A domain decomposition approach to exponential methods for time discretization of PDEs

Luca Bonaventura

MOX - Politecnico di Milano

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Outline of the talk

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- An **accuracy and efficiency** assessment of the simplest approaches for their application to PDEs

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- Some preliminary numerical results
- Conclusions and perspectives for future work

Basic idea of exponential methods

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- Exponential methods: turn this into a numerical method with errors and stability independent of Δt for linear problems
- Various extensions to nonlinear problems are available

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- **J. C. Schulze, P. J. Schmid, J. L. Sesterhenn, Exponential time integration using Krylov subspaces,** *Int. J. Num. Meth. Fluids*, **60, 591-609, 2009**

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$$\frac{d\mathbf{u}}{dt} = \mathbf{f}(\mathbf{u}) = \mathbf{f}(\mathbf{u}^n) + \mathbf{J}^n(\mathbf{u} - \mathbf{u}^n) + \mathbf{R}(\mathbf{u}) \quad t \in [t^n, t^{n+1}]$$

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- **One step, one stage, second order stiff solver with one evaluation of RHS**

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- **Krylov space dimension (and cost of time step) depend on the Courant number**
- **Alternative techniques for the computation of** $\exp(\Delta t \mathbf{A})\mathbf{v}$ **imply similar costs for large scale problems**

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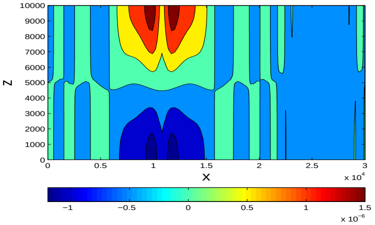

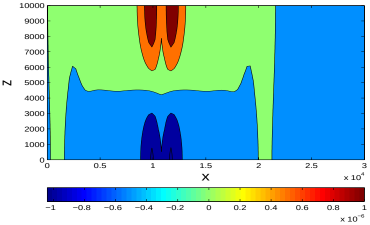

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	h error			
	LP	CN	EX2	EX3
Test 5	1.2e-2	9.1e-3	1.2e-3	1.1e-3
Test 6	5.9e-2	1.7e-2	3.8e-4	4.0e-4

A cost benefit analysis

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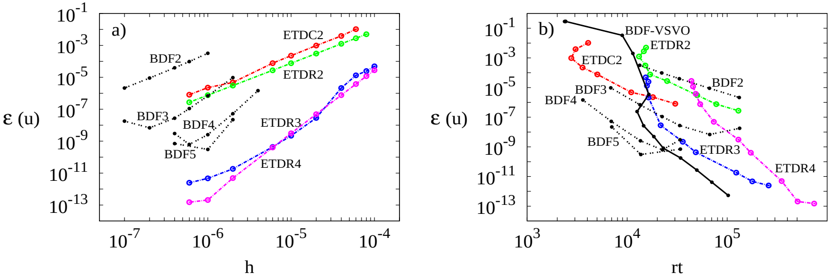

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- **Exponential of a sparse matrix is almost sparse (Iserles 2001)**
- **For $s-$banded $\mathbf{A} = (a_{i,j})$ with $|a_{i,j}| \leq \rho$, let $\exp(\mathbf{A}) = (e_{i,j})$.**

$$
\begin{aligned}
|e_{i,j}| &\leq \left(\frac{\rho s}{|i-j|}\right)^{\frac{|i-j|}{s}} \left[\mathrm{e}^{\frac{|i-j|}{s}} - \sum_{k=0}^{|i-j|-1} \frac{(|i-j/s|)^k}{k!} \right] \\
&\approx \left(\frac{\rho s}{|i-j|}\right)^{\frac{|i-j|}{s}} \frac{(|i-j|/s)^{|i-j|}}{|i-j|!}
\end{aligned}
$$

Application to PDE problems

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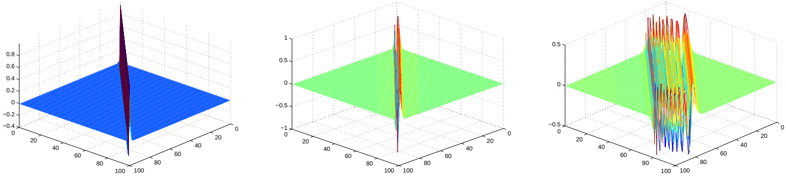

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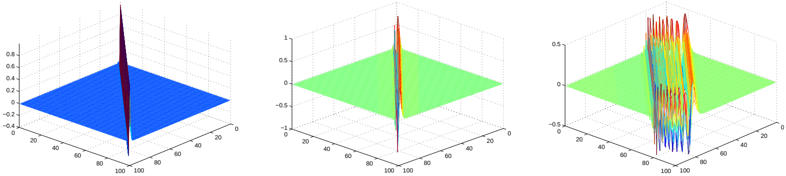

- **There is no real need to compute a global exponential matrix: Local Exponential Methods (LEM)**

LEM: a domain decomposition approach

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- **Decompose mesh in overlapping regions**

$$\mathcal{M} = \bigcup_{i=1}^{N} \mathcal{M}_i \quad \mathcal{M}_i = \mathcal{D}_i \cup \mathcal{B}_i$$

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- For $i = 1, \ldots, N$, solve local problem restricted to $\mathcal{M}_i$ by a local exponential method

$$\mathbf{u}^{n+1}_{\mathcal{M}_i} = \mathbf{u}^{n}_{\mathcal{M}_i} + \Delta t \phi\big(\mathbf{J}^{n}_{\mathcal{M}_i} \Delta t\big) \mathbf{f}(\mathbf{u}^{n}_{\mathcal{M}_i})_{\mathcal{M}_i}$$

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- Overwrite degrees of freedom belonging to $\mathcal{B}_i$

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- For small enough $\mathcal{D}_i$ local matrices can be stored...
- ... implying a major cost reduction if Jacobian is only recomputed every few time steps

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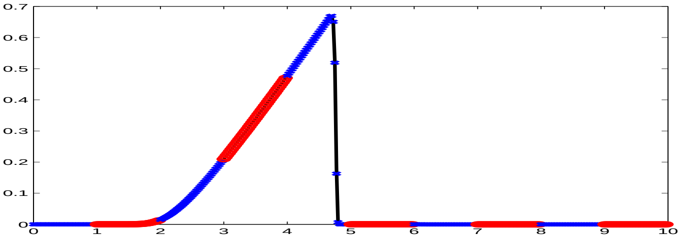

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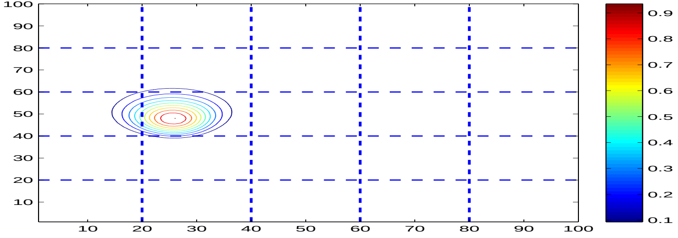

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- **Results as a function of the time step, the number D of subdomains employed and the number of grid points B in the buffer regions.**

	$D = 1$	$D = 2$	$D = 4$	$D = 5$	$D = 8$	$D = 10$
$C = 0.4, B = 5$	**25.89**	**13.46**	**10.63**	**10.30**	**10.09**	**10.42**
$C = 1, B = 15$	**30.07**	**9.02**	**4.70**	**4.72**	**4.78**	**5.49**
$C = 2, B = 20$	**26.77**	**6.57**	**2.87**	**3.24**	**3.05**	**3.13**

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- Computation of exponential matrix becomes trivially parallel, although an overhead is introduced due to overlapping buffers
- Preliminary numerical results show significant reduction of the computational cost
- Ongoing work: application of LEM to high order DG discretizations of various highly oscillating problems: atmospheric dynamics, Schrödinger equation