A numerical method for nonlinear diffusion + obstacle equation

Olivier Bokanowski
Laboratory Jacques Louis Lions
University Paris-Diderot (Paris 7)

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Numerical methods for PDEs: optimal control, games and image processing.

On the occasion of the 60th birthday of Maurizio Falcone

Plan

- I. Motivation
- II. Howard's algorithm
- III. Attempts & Numerical results

I. INTRODUCTION

Motivation

- American option (obstacle problem), or stopping time problems for optimal stochastic control:

$$\min(u_t - u_{xx}, u - g(x)) = 0$$

- Obstacle for treatment of state-constraints in optimal control:

$$\min(u_t + H(x, u_x), u - g(x)) = 0$$

$\Rightarrow$ Hamilton Jacobi Bellman (HJB) or Hamilton-Jacobi-Isaac (HJI) equations with obstacle terms

simple obstacle problem

- **American option pb:** $v(t,x) = \sup_{\tau \in \mathcal{T}_{[0,t]}} \mathbb{E}[g(X_\tau^{0,x})]$ (with $dX_\theta = \sigma dW_\theta$)

$$\min(v_t - \frac{\sigma^2}{2} v_{xx},\ v - g(x)) = 0, \quad t \in (0,T),\ x \in (0,1),$$
$$v(0,x) = v_0(x) \equiv g(x)$$

- We assume dirichlet boundary conditions to simplify
- **Explicit scheme:** (finite difference scheme)

$$\min\left(\frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{\sigma^2}{2}\left(\frac{-u_{i-1}^n + 2u_i^n - u_{i-1}^n}{\Delta x^2}\right),\ u_i^{n+1} - g_i\right) = 0,$$
$$1 \leq i \leq I$$

$$u_0^{n+1} = u_{I+1}^{n+1} = 0 \quad \text{(or given values)}$$

- **Linear case**

$$v_t - \frac{\sigma^2}{2} v_{xx} = 0$$

- **Explicit scheme:**

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{\sigma^2}{2}\left(\frac{-u_{i-1}^n + 2u_i^n - u_{i-1}^n}{\Delta x^2}\right) = 0 \quad 1 \leq i \leq I$$

hence

$$u_i^{n+1} = k u_{i-1}^n + (1-2k)u_i^n + k u_{i+1}^n \equiv (Su^n)_i \quad k := \frac{\sigma^2}{2}\frac{\Delta t}{\Delta x^2}.$$

- CONSISTENCY: $\frac{v^{n+1} - Sv^n}{\Delta t} \equiv O(\Delta t) + O(\Delta x^2)$
- STABILITY : **CFL condition** $\boxed{2k \leq 1} \Rightarrow \|U^{n+1}\|_\infty \leq \|U^n\|_\infty$

• **IMPLICIT scheme:**

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{\sigma^2}{2}\left(\frac{-u_{i-1}^{\mathbf{n+1}} + 2u_i^{\mathbf{n+1}} - u_{i-1}^{\mathbf{n+1}}}{\Delta x^2}\right) = 0 \quad 1 \leq i \leq I$$

$\Rightarrow AU^{n+1} = U^n$, with

$$A = \begin{bmatrix} 1+2k & -k & & \\ -k & \ddots & \ddots & \\ & & \ddots & -k \\ & & -k & 1+2k \end{bmatrix} \quad \text{and } k := \frac{\sigma^2}{2}\frac{\Delta t}{\Delta x^2} \geq 0.$$

• CONSISTENCY: idem, $O(\Delta t) + O(\Delta x^2)$

• STABILITY : **NO CFL condition !**

$$A \text{ "}\delta\text{-diag. dominant"} \Rightarrow \boxed{\|A^{-1}\|_\infty \leq \frac{1}{\delta} \leq 1} \Rightarrow \|U^{n+1}\|_\infty \leq \|U^n\|_\infty$$

• **American option, <u>implicit</u>** **Can we do the same ?**

Implicit finite difference scheme

$$\min\left(\frac{u_i^{n+1}-u_i^n}{\Delta t}-\frac{\sigma^2}{2}\left(\frac{-u_{i-1}^{n+1}+2u_i^{n+1}-u_{i-1}^{n+1}}{\Delta x^2}\right),\ u_i^{n+1}-g_i\right)=0,$$

$$1\le i\le I$$

After multiplication of the left part of the min by $\Delta t>0$, we get:

$$\min\Big(\underbrace{(1+2k)u_i^{n+1}-ku_{i-1}^{n+1}-ku_{i+1}^{n+1}}_{=(Au^{n+1})_i}-\underbrace{u_i^n}_{\equiv b_i},\ u_i^{n+1}-\underbrace{g(x_i)}_{\equiv g_i}\Big)=0$$

$$\Leftrightarrow\quad \text{find } x=U^{n+1},\quad \min((Ax-b)_i,x_i-g_i)=0,\quad 1\le i\le I$$

• STABILITY : **NO CFL condition !**

$$A\ \text{"}\delta\ge 1\text{-diag. dominant"}\Rightarrow \|U^{n+1}\|_\infty\le\max(\|U^n\|_\infty,\|g\|_\infty)$$

II. HOWARD'S ALGORITHM

• Nice discrete scheme, but nonlinear !

$$\boxed{\min(Bx - b, x - g) = 0, \quad x \in \mathbb{R}^N}$$

Obstacle PB

• More general: Merton's portfolio problem: [1]

$$v(T-t,x) := \operatorname*{ess\,sup}_{\alpha:(t,T)\to\mathcal{K}} \mathbb{E}[\varphi(X_T^{t,x,\alpha})|\mathcal{F}_t], \quad \boxed{\mathcal{K} = [0,1]}$$

$$\max_{a\in K}\left(v_t - \frac{1}{2}a^2x^2v_{xx} - (a\mu + (1-a)r)xv_x\right) = 0.$$

• **Implicit** finite difference scheme : we get a matrix B_a depending of the parameter a, and the implicit scheme

$$\boxed{\max_{a\in K}(B_ax - b_a) = 0, \quad x \in \mathbb{R}^N}$$

Can we solve this ?

[1] with $\frac{dX_\theta}{X_\theta} = (\mu\alpha + (1-\alpha)r)d\theta + \alpha\sigma dW_\theta$

Howard's algorithm (1958)

• **Definition 1.** For $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathcal{K}^N$, consider

$$B(\alpha)_{ij} := (B_{\alpha_i})_{ij} \quad \text{and} \quad b(\alpha)_i = (b_{\alpha_i})_i.$$

• Then $F(x) \equiv \min_{a\in\mathcal{K}}(B_a x - b_a) \equiv \min_{\alpha\in\mathcal{K}^N}(B(\alpha)x - b(\alpha))$
• Obstacle problem:

$$\mathcal{K} = \{0,1\},\ (B_0, b_0) = (B, b),\ (B_1, b_1) = (I, g)$$

• **Definition 2: Howard's algorithm (H)** for solving $F(x) = 0$:
Starting from a given $x^0 \in \mathbb{R}^N$, iterate for $k \geq 0$:

$$(\mathbf{H}) \begin{cases} \text{Compute } \alpha_i^{k+1} := argmin_{\alpha\in\mathcal{K}^N}(B(\alpha)x^k - b(\alpha))_i, \\ \text{Compute } x^{k+1} \text{ s.t. } B(\alpha^{k+1}x^{k+1}) - b(\alpha^{k+1}) = 0. \end{cases}$$

until some stopping criteria is satisfied.

Proposition

Howard's algorithm (H) and Newton's method (N) are the same

We cannot apply directly Newton's method since $F(x)$ is only Lipschitz and not twice differentiable.

- **Assumption (M):**

$$\begin{cases} (i)\ \alpha \to B(\alpha),\ \alpha \to b(\alpha) \text{ are continuous fonctions} \\ (ii)\ \forall \alpha \in \mathcal{K}^N,\ \ B(\alpha) \text{ is a monotone matrix}^2 \end{cases}$$

- **Ex.1** For the obstacle pb: B is an M-matrix $\Rightarrow$ **(M)**
- **Ex.2** For Merton's pb: Implicit scheme $\Rightarrow$ **(M)**

[2] B is a monotone matrix if B is invertible and $B^{-1} \geq 0$ componentwise

Theorem (1) (see [B., Maroso, Zidani 09'] for a direct proof)

Assume **(M)**,
(i) *there exists a unique* $x \in \mathbb{R}^N$ *s.t.* $F(x) = 0$;
(ii) $\forall x^0$, $\lim_{k\to\infty} x^k = x$. *(Furthermore* $x^k \le x^{k+1}$*)*
(iii) *The convergence is superlinear.*
(iv) *If* $\mathcal{K}$ *is discrete, the convergence is in at most* $Card(\mathcal{K})^N$ *iterations.*

Theorem (2)

For the obstacle pb, assume **(M)**, *the convergence is in at most N iterations !*

REFS:
- Rust & Santos (2004)
- Intermuller, Ito, Kunish
- B, Maroso, Zidani (2009)

Application to american options:

Limitation of the total number's of newton's iteration := bounded by the number of mesh points where the value takes off the payoff function.

Complement : Two player games

$$\text{find } x \in \mathbb{R}^N, \quad F(x) = \max_{b\in\mathcal{B}} \min_{a\in\mathcal{A}} (B_{a,b}x - b_{a,b}) \equiv 0$$

Newton's algo ? Fails here in general because no more convexity. Function F in general not slantly differentiable.

Generalized (Ho) algo.:

- Notations: $B(\alpha,\beta)$, $b(\alpha,\beta)$.
- $\boxed{F(x) = \max_{\beta\in\mathcal{B}^N} F^\beta(x)}$ where $F^\beta(x) := \min_{\alpha\in\mathcal{A}^N} (B(\alpha,\beta)x - b(\alpha,\beta))$
- Starting from a given $x^0 \in \mathbb{R}^N$, iterate for $k \geq 0$:

$$\begin{cases} \text{Compute } \beta^{k+1} := argmin_{\beta\in\mathcal{K}^N} F^\beta(x^k), \\ \text{Compute } x^{k+1} \text{ s.t. } F^{\beta^{k+1}}(x^{k+1}) = 0 \text{ or s.t. } \|F^{\beta^{k+1}}(x^{k+1})\| \leq \eta_k \end{cases}$$

until some stopping criteria is satisfied.

Theorem (B., Maroso, Zidani 2009)

Assuming **(M)** *(continuity plus $B(\alpha, \beta)$ all monotone matrices):*
(i) There exists a unique solution
(ii) Generalized Howard's algorithm converges
(iii) Bounded number of iterations if $\mathcal{A}, \mathcal{B}$ finite.

• Furthermore, if we solve only in an approximate way $\|F^{\beta^{k+1}}(x^{k+1})\| \leq \eta_k$ with $\sum_k \eta_k < \infty$, then the corresponding "approximate generalized Howard's algorithm" converges to the solution, and

$$-C\eta_k \leq x^k - x \leq C \sum_{j \geq k} \eta_j$$

• Open questions: linear ? superlinear convergence ? More efficient schemes using penalisation approach (Reisinger & Whitte) ?

III. TOWARDS SECOND ORDER

- Joint – on going – work with Kristian Debrabant
- Very useful discussions with Yves Achdou !

Attempt 1 : A Crank-Nicolson (CN) scheme :

$$\min\left(\frac{u_i^{n+1}-u_i^n}{\Delta t}+\frac{1}{2}(AU^n+AU^{n+1})_i,\ u_i^{n+1}-g_i\right)=0 \quad 1\le i\le I.$$

• **CONSISTENCY:** $O(\Delta t^2)+O(\Delta x^2)$ (for $\sigma=\sigma(x)$ and regular v).
To see this, there is an equivalent PDE: $\min(u_t+\mathcal{A}u, u_t)=0$.
Corresponding CN scheme is

$$\min\left(\frac{u_i^{n+1}-u_i^n}{\Delta t}+\frac{1}{2}(AU^n+AU^{n+1})_i,\ \frac{u_i^{n+1}-u_i^n}{\Delta t}\right)=0 \quad 1\le i\le I.$$

In practice, the constraint $u_i^{n+1}\ge u_i^n$ is equivalent to $u_i^{n+1}\ge g_i$.

• STABILITY : **NOT CLEAR !** Von Neumann L^2 stability result OK.
Stability results such as $\|B^n\|_\infty\le C$ do also hold where
$B:=(I+\frac{1}{2}\Delta tA)^{-1}(I-\frac{1}{2}\Delta tA)$ is the amplication matrix. (S.I. Serdjukova, 1964; Borovykh, Drissi, Spijker 2002, ...); ⇒ stability of the CN scheme for pure diffusion. But the L^∞ stability is an open question for the obstacle scheme.

• IMPLEMENTATION : Newton/Howard's algorithm for *M*-matrices

L^2 error		Error L^1		Error L^2		Error L^∞	
I	N	error	order	error	order	error	order
80	80	1.74E-02	0.00	2.16E-02	0.00	4.49E-02	0.00
160	160	3.14E-03	2.47	3.71E-03	2.54	5.23E-03	3.10
320	320	8.25E-04	1.93	9.68E-04	1.94	1.37E-03	1.93
640	640	2.06E-04	2.00	2.40E-04	2.01	3.34E-04	2.03
1280	1280	4.39E-05	2.23	5.06E-05	2.24	7.07E-05	2.24

Table: Crank-Nicolson scheme for a 1d-American obstacle problem

However, for lower *N* values (larger CFL numbers) the CN scheme is no more second order and goes back to first order behavior.

Attempt 2 : A Semi-Lagrangian (SL) scheme :

• Let us consider only the semi-discrete problem, let $h = \Delta t$. Then

$$u_i^{n+1} = S^1(u^n)_i := \frac{1}{2}(u^n(x_i - \sigma\sqrt{h}) + u^n(x_i + \sigma\sqrt{h}))$$

is a typical SL scheme of first order (order $O(h)$).

• Second order can be obtain with the "Platen's" scheme (coming from weak Taylor approximation in stochastic calculus): For $\sigma = const$:

$$u_i^{n+1} = S^2(u^n)_i := \frac{1}{6}(u^n(x_i - \sigma\sqrt{3h}) + 4u^n(x_i) + u^n(x_i + \sigma\sqrt{3h})).$$

• Hence a natural scheme for the obstacle diffusion problem could be:

$$u_i^{n+1} := \max(S^2(u^n)_i, g_i)$$

However, this can only be consistent of first order !

Attempt 3 : Gear (BDF2) obstacle scheme

We propose the following two-step implicit Gear scheme, for $n \geq 1$:

$$H(U^n)_i :\equiv \min\left(\frac{3U_i^{n+1} - 4U_i^n + U_i^{n-1}}{2\Delta t} + (AU^{n+1} + q(t_{n+1}))_i,\ U_i^{n+1} - g_i\right) = 0$$

• Second order consistency error, when v is regular, for $V_i^n = v(t_n, x_i)$:

$$\begin{aligned} H(V^n) &= \min(v_t + \mathcal{A}v, v - g)(t_{n+1}, x_i) \\ &\quad + O(\Delta t^2 \|v_{3t}\|_\infty) + O(\Delta x^2(\|v_{3x}\|_\infty + \|v_{4x}\|_\infty)). \qquad (1) \end{aligned}$$

• Corresponding discrete obstacle pb solved by Howard/Newton method (efficient)

Gear (BDF2) obstacle scheme - Numerical results

L^2 error		Error L^1		Error L^2		Error L^∞	
I	N	error	order	error	order	error	order
80	8	8.23E-03	0.00	1.25E-02	0.00	3.59E-02	0.00
160	16	9.64E-04	3.09	1.28E-03	3.28	2.21E-03	4.02
320	32	4.20E-04	1.20	5.44E-04	1.24	8.88E-04	1.31
640	64	1.56E-04	1.43	1.96E-04	1.47	3.04E-04	1.55
1280	128	5.01E-05	1.64	6.15E-05	1.67	9.21E-05	1.72
2560	256	1.42E-05	1.82	1.72E-05	1.84	2.50E-05	1.88

Table: BDF2-Gear scheme for American option - "Large" CFL number

$\Rightarrow$ GOOD !

BDF3 obstacle scheme - Numerical results

A three-step (BDF3) implicit scheme, for $n \geq 2$:

$$\min\left(\frac{\frac{11}{6}U_i^{n+1} - 3U_i^n + \frac{3}{2}U_i^{n-1} - \frac{1}{3}U_i^{n-2}}{\Delta t} + (AU^{n+1} + q(t_{n+1}))_i,\right.$$

$$\left. U_i^{n+1} - g_i\right) = 0$$

(Initial steps U^0, U^1, U^2 of second order)

I	N	Error L^1		Error L^2		Error L^∞	
		error	order	error	order	error	order
80	16	1.81E-02	0.00	2.21E-02	0.00	4.39E-02	0.00
160	32	3.67E-03	2.30	4.34E-03	2.35	6.06E-03	2.86
320	64	1.06E-03	1.79	1.24E-03	1.81	1.68E-03	1.85
640	128	2.09E-04	2.34	2.43E-04	2.35	3.42E-04	2.30
1280	256	1.89E-05	3.47	2.86E-05	3.09	6.07E-05	2.49

Table: BDF3 scheme for the American option pb

Complement : L^2 stability analysis for BDF2

With $q \equiv 0$, the scheme has the following form:

$$\min\left((I + \frac{2}{3}\Delta t A)\, U^{n+1} - \frac{4}{3} U^n + \frac{1}{3} U^{n-1}),\ U^{n+1} - g \right) = 0$$

The exact solution satisfies an estimate in the following form:

$$\min\left((I + \frac{2}{3}\Delta t A)\, V^{n+1} - \frac{4}{3} V^n + \frac{1}{3} V^{n-1} - \Delta t\ \bar{\epsilon}_n,\ V^{n+1} - g \right) = 0$$

where $\bar{\epsilon}_n$ is a consistency error (hopefully of order $\Delta t^2 + \Delta x^2$)

Let $\langle\ ,\ \rangle$ denotes the scalar product in $\mathbb{R}^I$.

Lemma

For any matrix B, the following equivalence holds:

$$\min(Bx - b, x - g) = 0 \quad \Leftrightarrow \quad x \geq g \text{ and } \Big(\langle Bx - b, v - x \rangle \geq 0,\ \forall v \geq g \Big)$$

Remark: It is known that if B is a positive definite symmetric matrix, the above assertion is furthermore equivalent to :

$$\Leftrightarrow \quad x \text{ solves } \min_{x \geq g} \frac{1}{2} \langle x, Bx \rangle - \langle b, x \rangle$$

Energy like estimate: By using the "variational formulation", a similar analysis as for a Gear scheme can be done:

Assumption (H):

$$\langle x, Ax \rangle \geq 0, \quad \forall x \in \mathbb{R}^I.$$

Proposition (Stability of the Gear BDF2 obstacle scheme)

Let $e^n := v^n - u^n$ and let $\Delta t > 0$ be sufficiently small. Under assumption (H), then there exists a constant C_1 independant of n such that for all $t_n \leq T$,

$$\|e^n\|_2^2 + \sum_{k=1}^{n} \frac{2\Delta t}{3} \langle e^n, Ae^n \rangle \leq C_1 \left(\|e^0\|^2 + \|e^1\|^2 + \Delta t \sum_{k=1,\ldots,n} \|\bar{\epsilon}_n\|^2 \right).$$

Roughly speaking,

$$\Rightarrow \|e^n\|_2^2 \leq Const\, \Delta t \sum_{k=1,\ldots,n} \|\bar{\epsilon}_n\|^2.$$

Conclusion.

• We propose a BDF like scheme for obstacle problems. An L^2 stability estimates holds. For the moment, does not gives an error estimate.

• Perform a rigorous L^∞ stability analysis for the BDF2 - Gear scheme for diffusion + obstacle problem.
• First order HJ + obstacle : find efficient really second order schemes with rigourous analysis.