SEMIDISCRETE AND FINITE DIFFERENCES APPROXIMATION OF MEAN FIELD GAMES

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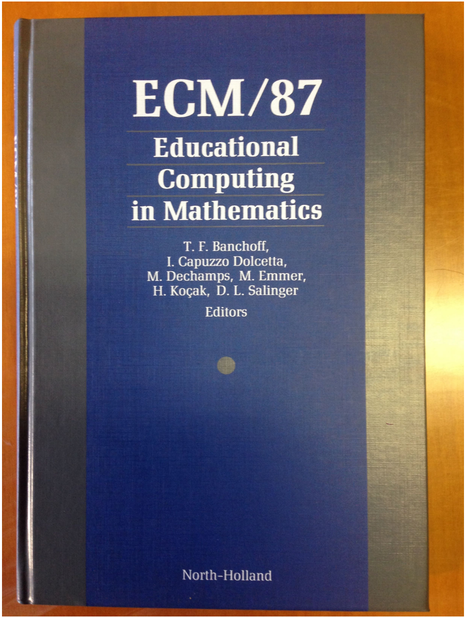
Numerical methods for PDEs:
optimal control, games and image processing

Conference on the occasion of Maurizio Falcone's 60th birthday

Roma, December 4-5, 2014

1987


ECM/87
Educational
Computing
in Mathematics
T. F. Banchoff,
I. Capuzzo Dolcetta,
M. Dechamps, M. Emmer,
H. Koçak, D. L. Salinger
Editors
North-Holland

1987

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INTRODUCTION

This volume contains contributions to the International Conference ECM/87 "Educational Computing in Mathematics" held at the University of Rome "La Sapienza", June 4-6, 1987.

The idea of the conference was to share experiences on computer aided teaching and also to point out some topics where current mathematical research takes advantage of the facilities offered by a computer. This idea originated in 1986 among the members of the EEC (European Economic Community) project "Utilisation des moyens informatiques dans l'enseignement des mathématiques" and the CNR (National Council of Research) supported "Progetto strategico: nuove tecnologie per la didattica". About 120 participants from various European countries and the U.S.A. attended the Conference which was organized in invited and contributed talks, software demonstration sessions and a symposium on "The Italian situation in computer aided teaching" coordinated by M. Falcone, with the interventions of C. Bohm, C. Pucci, and V. Villani, President of the Unione Matematica Italiana.

These proceedings contain the following three sections: Mathematical Experiments, Teaching Experiences, and Software Reports, covering all invited talks as well as a selection of contributed papers and software demonstration reports.

1988

ECM/87 - Educational Computing in Mathematics
T.F. Banchoff et al. (editors)
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THE LABORATORY OF MATHEMATICS: COMPUTERS AS AN INSTRUMENT FOR TEACHING CALCULUS

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Personal computers has been extensively used in the courses of Calculus and Advanced Calculus at the University of Rome in the last four years. In particular they have been used to illustrate mathematical phenomena as well as to solve problems. This fact obliged teachers to change the contents and the organization of traditional courses. We shall present some recent developments of this experiment showing examples of the results which were obtained and discussing in details some related questions: what kind of organization is the best ? what language is more adapt to the target ? which are the essential features requested for a demonstration software in this field ? how final examinations can be organized ?

1. THE FIRST EXPERIENCE

Starting from 1983 computers have been introduced in the courses of Calculus and Advanced Calculus at the University of Rome. The main reasons to introduce them are related to the importance of computers in modern mathematics: first of all for the possibility they offer to solve difficult mathematical problems whose (theoretical) solution cannot be exhibited in an explicit form, and secondary for the hints that a computer simulation can give in the study of many mathematical phenomena. It is quite important to notice that the technological (r)evolution of the

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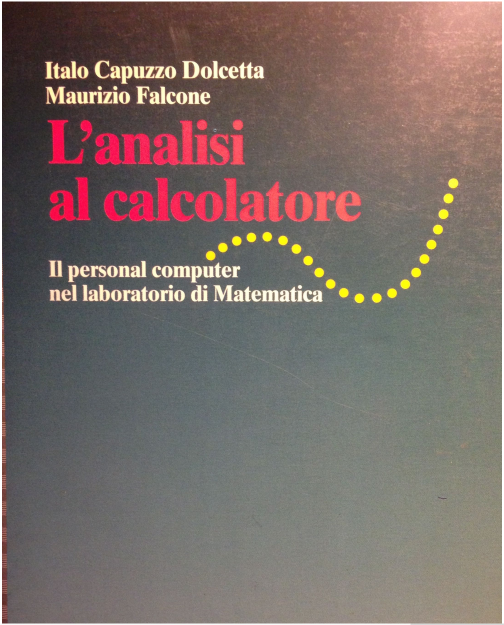

Italo Capuzzo Dolcetta
Maurizio Falcone
L'analisi
al calcolatore
Il personal computer
nel laboratorio di Matematica

1985-89

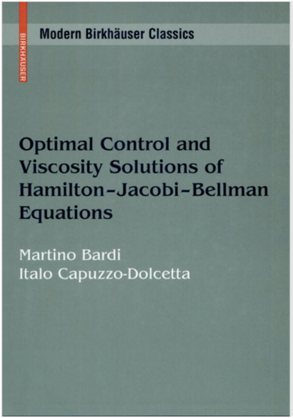
ANNALES DE L'I. H. P., SECTION C

I. CAPUZZO DOLCETTA
M. FALCONE

Discrete dynamic programming and viscosity solutions of the bellman equation

Annales de l'I. H. P., section C, tome S6 (1989), p. 161-183.
<http://www.numdam.org/item?id=AIHPC_1989__S6__161_0>

1997


Modern Birkhäuser Classics
BIRKHÄUSER
Optimal Control and
Viscosity Solutions of
Hamilton–Jacobi–Bellman
Equations
Martino Bardi
Italo Capuzzo-Dolcetta

1997

2008

A classical optimization problem

Given a time interval $[0, T]$ consider the classical Mayer type problem

$$\inf \int_t^T \left[\frac{1}{2} |\dot{X}_s|^2 + L(X_s) \right] ds + G(X_T) \tag{1}$$

where $X := X^{t,x}$ is any curve in the Sobolev space $W^{1,2}([t, T]; \mathbb{R}^d)$ such that $X_T = x \in \mathbb{R}^d$ for $t \in [0, T]$.

Well-known that if $L : \mathbb{R}^d \times [0, T] \to \mathbb{R}$, $g : \mathbb{R}^d \to \mathbb{R}$ are continuous and bounded, then the **value function** of problem (1) above, i.e.

$$u(t, x) = \inf \left\{ \int_t^T \left[\frac{1}{2} |\dot{X}_s|^2 + L(X_s) \right] ds + G(X_T) \, ; \; X \in W^{1,2}([0, T]; \mathbb{R}^d) \right\}$$

is the unique bounded continuous **viscosity solution** of the backward Cauchy problem

$$\begin{cases} -\partial_t u(t,x) + \frac{1}{2}\left|\nabla_x u(t,x)\right|^2 = L(x) & \text{in } (0,T)\times\mathbb{R}^d, \\ u(T,x) = G(x) & \text{in } \mathbb{R}^d \end{cases} \tag{2}$$

of Hamilton-Jacobi type.
The proof that u solves (2) in viscosity sense is a simple consequence of the following identity, the **Dynamic Programming Principle**:

$$u(t,x) = \inf\left\{ u(s, X^{t,x}(s)) + \int_s^t L(X_s)\,ds \;; \quad X \in W^{1,2}([0,T];\mathbb{R}^d) \right\}$$

valid for any given $(t,x) \in (0,T)\times\mathbb{R}^d$ and any $s \in [t,T]$.

Uniqueness of solution of (2) is a non trivial, fundamental result in viscosity solutions theory (Lions 1982).

As for **optimal curves**, easy to check that $\overline{X}^{t,x}$ is optimal for the initial setting (t, x) if and only if

$$u(t,x) = u(s, \overline{X}^{t,x}(s)) + \int_s^T L(\overline{X}^{t,x}(\tau))\, d\tau \ \text{ for all } \ s \in [t, T]$$

Moreover, if u is smooth enough, the **velocity field** of the **optimal paths** is the spatial gradient of the solution of the HJ equation.

More precisely,

The Verification Lemma

Lemma

Let $X^*(t)$ be such that

$$\dot{X}^*(s) = -\nabla_x u(s, X^*(s)) \;\; for \;\; s \in [t, T] \;\; , \;\; X^*(t) = x$$

Then,

$$\int_t^T \left[\frac{1}{2}|\dot{X}^*(s)|^2 + L(X^*(s))\right] ds + G(X^*(T)) =$$

$$= \inf \int_t^T \left[\frac{1}{2}|\dot{X}_s|^2 + L(X_s)\right] ds + G(X_T)$$

A deterministic mean field game problem

A very interesting new class of **optimal control** or **game** problems concerning the optimal decision policy of an agent acting in a scenario comprising a **continuum** of similar agents has become recently object of interest after the 2006/07 papers by Lasry and Lions. Related ideas have been developed independently in the engineering literature, and at about the same time, by Huang, Caines and Malhamé. See also for more recent developments:

- P.-L. Lions, Cours au Collège de France www.college-de-france.fr.
- Camilli, Fabio; Capuzzo Dolcetta, Italo; Falcone, Maurizio Preface [Special issue on mean field games]. Netw. Heterog. Media 7 (2012)
- Bardi, Martino; Caines, Peter E.; Capuzzo Dolcetta, Italo Preface: DGAA special issue on mean field games. Dyn. Games Appl. 3 (2013)
- Bardi, Martino; Caines, Peter E.; Capuzzo Dolcetta, Italo Preface: DGAA 2nd special issue on mean field games. Dyn. Games Appl. 4 (2014)

Assume that the running cost $L(X_s)$ depends also on an **exhogenous variable** $m(s, X_s)$ modeling the **density of population** of agents at state X_s at time s. The new cost criterion is then

$$\inf \int_t^T \left[\frac{1}{2} |\dot{X}_s|^2 + L(X_s, m(s, X_s)) \right] ds + G(X_T, m(T, X_T)) \qquad (3)$$

Here, m is a **non-negative** function valued in $[0, 1]$ such that $\int_{\mathbb{R}^d} m(s, x)\, dx = 1$ for all s.

The time evolution of m starting from an initial configuration $m(0, x)$ is governed by the **continuity equation**

$$\partial_t m(t, x) - div\,(m(t, x) D_x u(t, x)) = 0 \quad \text{in } (0, T) \times \mathbb{R}^d$$

Note that in the cost criterion the evolution of the measure m enters as a parameter. The value function of the agent is then given by

$$\inf \int_t^T \left[\frac{1}{2} |\dot{X}_s|^2 + L(X_s, m(s, X_s)) \right] ds + G(X_T, m(T, X_T)) \qquad (4)$$

His optimal control is, at least heuristically, given in feedback form by $\alpha^*(t, x) = -\nabla_x u(t, x)$.

Now, if all agents argue in this way, their repartition will move with a velocity which is due to the drift term $\nabla_x u(t, x)$.
This leads eventually to the continuity equation.

We are therefore led to consider the following system of nonlinear evolution pde's for the unknown functions $u = u(t,x)$, $m = m(t,x)$:

$$-\frac{\partial u}{\partial t} + \frac{1}{2}|\nabla u|^2 = L(x,m) \quad \text{in } (0,T) \times \mathbb{R}^d \tag{5}$$

$$\frac{\partial m}{\partial t} - div\,(m\,\nabla u) = 0 \quad \text{in } (0,T) \times \mathbb{R}^d \tag{6}$$

with the initial and terminal conditions

$$m(0,x) = m_0(x), \quad u(T,x) = G(x, m(T,x)) \quad \text{in } \mathbb{R}^d \tag{7}$$

Three crucial structural features:

- first equation **backward**, second one **forward** in time
- the operator in the **continuity** equation is the **adjoint of the linearization** at u of the operator in the HJ operator in the first equation
- nonlinearity in the HJB equation is **convex** with respect to $|\nabla u|$
- Hamiltonian structure

The planning problem

An interesting variant of the **(MFG)** system has been also proposed by Lions for modeling the presence of a **regulator prescribing a target density to be reached at final time**:

$$\frac{\partial u}{\partial t} + \frac{1}{2}|\nabla u|^2 = L(x, m) \quad \text{in } (0, T) \times \mathbb{R}^d$$

$$\frac{\partial m}{\partial t} - div\,(m \nabla u) = 0 \quad \text{in } (0, T) \times \mathbb{R}^d$$

with the initial and terminal conditions

$$m(0, x) = m_0(x) \geq 0, \quad m(T, x) = m_T(x), \quad \text{in } \mathbb{R}^d$$

No side conditions on u.

For $L \equiv 0$, the above is the equivalent formulation of **Monge-Kantorovich optimal mass transport** problem considered by Benamou-Brenier (2000), see also Achdou-Camilli-CD SIAM J. Control Optim. (2011), Porretta, pOn the planning problem for the mean field games system. Dyn. Games Appl. 4 (2014),.

Stochastic mean field game models

The presence of Brownian random effects in the evolution of the state of the system gives rise to the the following system **((MFG))** of second order evolution pde's:

$$-\frac{\partial u}{\partial t} - \nu \Delta u + \frac{1}{2}|\nabla u|^2 = L(x, m) \quad \text{in } (0, T) \times \mathbb{R}^d \tag{8}$$

$$\frac{\partial m}{\partial t} - \nu \Delta m - div(m \nabla u) = 0 \quad \text{in } (0, T) \times \mathbb{R}^d \tag{9}$$

with the initial and terminal conditions

$$m(0, x) = m_0(x), \quad u(T, x) = G(x, m(T, x)) \quad \text{in } \mathbb{R}^d \tag{10}$$

ν is a positive number.

A semi-discrete approach to deterministic (MFG)

We describe next a **semi-discretization** approach to the deterministic mean field game system:

$$-\frac{\partial u}{\partial t} + \frac{1}{2}|\nabla u|^2 = L(x, m) \quad \text{in } (0, T) \times \mathbb{R}^d$$

$$\frac{\partial m}{\partial t} - div\ (m \nabla u) = 0 \quad \text{in } (0, T) \times \mathbb{R}^d$$

with the initial and terminal conditions

$$m(0, x) = m_0(x), \quad u(T, x) = G(x, m(T, x)) \quad \text{in } \mathbb{R}^d$$

To begin let us recall first the semi-discrete approximation to equations of Hamilton-Jacobi type introduced in CD (1983), see also CD-Ishii (1984) and CD-Falcone (1989), a prototype of semi-Lagrangian approximation of convex HJ equations.

A very good updated source for this is of course:

Falcone M., Ferretti R., **Semi-Lagrangian Approximation Schemes for Linear and Hamilton-Jacobi Equations**, Society for Industrial and Applied Mathematics, Philadelphia, USA (2014)

Fix $\Delta t > 0$, set $K = [\frac{T}{\Delta t}]$ and for $n = 0, 1, ..., K-1$ consider piecewise constant controls

$$\alpha = (\alpha_k)_{k=n}^{K-1} \in \mathbb{R}^{d\times(K-n)}$$

To each α there is an associated discrete dynamics $X_k^{x,n}[\alpha]$ obtained by the recurrence

$$X_n = x \, ; \quad X_{k+1} = X_k - \Delta t \alpha_k = x - \Delta t \sum_{i=n}^{k} \alpha_i \quad \text{for} \quad k = n, ..., K-1$$

A semi-Lagrangian approximation

the **discrete cost criterion** :

$$J_{\Delta t}(\alpha; x, n) = \Delta t \sum_{k=n}^{K-1} \left[\frac{1}{2}|\alpha_k|^2 + L(k\Delta t, X_k)\right] + G(X_K)$$

the **discrete value function**:

$$u_{\Delta t}(n, x) = \inf_{\alpha} J_{\Delta t}(\alpha; x, n) \ ; \quad u_{\Delta t}(K, x) = G(x)$$

- the **discrete Hamilton-Jacobi-Bellman equation**

$$u_{\Delta t}(n, x) = \inf_{\alpha \in \mathbb{R}^d} \left[u_{\Delta t}(n+1, x - \Delta t\, \alpha) + \frac{1}{2} \Delta t |\alpha|^2 \right] + \Delta t\, L(nh, x)$$

for$n = 1, ..., K - 1$ and, for $n = K$, the terminal condition

$$u_{\Delta t}(K, x) = G(x)$$

- **synthesis** : take the argmin in the discrete equation; note that this does not require any regularity at the discrete level and produces suboptimal controls for the original problem

Assume that $L : \mathbb{R}^d \times [0,T] \to \mathbb{R}$, $g : \mathbb{R}^d \to \mathbb{R}$ are continuous and

$$\|L(t,.)\|_{C^2} \le C \quad \forall t \in [0,T], \qquad \|g\|_{C^2} \le C$$

and set $\hat{u}_{\Delta t}(t,x) = u_{\Delta t}([\frac{t}{h}],x)$. Then,

Theorem

- **uniform semiconcavity**: $u_{\Delta t}(n,x+y) - 2u_{\Delta t}(n,x) + u_{\Delta t}(n,x-y) \le C|y|^2$, C *independent of* h
- **uniform convergence**: *as* $\Delta t \to 0^+$, $\hat{u}_{\Delta t}$ *converge locally uniformly in* $[0,T] \times \mathbb{R}^d$ *to the unique viscosity solution of*

$$-\frac{\partial u}{\partial t} + \frac{1}{2}|\nabla u|^2 = L(x) \quad , \quad u(T,x) = G(x)$$

moreover, $||\hat{u}_{\Delta t} - u|| \le C\Delta t$

- **regularity**: $u \in W^{1,\infty}([0,T] \times \mathbb{R}^d)$, u *is semiconcave w.r.t* x

Approximation of the continuity equation

We describe now, following Camilli-Silva (2012) an approximation scheme for the continuity equation :

$$\frac{\partial m}{\partial t} - div\ (m \nabla u) = 0 \quad m(0,x) = m_0(x)$$

Denote by $\mathcal{P}_1$ the set of probability measures m on $\mathbb{R}^d$ s.t

$$\int_{\mathbb{R}^d} |x| dm(x) < +\infty$$

endowed with Kantorovic-Rubinstein-Wasserstein distance

$$d_1(m_1, m_2) = \sup \left\{ \int_{\mathbb{R}^d} f(x)\, d(m_1 - m_2)(x) \ : f \text{ is -1 Lipschitz} \right\}$$

Precisely, the **optimal discrete flow** starting from x is defined by

$$\Phi_0^{\Delta t}(x) = x \ , \ \Phi_{k+1}^{\Delta t}(x) = \Phi_k^{\Delta t}(x) - \Delta t \nabla u_{\Delta t}(k+1, \Phi_k^{\Delta t}(x)), k = 1, ..., K-1$$

Define now $m_{\Delta t}(k) := \Phi_k[m_0]$ as the **push-forward** of m_0 through the discrete flow, i.e. by asking that, for $k = 1, ..., K$,

$$\int_{\mathbb{R}^d} \Psi(x) dm_{\Delta t}(k) = \int_{\mathbb{R}^d} \Psi(\Phi_k^{\Delta t}(x)) m_0(x)\, dx$$

for any $\Psi \in C(\mathbb{R}^d)$.

Theorem

As $\Delta t \to 0^+$, the discrete measures $m_{\Delta t}$ converge to a measure m in $C([0, T]; \mathcal{P}_1)$ which solves the Fokker-Planck equationin the sense of distributions.

The semi-discrete scheme for the (MFG)system

The complete semi-discrete scheme is

$$u_{\Delta t}(k,x) = \inf_{\alpha\in\mathbb{R}^d}\left[u_{\Delta t}(k+1, x-\Delta t\,\alpha)+\frac{1}{2}\Delta t|\alpha|^2\right]+\Delta t\, L(x, m_h(k))\ , n = 1, ..., K$$

$$m_{\Delta t}(k) = \Phi_k^{\Delta t}[m_0] \quad , m_{\Delta t}(0) = m_0 \in \mathcal{P}_1$$

$$u_{\Delta t}(K,x) = G(x, m_{\Delta t}(K))$$

Remember that the flow $\Phi_k^{\Delta t}[m_0]$ is constructed via the optimization procedure dictated by the solution of the **discrete** H-J-B equation.

The following well-posedness result due to Camilli-Silva (2012) holds:

Theorem

For sufficiently small time step Δt:

- *the discrete system has a solution* $(u_{\Delta t}, m_{\Delta t}) \in C([0,T] \times \mathbb{R}^d) \times C([0,T]; \mathcal{P}_1)$

  If, in addition, for all $m_1, m_2 \in \mathcal{P}_1, m_1 \neq m_2$
- $\int_{\mathbb{R}^d} (L(x,m_1) - L(x,m_2))\, d(m_1 - m_2)(x) > 0$
- $\int_{\mathbb{R}^d} (G(x,m_1) - G(x,m_2))\, d(m_1 - m_2)(x) \geq 0$

  then the solution is unique.

As $\Delta t \to 0$*:*

- $u_{\Delta t}$ *converges to* u *locally uniformly to* u*,*
- $m_{\Delta t}$ *converges to* m *in* $C([0,T]; \mathcal{P}_1)$*,*

  where (u, m) *is the unique solution of system* **(MFG)**

Further research on this line: Carlini, E.; Silva, F. J. A fully discrete semi-Lagrangian scheme for a first order mean field game problem. SIAM J. Numer. Anal. 52 (2014)

Finite difference schemes

A different approximation approach using finite differences for numerical solution of

$$\frac{\partial u}{\partial t} - \nu \Delta u + H(x, \nabla u) = F(m(x)) \quad \text{in } (0, T) \times \mathbb{T}^d$$

$$\frac{\partial m}{\partial t} + \nu \Delta m + \operatorname{div}\left(m \frac{\partial H}{\partial p}(x, \nabla u) \right) = 0 \quad \text{in } (0, T) \times \mathbb{T}^d$$

- Y. Achdou-I.CD, Mean Field Games: Numerical Methods, SIAM J. Numerical Analysis (2010)
- Y. Achdou-F. Camilli-I.CD, Mean field games: numerical methods for the planning problem, SIAM J. Control and Optimization (2011)
- Y. Achdou-F. Camilli-I.CD, Mean field games: convergence of a finite difference method, SIAM J. Numerical Analysis (2013)

Finite difference schemes

In those papers:
finite difference methods basically relying on monotone approximations of the Hamiltonian and on a suitable weak formulation of the Fokker-Planck equation, both for infinite, finite horizon and planning problems with mean field games. These schemes were shown to have several important features:

- existence and uniqueness for the discretized problems obtained by similar arguments as those used in the continuous case by Lasry-Lions,
- they are robust when $\nu \to 0$ (the deterministic limit of the models),
- bounds on the solutions (especially on the Lipschitz norm of $u(t, \cdot)$), which are uniform in the grid step proved under reasonable assumptions on the data,
- convergence of the schemes in the reference case $H(x, p) = c(x) + |p|^\beta$ where $\beta > 1$, c is a smooth periodic function and F satisfies monotonicity assumptions which are standard in MFG theory,
- fast algorithms for solving the discrete nonlinear systems

Finite difference schemes for the (MFG) planning problem

Take $d = 2$, for simplicity of notation

- Let $\mathbb{T}_h$ be a uniform grid on the torus $\mathbb{T}^2$ with mesh step h, and x_{ij} be a generic point in $\mathbb{T}_h$
- Uniform time grid: $\Delta t = T/N_T$, $t_n = n\Delta t$
- The values of u and m at $(x_{i,j}, t_n)$ are approximated by $U^n_{i,j}$ and $M^n_{i,j}$

Notations

- the discrete Laplace operator:

$$(\Delta_h W)_{i,j} = -\frac{1}{h^2}(4W_{i,j} - W_{i+1,j} - W_{i-1,j} - W_{i,j+1} - W_{i,j-1})$$

- right-sided finite difference formulas for $\frac{\partial W}{\partial x_1}(x_{i,j})$ and $\frac{\partial W}{\partial x_2}(x_{i,j})$:

$$(D_1^+ W)_{i,j} = \frac{W_{i+1,j} - W_{i,j}}{h}, \quad \text{and} \quad (D_2^+ W)_{i,j} = \frac{W_{i,j+1} - W_{i,j}}{h}$$

- the set of 4 finite difference formulas at $x_{i,j}$:

$$[D_h W]_{i,j} = \left((D_1^+ W)_{i,j}, (D_1^+ W)_{i-1,j}, (D_2^+ W)_{i,j}, (D_2^+ W)_{i,j-1}\right)$$

Discrete HJB equation

$$\frac{\partial u}{\partial t} - \nu \Delta u + H(x, \nabla u) = F[m]$$
$$\downarrow$$
$$\frac{U^{n+1}_{i,j} - U^n_{i,j}}{\Delta t} - \nu(\Delta_h U^{n+1})_{i,j} + g(x_{i,j}, [D_h U^{n+1}]_{i,j}) = (F_h[M^n])_{i,j}$$

where

$$g(x_{i,j}, [D_h U^{n+1}]_{i,j})$$
$$= g\left(x_{i,j}, (D_1^+ U^{n+1})_{i,j}, (D_1^+ U^{n+1})_{i-1,j}, (D_2^+ U^{n+1})_{i,j}, (D_2^+ U^{n+1})_{i,j-1}\right),$$

for instance,

$$(F_h[M])_{i,j} = F[m_h](x_{i,j}),$$

where m_h is the piecewise constant function on $\mathbb{T}$ taking the value $M_{i,j}$ in the square $|x - x_{i,j}|_\infty \leq h/2$.

Assumptions on the discrete Hamiltonian g

Typical assumptions on g:

$$(q_1, q_2, q_3, q_4) \to g\,(x, q_1, q_2, q_3, q_4)$$

- **monotonicity:** g is nonincreasing with respect to q_1 and q_3 and nondecreasing with respect to to q_2 and q_4
- **consistency:**

$$g\,(x, q_1, q_1, q_3, q_3) = H(x, q), \quad \forall x \in \mathbb{T}, \forall q = (q_1, q_3) \in \mathbb{R}^2$$

- **differentiability:** g is of class C^1, and

$$\left| \frac{\partial g}{\partial x}\Big(x, (q_1, q_2, q_3, q_4)\Big) \right| \leq C(1 + |q_1| + |q_2| + |q_3| + |q_4|)$$

- **convexity:** $(q_1, q_2, q_3, q_4) \to g\,(x, q_1, q_2, q_3, q_4)$ is convex

The discrete version of (FP)

The discrete version of **(FP)**

$$\frac{\partial m}{\partial t} + \nu \Delta m + \mathrm{div}\left(m \frac{\partial H}{\partial p}(x, \nabla u) \right) = 0$$

is built in such a way that

- each time step leads to a linear system governed by a matrix
  - whose diagonal coefficients are negative,
  - whose off-diagonal coefficients are nonnegative,

  in order to hopefully use some **discrete maximum principle.**
- The argument for uniqueness should hold in the discrete case, so **the discrete Hamiltonian** g **should be used as well.**

Discretize the weak formulation of **(FP)**

$$-\int_{\mathbb{T}} \mathrm{div}\left(m\frac{\partial H}{\partial p}(x,\nabla u)\right) w = \int_{\mathbb{T}} m\frac{\partial H}{\partial p}(x,\nabla u)\cdot\nabla w$$

by

$$-h^2\sum_{i,j}\mathcal{B}_{i,j}(U,M)W_{i,j} := h^2\sum_{i,j} M_{i,j}\nabla_q g(x_{i,j},[D_hU]_{i,j})\cdot[D_hW]_{i,j}$$

which leads to

$$h\,\mathcal{B}_{i,j}(U,M) = \left(\begin{array}{c} \left(\begin{array}{c} M_{i,j}\frac{\partial g}{\partial q_1}(x_{i,j},[D_hU]_{i,j}) - M_{i-1,j}\frac{\partial g}{\partial q_1}(x_{i-1,j},[D_hU]_{i-1,j}) \\ +M_{i+1,j}\frac{\partial g}{\partial q_2}(x_{i+1,j},[D_hU]_{i+1,j}) - M_{i,j}\frac{\partial g}{\partial q_2}(x_{i,j},[D_hU]_{i,j}) \end{array}\right) \\ +\left(\begin{array}{c} M_{i,j}\frac{\partial g}{\partial q_3}(x_{i,j},[D_hU]_{i,j}) - M_{i,j-1}\frac{\partial g}{\partial q_3}(x_{i,j-1},[D_hU]_{i,j-1}) \\ +M_{i,j+1}\frac{\partial g}{\partial q_4}(x_{i,j+1},[D_hU]_{i,j+1}) - M_{i,j}\frac{\partial g}{\partial q_4}(x_{i,j},[D_hU]_{i,j}) \end{array}\right) \end{array}\right)$$

This yields the semi-implicit scheme:

$$\frac{U_{i,j}^{n+1} - U_{i,j}^{n}}{\Delta t} - \nu(\Delta_h U^{n+1})_{i,j} + g(x_{i,j}, [D_h U^{n+1}]_{i,j}) = (F_h[M^n])_{i,j}$$

$$0 = \frac{M_{i,j}^{n+1} - M_{i,j}^{n}}{\Delta t} + \nu(\Delta_h M^n)_{i,j}$$

$$+\frac{1}{h}\left(\begin{array}{c} \left(\begin{array}{l} M_{i,j}^n \frac{\partial g}{\partial q_1}(x_{i,j}, [D_h U^{n+1}]_{i,j}) - M_{i-1,j}^n \frac{\partial g}{\partial q_1}(x_{i-1,j}, [D_h U^{n+1}]_{i-1,j}) \\ +M_{i+1,j}^n \frac{\partial g}{\partial q_2}(x_{i+1,j}, [D_h U^{n+1}]_{i+1,j}) - M_{i,j}^n \frac{\partial g}{\partial q_2}(x_{i,j}, [D_h U^{n+1}]_{i,j}) \end{array} \right) \\ + \left(\begin{array}{l} M_{i,j}^n \frac{\partial g}{\partial q_3}(x_{i,j}, [D_h U^{n+1}]_{i,j}) - M_{i,j-1}^n \frac{\partial g}{\partial q_3}(x_{i,j-1}, [D_h U^{n+1}]_{i,j-1}) \\ +M_{i,j+1}^n \frac{\partial g}{\partial q_4}(x_{i,j+1}, [D_h U^{n+1}]_{i,j+1}) - M_{i,j}^n \frac{\partial g}{\partial q_4}(x_{i,j}, [D_h U^{n+1}]_{i,j}) \end{array} \right) \end{array} \right)$$

A model convergence result

Assume that

- $H(x,p) = c(x) + |p|^\beta$ with $\beta > 2$, c is C^1 and periodic
- u_0 and m_T are smooth,
- F satisfies the monotonicity condition

$$\int (F[m](x) - F[\tilde{m}](x))(dm(x) - d\tilde{m}(x)) \leq 0 \Rightarrow m = \tilde{m}.$$

- F is regularizing, namely continuously maps the set of probability measures on $\mathbb{T}^2$ (endowed with the weak * topology) to a bounded subset of $Lip(\mathbb{T}^2)$ (typically a nonlocal convolution operator),
- the numerical Hamiltonian satisfies standard conditions

then:

A model convergence result

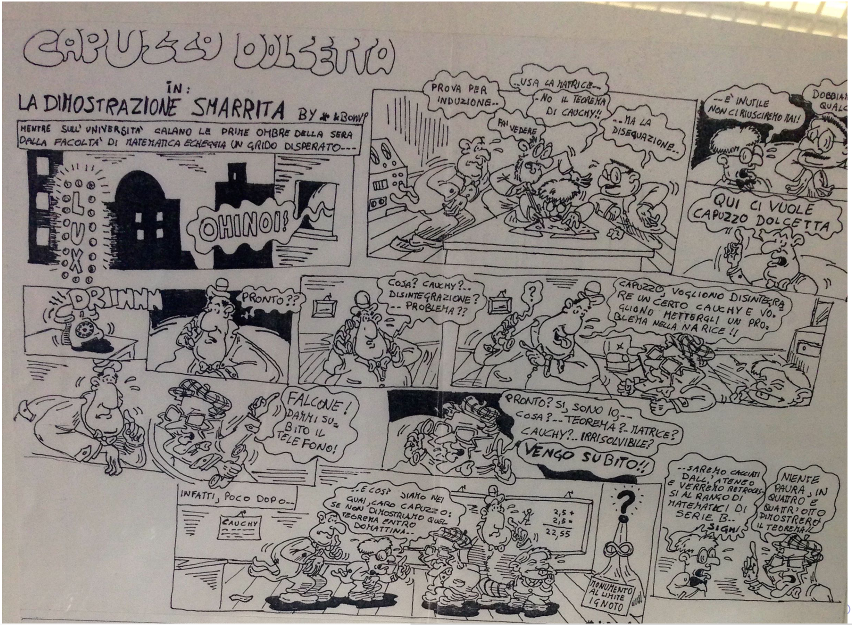
Theorem

Under the assumptions made the semi-implicit scheme has a unique solution $U^n_{i,j}, M^n_{i,j}$ *and*

$$\lim_{h,\Delta t\to 0} \sup_{i,j,n} |u(x_{i,j}, t_n) - U^n_{i,j})| = 0$$

$$\lim_{h,\Delta t\to 0} \sup_{i,j,n} |m(x_{i,j}, t_n) - M^n_{i,j})| = 0$$

l'imprescindibile (1982?):



l'imprescindibile (1982?):