Some issues in the semi-Lagrangian treatment of second-order balance laws

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joint works with L. Bonaventura

ROMA TRE UNIVERSITÀ DEGLI STUDI

Outline

Basic concepts on SL schemes – hyperbolic case

Model equation: linear, constant-coefficient advection

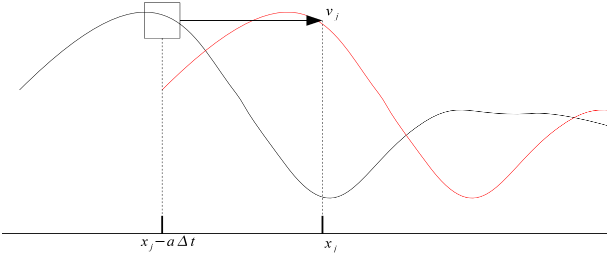
$$\begin{cases} u_t(x,t) + a u_x(x,t) = 0 \\ u(x,0) = u_0(x) \end{cases}$$

Representation formula

$$u(x,t) = u_0(x - at)$$

Semi-Lagrangian (SL) schemes stem from the so-called **Courant–Isaacson–Rees (CIR) method** ('52) which discretizes the representation formula (instead of the equation)

General principle of SL schemes



Advection of the solution **along characteristics**:

$$u(x_j, t_{n+1}) = u(x_j - a\Delta t, t_n)$$

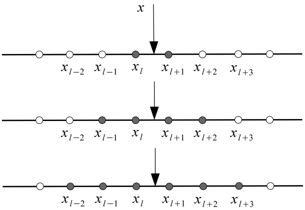
A numerical strategy $I[V]$ is used to **reconstruct the value at** $x_j - a\Delta t$.

Construction of SL schemes

Semi-Lagrangian (SL) discretization

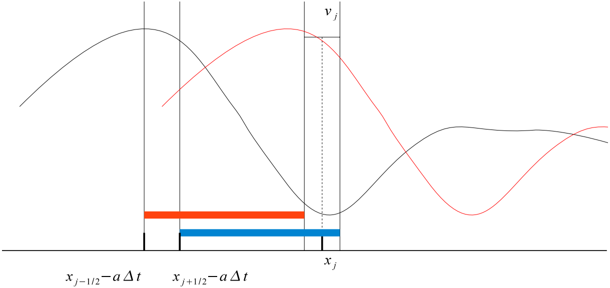
$$v_j^{n+1} = I[V^n](x_j - a\Delta t)$$

The most classical choice for the interpolation $I[V]$ is a **symmetric Lagrange interpolation** on a structured uniform mesh:



Various other options are possible, among which **Galerkin projection**

General principle of FFSL schemes



Mass balance in the **control volume** $\Omega_j = [x_{j-1/2}, x_{j+1/2}]$:

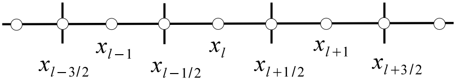
$$\int_{\Omega_j} u(t_{n+1})dx = \int_{\Omega_j} u(t_n)dx + \\ + \int_{x_{j-1/2}-a\Delta t}^{x_{j-1/2}} u(t_n)dx - \int_{x_{j+1/2}-a\Delta t}^{x_{j+1/2}} u(t_n)dx$$

Construction of FFSL schemes

Flux-Form Semi-Lagrangian (FFSL) discretization

$$v_j^{n+1} = v_j^n + \frac{1}{\Delta x}\left(\int_{x_{j-1/2}-a\Delta t}^{x_{j-1/2}} R[V^n](x)dx - \int_{x_{j+1/2}-a\Delta t}^{x_{j+1/2}} R[V^n](x)dx\right)$$

As in a conventional **FV scheme**, the reconstruction $R[V]$ **preserves the average** on each cell $[x_{j-1/2}, x_{j+1/2}]$ and **reconstructs with high order** smooth solutions



Theoretical results

- **SL**: **multidimensional and high-order implementations** can be easily constructed; theoretical results available for **Galerkin projection and symmetric Lagrange interpolation** in the variable coefficient case

Theoretical results

- **SL**: **multidimensional and high-order implementations** can be easily constructed; theoretical results available for **Galerkin projection and symmetric Lagrange interpolation** in the variable coefficient case
- **FFSL**: **more complex in higher dimension**; theoretical results available in one space dimension by exploiting the **relationship with the SL case**

Stochastic representation formula

The generalization of the concept of characteristics to **diffusion operators in trace form** may be performed in a stochastic framework

- **Feynman–Kac formula**: generalizes the representation formula by characteristics to the second-order case. Its use in numerical schemes was first proposed by Kushner in the 70s.
- It is a **stochastic representation formula**, but since it represents the solution as an expectation, its result is purely deterministic.
- It can be extended to more general situations, including **second-order Dynamic Programming Equations** ([KD01, CF95])
- Numerical implementation is performed via the **stochastic weak Euler scheme**, some high-order extension [F10]

Deterministic interpretation (1)

A deterministic construction of this technique may be given via the **Taylor expansion**:

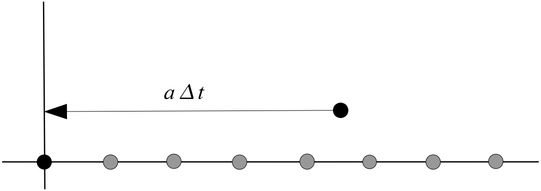
$$\begin{aligned} u(x_j + a\Delta t + \sigma\sqrt{2\Delta t}) &= u(x_j) + \Delta t\, a u_x(x_j) + \sqrt{2\Delta t}\,\sigma u_x(x_j) + \\ &\quad + \Delta t\, \sigma^2 u_{xx}(x_j)\sigma + O(\Delta t^{3/2}) + O(\Delta t^2) \end{aligned}$$

$$\begin{aligned} u(x_j + a\Delta t - \sigma\sqrt{2\Delta t}) &= u(x_j) + \Delta t\, a u_x(x_j) - \sqrt{2\Delta t}\,\sigma u_x(x_j) + \\ &\quad + \Delta t\, \sigma^2 u_{xx}(x_j)\sigma - O(\Delta t^{3/2}) + O(\Delta t^2) \end{aligned}$$

Taking the mean value we get:

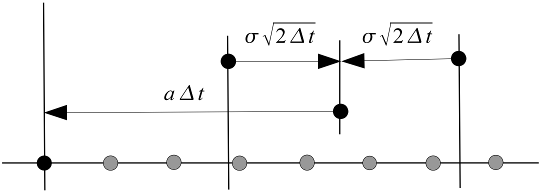
$$\begin{aligned} \frac{1}{2}\left[u(x_j + a\Delta t + \sigma\sqrt{\Delta t}) + u(x_j + a\Delta t - \sigma\sqrt{\Delta t})\right] = u(x_j) + \\ + \Delta t\left[a u_x(x_j) + \sigma^2 u_{xx}(x_j)\sigma\right] + O(\Delta t^2) \end{aligned}$$

Deterministic interpretation (2)



- A first **upwinding** of magnitude $a\,\Delta t$ follows the **advection** (if advection occurs)

Deterministic interpretation (2)



- A first **upwinding** of magnitude $a\,\Delta t$ follows the **advection** (if advection occurs)
- A second **symmetric upwinding** of magnitude $\sigma\sqrt{2\Delta t}$ is related to the **diffusion**

Operators in divergence form

The stochastic framework of the Feynman–Kac formula is **unsuitable to treat operators in divergence form**. To study the extension of this technique, we use the

Model equation: linear, second-order balance law

$$u_t + au_x = (\nu(x)u_x)_x \tag{1}$$

(formally, this equation can be put in the **advective form**

$$u_t + (a - \nu_x(x))u_x = \nu(x)u_{xx}$$

although a SL discretization of this latter equation would **lose the conservative character and require the knowledge of** ν_x)

Operators in divergence form

In the model problem proposed:

- Variable-coefficient **advection terms** may be included without technical problems if **first-order consistency** is enough
- The diffusivity has **no dependence on** t (again, it can be extended to **time dependent diffusivity** if first-order schemes are OK)
- For the moment, we work in a **single space dimension** (extension to higher dimensions is possible, but not trivial for the conservative scheme)

General construction of the schemes

At the level of **consistency**, the analysis [BF14] can be based on the

General structure

$$u(x_j, t_{n+1}) \approx A_j^+ u(x_j - a\Delta t + \delta_j^+, t_n) + A_j^- u(x_j - a\Delta t - \delta_j^-, t_n). \quad (2)$$

- We have **four free parameters** to be consistent with the evolution operator
- (2) should be regarded as a **time discretization** (no space reconstruction is introduced yet)
- In general, **we don't expect that the consistency rate could go beyond the unity**

Consistency

First order consistency conditions for the approximation (2)

$$\begin{cases} A_i^+ + A_i^- = 1 + O(\Delta t^2) \\ A_i^+\delta_i^+ - A_i^-\delta_i^- = \Delta t\, \nu_x(x_i) + O(\Delta t^2) \\ A_i^+{\delta_i^+}^2 + A_i^-{\delta_i^-}^2 = 2\Delta t\, \nu(x_i) + O(\Delta t^2) \\ A_i^+{\delta_i^+}^3 - A_i^-{\delta_i^-}^3 = O(\Delta t^2). \end{cases}$$

- In the case of **constant viscosity**, we obtain again

$$\delta_j^+ = \delta_j^- = \sqrt{2\Delta t\, \nu}, \quad A_j^+ = A_j^- = 1/2$$

- Extension to a **generic dimension**: relatively straightforward [BF14]
- Extension to **higher order consistency**: some existing works for trace operators [F10], very difficult in general

Stability

Stability **not proved in general**, except for

- **Monotone space discretizations** (in this case the scheme is monotone and L^∞ stable)
- **Constant diffusivity** (in this case the scheme is the convex combination of stable schemes)

Advective scheme – abstract formulation

Equation (1) can be treated in this form way by **keeping** $A_j^{\pm} = 1/2$ **and defining two (different) displacements** $\delta_j^{\pm}$:

$$u(x_j, t_{n+1}) \approx \frac{1}{2} u(x_j - a\Delta t + \delta_j^+, t_n) + \frac{1}{2} u(x_j - a\Delta t - \delta_j^-, t_n)$$

with the $\delta^{\pm}$ defined as **solutions of**

$$\delta_j^{\pm} = \sqrt{2\Delta t \, \nu \left(x_j \pm \delta_j^{\pm} \right)}$$

(the resulting time discretization is **first-order, nonconservative**). Note that the **difference between** δ_j^+ **and** δ_j^- generates the **additional advection term**:

$$\frac{1}{2}(\delta_j^+ - \delta_j^-) = \Delta t \, \nu_x(x_j) + O\left(\Delta t^2\right).$$

Fully discrete advective scheme

Fully discrete scheme

$$v_j^{n+1} = \frac{1}{2} I[V^n](x_j + \delta_j^+) + \frac{1}{2} I[V^n](x_j - \delta_j^-)$$

where (under reasonable assumptions) the displacements $\delta_j^\pm$ can be computed **via the iteration**

$$\delta_j^{\pm(k+1)} = \sqrt{2\Delta t \, \nu \left(x_j \pm \delta_j^{\pm(k)} \right)}$$

- For a space interpolation of degree r, the **consistency error** is

$$L(\Delta x, \Delta t) = O\left(\Delta t + \frac{\Delta x^{r+1}}{\Delta t} \right)$$

- **Few iterations required** to compute $\delta_j^\pm$ without degrading the consistency rate

Flux-form scheme

...omissis...

Nonlinear conservation laws – construction of the scheme

Model equation: nonlinear conservation law

$$u_t + f(u)_x = \nu u_{xx}$$

The advective form of the equation is

$$u_t + f'(u)u_x = \nu u_{xx},$$

which is naturally discretized by the

Non-conservative scheme

$$\begin{aligned} v_j^{n+1} &= \frac{1}{2} I[V^n](x_j - \Delta t\, f'(v_j^{n+1}) - \sqrt{2\Delta t\, \nu}) + \\ &+ \frac{1}{2} I[V^n](x_j - \Delta t\, f'(v_j^{n+1}) + \sqrt{2\Delta t\, \nu}) \end{aligned}$$

v_j^{n+1} and the speed of propagation are unknown (the scheme is implicit).

Convergence of the iterative solver

The scheme computes the solution V^{n+1} as a solution of **a system of scalar fixed point equations**:

$$V = T(V),$$

in which T depends on V^n and has **decoupled** components. Partial derivatives of the transformation T are given by

$$\frac{\partial T_j}{\partial v_j^{n+1}} = -\frac{\Delta t\, f''(v_j^{n+1})}{2} \Big[I'[V^n] \Big(x_j - \Delta t\, f'(v_j^{n+1}) - \sqrt{2\Delta t\, \nu} \Big) + \\ + I'[V^n] \Big(x_j - \Delta t\, f'(v_j^{n+1}) + \sqrt{2\Delta t\, \nu} \Big) \Big]$$

and are therefore “small” (i.e., T **is a contraction**) for Δt small enough.

A unilateral bound on the incremental ratio

Despite being non-conservative, an indication on the **correct entropic behaviour of the scheme** may be obtained by a **unilateral bound on the incremental ratio** of the numerical solution. If $\bar{D}_j$ is a suitably defined incremental ratio at the foot of the characteristic ending at (x_j, t_n), then

$$\frac{v_{j+1}^{n+1} - v_j^{n+1}}{\Delta x} = \frac{\bar{D}_j^n}{1 + \Delta t \bar{D}_j^n f''(\eta)}$$

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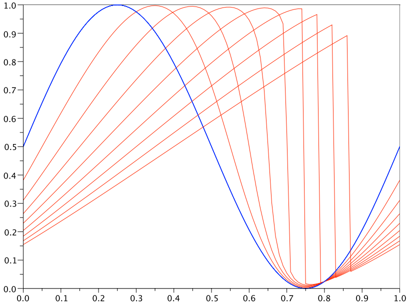
which entails that, **uniformly wrt** ν:

- for a **convex flux** ($f'' > 0$), the maximum value of the derivative (if positive) decreases strictly from t_n to t_{n+1}
- for a **concave flux** ($f'' < 0$), the minimum value of the derivative (if negative) increases strictly from t_n to t_{n+1}

1-D numerical tests – Burgers' equation

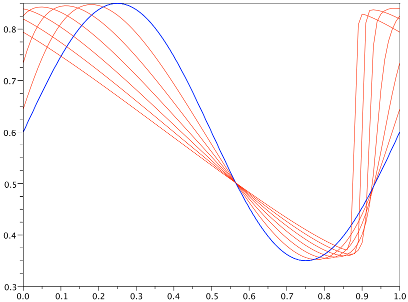
$$u_t + \left(\frac{u^2}{2}\right)_x = \nu u_{xx}, \quad t \in [0, 0.8], \quad \nu = 10^{-3}$$



100 nodes, $\Delta t = 0.05$, Courant number $\lesssim 5$

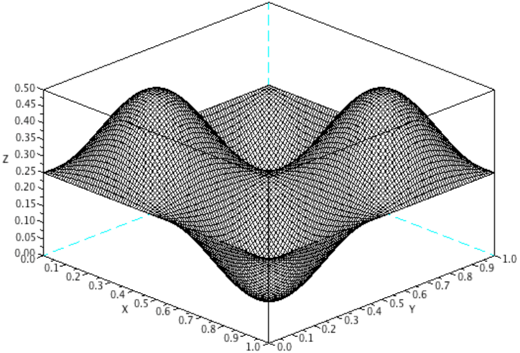
1-D numerical tests – LWR equation

$$u_t + \left(\frac{u - u^2}{2}\right)_x = \nu u_{xx}, \quad t \in [0, 1.2], \quad \nu = 10^{-3}$$



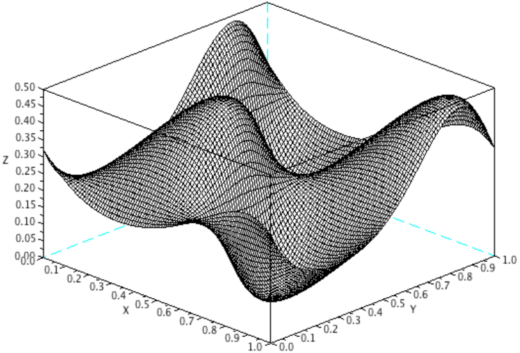
100 nodes, $\Delta t = 0.05$, Courant number $\lesssim 2.5$

2-D numerical tests – Burgers' system



$\nu = 10^{-3}$, 100×100 nodes, $\Delta t = 0.1$, Courant number $\lesssim 10$

2-D numerical tests – Burgers' system



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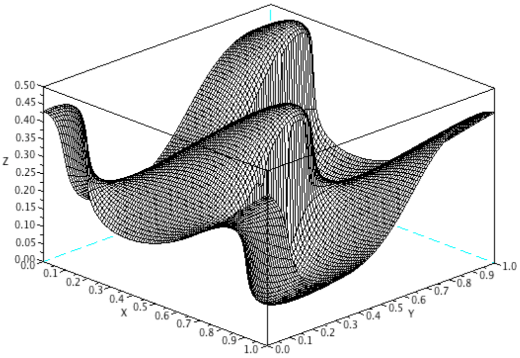

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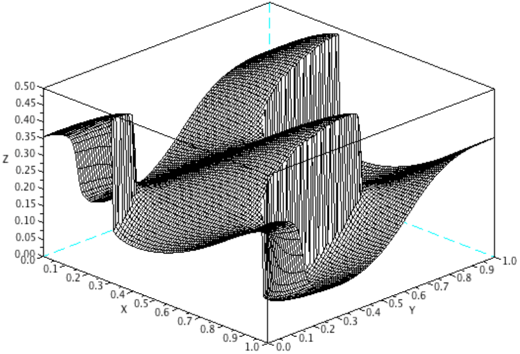

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Conclusions

- Construction of a general approach to treat diffusion terms in SL schemes, **both** in trace and divergence form

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Conclusions

- Construction of a general approach to treat diffusion terms in SL schemes, **both** in trace and divergence form
- Nonconservative structure, but conservative version is in progress (1D ok, multi-D ?)
- Various extensions to nonlinear problems: porous media eqn, turbulent viscosity for atmospherical models, viscous nonlinear conservation laws
- Flux-form version of the scheme for nonlinear conservation laws in progress
- Good overall numerical accuracy, but only first-order convergence wrt time

References

L. Bonaventura and R. Ferretti, *Semi-Lagrangian methods for parabolic problems in divergence form*, SIAM J. Sci. Com., to appear

F. Camilli and M. Falcone, *An approximation scheme for the optimal control of diffusion processes*, M^2AN Math. Model. Numer. Anal., **29** (1995), 97 – 122

M. Falcone and R. Ferretti, Semi-Lagrangian approximation schemes for linear and Hamilton–Jacobi equations, SIAM, Philadelphia, 2013

R. Ferretti, *A technique for the high-order treatment of diffusion terms in semi-Lagrangian schemes*, Commun. Comp. Phys. **8** (2010), 445 – 470

H. Kushner and P. Dupuis, Numerical methods for stochastic control problems in continuous time, Springer, New York, 2001

Thank you for your attention!

Thank you for your attention!

(but...)

Part 2

Highlights from my (scientific) life with Maurizio

"You are Ferretti of the Falcone–Ferretti paper? Wow."
(Santiago de Compostela, 2005)

Literature

- 1 book
- 6 papers on journals
- 5 proceedings
- 2 editorials

Conferences

- Uncountable set

Conferences

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Conferences

- Uncountable set, but
- 2 conferences with social ballroom dancing

Conferences

- Uncountable set, but
- 2 conferences with social ballroom dancing
- 1 conference reached by sailing boat

Conferences

- Uncountable set, but
- 2 conferences with social ballroom dancing
- 1 conference reached by sailing boat
- 1 conference staying in a double bed room

Hints about Maurizio's jargon (1)

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"...il signor Severino Pinzelleri..."

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"...il signor Severino Pinzelleri..."

- Verbatim translation: "...Mr. Severino Pinzelleri..."

Hints about Maurizio's jargon (1)

"...il signor Severino Pinzelleri..."

- Verbatim translation: "...Mr. Severino Pinzelleri..."
- Actual translation: "...Mr. John Smith..." (antonomasia)

Hints about Maurizio's jargon (2)

"...con tutti i cazzimbiccheri..."

Hints about Maurizio's jargon (2)

"...con tutti i cazzimbiccheri..."

- Verbatim translation: untranslatable

Hints about Maurizio's jargon (2)

"...con tutti i cazzimbiccheri..."

- Verbatim translation: untranslatable
- Actual translation: "...with all the bells'n'whistles..."

Hints about Maurizio's jargon (3)

"Salutace Biancaneve!"

Hints about Maurizio's jargon (3)

"Salutace Biancaneve!"

- Verbatim translation: "Best regards to Snow White!"

Hints about Maurizio's jargon (3)

"Salutace Biancaneve!"

- Verbatim translation: "Best regards to Snow White!"
- Actual translation: "Don't even think of it, no hope! "

Hints about Maurizio's jargon (4)

"Questo è un problema intricato"

Hints about Maurizio's jargon (4)

"Questo è un problema intricato"

- Verbatim translation: "This is an intricate problem"

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- Actual translation: "This is an intricate problem.

Hints about Maurizio's jargon (4)

"Questo è un problema intricato"

- Verbatim translation: "This is an intricate problem"
- Actual translation: "This is an intricate problem. You do the computations"

Hints about Maurizio's jargon (5)

(as a chairman) "What about error estimates?"

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- Verbatim translation: not required

Hints about Maurizio's jargon (5)

(as a chairman) "What about error estimates?"

- Verbatim translation: not required
- Actual translation: "Come on, guys. Doesn't *really* anyone have a question?"

Hints about Maurizio's jargon (6)

"Vediamoci alle... (ora t_0)"

Hints about Maurizio's jargon (6)

"Vediamoci alle... (ora t_0)"

- Verbatim translation: "Let's meet at... (time t_0)"

Hints about Maurizio's jargon (6)

"Vediamoci alle... (ora t_0)"

- Verbatim translation: "Let's meet at... (time t_0)"
- Actual translation: "I will be there at some time $t \in [t_0 + 30m, +\infty)$"

Cheers, Maurizio!