Independent and Patchy sub-domains in a Hamilton-Jacobi Equation

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and several Maurizio's collaborators

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Numerical methods for PDEs
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RICAM
JOHANN·RADON·INSTITUTE
FOR COMPUTATIONAL AND APPLIED MATHEMATICS

ITN SADCO
Initial Training Network
Sensitivity Analysis for Deterministic Controller Design

A ’sparse’ story

1: Rome: 2011

Patchy Decomposition

Patchy Decomposition

- Cacace, Cristiani, Falcone and Picarelli, *A patchy dynamic programming scheme for a class of Hamilton-Jacobi-Bellman equation*, SIAM J. Sc. Comp. (2012)

- *Reconstruction of some "Sub-Domains of Invariance" through the resolution of the problem on a course grid passing by the synthesis of the controls*
- **Goal:** solve the problem in parallel on a fine grid
- Good point: Some cases of interest where this idea works well
- **Open questions:**
  - Convergence, error introduced,
  - Extension of this idea to a wider class of problems

Patchy Decomposition: example

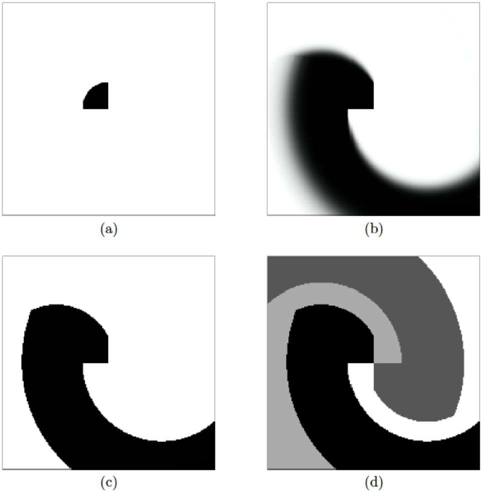


Figure: Some steps of the *Patchy Algorithm* (thanks to the authors)

2: London: 2013

Decomposition for Differential Games

Decomposition for Differential Games

(with Vinter, preprint 2014)

Let us consider, for an H not necessarily convex

$$\begin{cases} \lambda v(x) + H(x, Dv(x)) = 0 & x \in \Omega \\ v(x) = g(x) & x \in \Gamma \end{cases}$$

Considered a **decomposition of the boundary** $\Gamma := \bigcup_{i \in \mathbb{I}} \Gamma_i$, with $\mathcal{I} := \{1, ...m\} \subset \mathbb{N}$, we call $v_i : \bar{\Omega} \to \mathbb{R}$ a Lipschitz continuous viscosity solution of the problem

$$\begin{cases} \lambda v_i(x) + H\left(x, Dv_i(x)\right) = 0 & x \in \Omega \\ v_i(x) = g_i(x) & x \in \Gamma \end{cases}$$

where the functions $g_i : \Gamma \to \mathbb{R}$ is a regular function such that

$$\begin{aligned} g_i(x) = g(x), &\ \text{if } x \in \Gamma_i, \\ g_i(x) > g(x), &\ \text{otherwise.} \end{aligned}$$

Define

$$\begin{aligned} I(x) &:= \{i \in \{1,\dots,m\} | v_i(x) = \min_j v_j(x)\}, \\ \Sigma &:= \{x \in \mathbb{R}^N | card(I(x)) > 1\}. \end{aligned}$$

Theorem

Assume the following condition satisfied: for arbitrary $x \in \Sigma$, any convex combination $\{\alpha_i \,|\, i \in I(x)\}$ and any collection of vectors $\{p_i \in \partial^L v_i(x) \,|\, i \in I(x)\}$ we have

$$\lambda \bar{v} + H\left(x, \sum_i \alpha_i p_i\right) \leq 0\,. \qquad \text{(E)}$$

Then, for all $x \in \mathbb{R}^N \backslash \mathcal{T}$,

$$v(x) \;=\; \bar{v}(x) \;:=\; \min_i \{v_1(x), \dots, v_m(x)\}.$$

3: Paris: 2014

Independent subdomains reconstruction

Differential Games Problem

Let the dynamics be given by

$$\begin{cases} \dot{y}(t) = f(y(t), a(t), b(t)), & a.e. \\ y(0) = x, \end{cases}$$

$x \in \Omega \subseteq \mathbb{R}^n$ open, $a, b \in \mathcal{A}, \mathcal{B} = \{\mathbb{R}^+ \to A, \; measureable\}$, A, B compact sets. A solution is a trajectory $y_x(t, a(t), b(t))$.

The goal is to find the *sup – inf* **optimum** over $\mathcal{A}$, $\mathcal{B}$ of

$$J_x(a, b) := \int_0^{\tau_x(a,b)} l(y_x(s, a(s), b(s)), a(s), b(s)) e^{-\lambda s} ds$$
$$+ e^{-\lambda \tau_x(a,b)} g(y_x(\tau_x(a, b))), \quad \lambda \geq 0,$$

where $\tau_x(a, b) := \min \{t \in [0, +\infty) \mid y_x(t, a(t), b(t)) \notin \Omega\}$.

the value function of this problem is

$$v(x) := \sup_{\phi \in \Phi} \inf_{a \in \mathcal{A}} J_x(a, \phi(a)),$$

$$\Phi := \{\phi : \mathcal{A} \to \mathcal{B} : t > 0, a(s) = \tilde{a}(s) \text{ for all } s \leq t \\ \text{implies } \phi[a](s) = \phi[\tilde{a}](s) \text{ for all } s \leq t\}.$$

we will assume the *Isaacs' conditions* verified.

Theorem

The value function of the problem is a viscosity solution of the HJ equation associated with

$$H(x, p) := \min_{b \in \mathcal{B}} \max_{a \in \mathcal{A}} \{-f(x, a, b) \cdot p - l(x, a, b)\}.$$

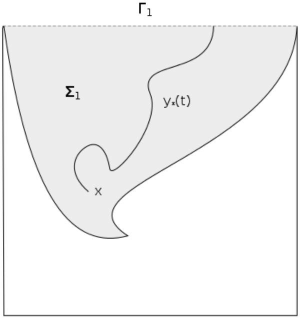
Independent Sub-Domains

Definition

A closed subset $\Sigma \subseteq \overline{\Omega}$ is an independent sub-domain of the problem (11) if, given a point $x \in \Sigma$ and an **optimal control** $(\overline{a}(t), \bar{\phi}(\overline{a}(t))$

(i.e. $J_x(\overline{a}, \bar{\phi}(\bar{a})) \leq J_x(a, \bar{\phi}(a))$ for every choice of $a \in \mathcal{A}$, and $J_x(\bar{a}, \bar{\phi}(\bar{a})) \geq J_x(\bar{a}, \phi(\bar{a}))$ for every choice of $\phi \in \Phi$),

the trajectory $y_x(\bar{a}(t), \bar{\phi}(\bar{a}(t))) \in \Sigma$ for $t \in [0, \tau_x(\bar{a}, \bar{\phi}(\bar{a}))]$.



Independent Domains Decomposition

Proposition

Given a collection of $n-1$ dimensional subsets $\{\Gamma_i\}_{i=\mathcal{I}}$ such that $\Gamma = \cup_{i=1}^{m}\Gamma_i$, the sets defined as

$$\Sigma_i := \left\{x \in \overline{\Omega} | v_i(x) = v(x)\right\}, \quad i = 1, ..., m$$

where v_i, v are defined accordingly to Theorem (1), are independent sub-domains of the original problem.

Proof.

By contradiction using the DPP. □

Example of Reconstruction (I)

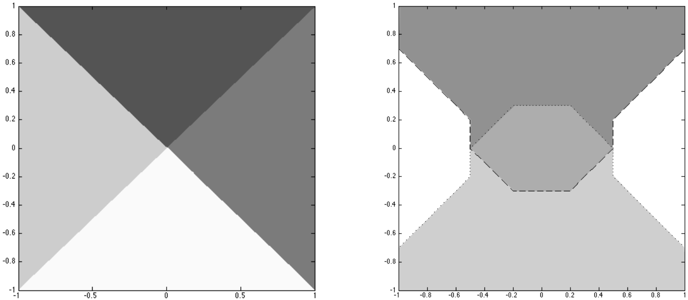


Figure: Distance function: Exact decomposition and two (of the four) approximated independent subsets found with a course grind of 15^2 points.

Example of Reconstruction (II)

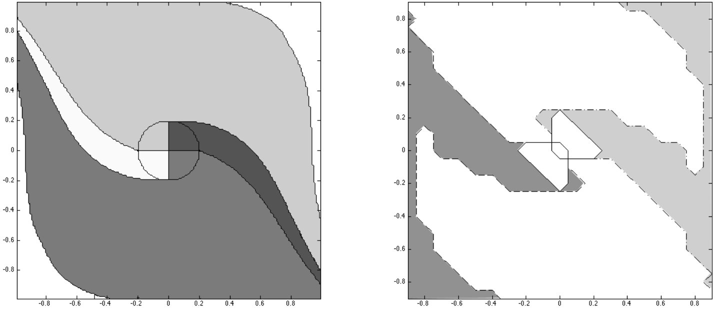


Figure: Van Der Pol: Exact decomposition and two (of the four) approximated independent subsets with a course grind of 15^2 points.

Conclusions

- In this last years the Patchy approach **aroused a large interest** in the Numerical HJ community
- Patchy approach is showing to be **effective** in various (non trivial) situations
- Independent domains reconstruction seems to be a good modification/tool to have a proof of convergence

- add **sparsity**? $\rightarrow$ Linz (Austria)?

The other side of the coin..

Thank you.