariate–NBU of the first type (shortly, B–MNBU1) iff $B_{\overline{F}} \in \mathcal{A}_1^+$;
- $\overline{F}$ is B–multivariate–NBU of the second type (shortly, B–MNBU2) iff $B_{\overline{F}} \in \mathcal{A}_2^+$;
- $\overline{F}$ is B–multivariate–IFR (shortly, B–MIFR) iff $B_{\overline{F}} \in \mathcal{A}_3^+$.

A remark

Any k–dimensional marginal of $\overline{F}$ $(k = 2, \ldots, d-1)$ has the same multivariate ageing property of $\overline{F}$, as formalised here.

Theorem

Let $\overline{F}^{(k)}$ be the k–dimensional marginal of $\overline{F}$ $(k \in \{2, \ldots, d\})$.
If $\overline{F}$ is B–MNBU1 (respectively, B–MNBU2 or B–MIFR),
then $\overline{F}^{(k)}$ is B–MNBU1 (respectively, B–MNBU2 or B–MIFR).

(Durante, Foschi and Spizzichino, 2010)

Probabilistic interpretations

Theorem

(a) $\overline{F}$ *is* B*–MNBU1 iff for all* $i \in \{1, \dots, d\}$, $\mathbf{x} \in \mathbb{R}^d$ *and* $\tau > 0$,

$$\mathbb{P}\left(X_1 > x_1, \dots, X_i > x_i + \tau, \dots X_d > x_d \mid X_i > x_i\right) \\ \geq \mathbb{P}\left(X_i > x_1 + \dots + x_i + \dots + x_d + \tau \mid X_i > x_i\right) .$$

(b) $\overline{F}$ *is* B*–MNBU2 iff for all* $i, j \in \{1, \dots, d\}$, $i \neq j$, $\hat{\mathbf{x}}_j = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_d) \in \mathbb{R}^{d-1}$ *and* $\tau > 0$,

$$\mathbb{P}\left(X_j > \tau \mid \hat{\mathbf{X}}_j > \hat{\mathbf{x}}_j\right) \geq \mathbb{P}\left(X_i > \tau + x_i \mid \hat{\mathbf{X}}_j > \hat{\mathbf{x}}_j\right) .$$

(Durante, Foschi and Spizzichino, 2010)

In the univariate case...

Let X be a lifetime, $X \sim \overline{G}$. Then $\overline{G}$ is NBU iff for all $x, \tau \in \mathbb{R}$

$$\mathbb{P}(X > \tau) \geq \mathbb{P}(X > \tau \mid X > x)\,.$$

Probabilistic interpretations

Some of these properties may then be expressed as comparisons between residual lifetimes, conditionally on a same history.

- $\overline{F}$ is B-MNBU2 iff

$$\left[X_i \mid \hat{\mathbf{X}}_i > \hat{\mathbf{x}}_i\right] \geq_{st} \left[X_j - x_j \mid \hat{\mathbf{X}}_i > \hat{\mathbf{x}}_i\right] ,$$

  for all $i, j \in \{1, \dots, d\}$, $i \neq j$, and for every $\mathbf{x} \in \mathbb{R}^d$

- $\overline{F}$ is B–MIFR iff

$$[X_i - x_i \mid \mathbf{X} > \mathbf{x}] \geq_{st} [X_j - x_j \mid \mathbf{X} > \mathbf{x}] ,$$

  for all $i, j \in \{1, \dots, d\}$, for every $\mathbf{x} \in \mathbb{R}^d$ such that $x_i \leq x_j$.

Preservation of ageing under mixtures

Theorem

Let $(\overline{F}_\theta)_{\theta\in\Theta}$ be a family of sf's, λ be a distribution on Θ and $\overline{F}$ the mixture of $(\overline{F}_\theta)_{\theta\in\Theta}$ with respect to λ, given, for every $\mathbf{x} \in \mathbb{R}^d$, by

$$\overline{F}(\mathbf{x}) = \int_\Theta \overline{F}_\theta(\mathbf{x})\mathrm{d}\lambda(\theta)\,.$$

The following statements hold:

(a) *if $\overline{F}_\theta$ is B–MNBU1 for every $\theta \in \Theta$, then $\overline{F}$ is B–MNBU1;*

(b) *if $\overline{F}_\theta$ is B–MNBU2 for every $\theta \in \Theta$, then $\overline{F}$ is B–MNBU2;*

(c) *if $\overline{F}_\theta$ is B–MIFR for every $\theta \in \Theta$, then $\overline{F}$ is B–MIFR.*

(Durante, Foschi and Spizzichino, 2010)

Preservation of ageing under mixtures–2

Theorem

Suppose that $\overline{F}$ is the survival function of conditionally iid lifetimes given a common factor Θ with prior distribution λ.
If $\overline{G}(\cdot \mid \theta)$ is NBU (respectively, IFR), then $\overline{F}$ is B–MNBU2 (respectively, B–MIFR).

(Durante, Foschi and Spizzichino, 2010)

As a consequence...mixtures of i.i.d. lifetimes that are NBU (respectively, IFR) conditionally on the same factor Θ, are also multivariate NBU (respectively, IFR).

Conclusions

Several notions of multivariate ageing for a vector $\mathbf{X}$ of exchangeable lifetimes have been introduced by using the multivariate ageing function, a semi–copula expressing properties of the level curves of the survival function $\overline{F}_{\mathbf{X}}$.

The notions introduced:

- are preserved under marginalization;
- are preserved under mixtures;
- have some interesting probabilistic interpretations.

Ageing and Dependence

This new approach seems especially suitable for introducing multivariate statistical models interpreting ageing in a very flexible way.

Theorem

Let $\overline{F} = K(\overline{G}, \dots, \overline{G})$ be a joint survival function.

(a) *If $K \in \mathcal{A}_1^+$ and $\overline{G}$ is NBU, then $\overline{F}$ is B–MNBU1.*

(b) *If $K \in \mathcal{A}_2^+$ and $\overline{G}$ is NBU, then $\overline{F}$ is B–MNBU2.*

(c) *If $K \in \mathcal{A}_3^+$ and $\overline{G}$ is IFR, then $\overline{F}$ is B–MIFR.*

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