Similarity solutions for the silo filling problem in granular matter theory: a numerical study

Stefano FINZI VITA
(in collaboration with Matteo Fois)

**Numerical methods for PDEs**
(on the occasion of M. Falcone’s 60th birthday)
Rome, December 4, 2014
Outline

1 Introduction
   - The Hadeler-Kuttler model for the filling of a silo
   - Existence and characterization of similarity solutions

2 Finite element approximation of similarity solutions

3 A finite difference scheme for the filling process

4 Numerical experiments
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Granular matter

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- Connections: traffic flows, crowd dynamics, mass transportation, mean field games
The silo problem

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- Granular materials adapt their shape to the container (like a fluid), but the free surface strictly depends on the way they are poured (intensity and dislocation of the sources)

- Moreover, the pressure on the bottom does not grow linearly with the height of the pile, since part of it discharges through arcs of grains against the walls. This has to be well considered in the construction of the container. Unpleasant consequence: silos can suddenly explode and collapse (thousands of cases in the USA every year)
A recent case in Italy  (September 27, 2014)

**BONEMERSE (CREMONA)**

**Cremona, crolla un silo**  
*Morti due operai sepolti dal mais*

Trovati i corpi di due operai di 48 e 54 anni. Erano impegnati a controllare il grosso silo che conteneva circa 700 tonnellate di mais.
The HK model for granular matter in a silo

[Hadeler-Kuttler, '99]

\begin{align*}
\nu_t &= \beta \nabla \cdot (\nu \nabla u) - \gamma (\alpha - |\nabla u|) \nu + f \quad \text{in } \Omega \times (0, T] \\
\nu_t &= \gamma (\alpha - |\nabla u|) \nu \quad \text{in } \Omega \times (0, T] \\
u(x, 0) &= u_0(x) , \quad \nu(x, 0) = 0 \quad \text{in } \Omega \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial \Omega \times (0, T] \\
\end{align*}

(1)

- $\Omega$: cross-section (base) of the silo
- $f$: (small) vertical source, with support $S_f \subset \Omega$
- $u$: standing layer; $\nu$: rolling layer;
- $\alpha$: angle of repose; $\beta, \gamma$: mobility and collision rates of grains
BC for the silo: a motivation

Mass conservation in the absence of the source \( (f \equiv 0) \) implies:

\[
0 = \frac{d}{dt} \int_{\Omega} (u + v) \, dx = \int_{\Omega} \beta \nabla \cdot (v \nabla u) = \int_{\partial \Omega} \beta v \frac{\partial u}{\partial n} \, d\sigma ,
\]

which suggests

\[
v \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega .\tag{2}
\]

On the other hand, the \( v \) layer flows in the direction of \(-\nabla u\), that is \( \partial \Omega \) is an outflow region for it, and we cannot prescribe BC on it. Then (2) reduces to a pure homogeneous Neumann condition on \( u \).
Similarity solutions

Definition

We call a pair of functions \((U(x), V(x))\) a similarity solution of (1) if there exist a time \(t_0\) and a constant \(c\) such that the functions

\[
    u(x, t) = U(x) + c(t - t_0) , \quad v(x, t) = V(x)
\]

solve the system for any \(t \geq t_0\).

In other words, after a certain time the free surface of the standing layer keep growing without changing its shape anymore, at a constant rate \(c\), while the rolling layer stabilizes itself.

Theorem

If \(f = f(x)\) (the source is constant in time), there exists a similarity solution \((U(x), V(x))\) of (1) (\(U\) unique up to an additive constant), with

\[
    c = \frac{1}{|\Omega|} \int_{\Omega} f \, dx \quad \text{(average precipitation)}
\]
Proof

Replacing (3) in (1) we get

\[
\begin{cases}
0 = \beta \nabla \cdot (V \nabla U) - \gamma (\alpha - |\nabla U|) V + f \\
c = \gamma (\alpha - |\nabla U|) V
\end{cases}
\]

(5)

Summing up the two equations we obtain

\[c = \beta \nabla \cdot (V \nabla U) + f;\]

from Green formula and boundary condition, it follows by integration

\[0 = -c|\Omega| + \int_{\Omega} f \, dx \quad \Rightarrow \quad c = \frac{1}{|\Omega|} \int_{\Omega} f \, dx.\]

(6)
Proof (2)

If now we set $W = V \nabla U$ and $g = (f - c)/\beta$, we see that $W$ has to solve

$$\begin{align*}
\nabla \cdot W &= -g \quad \text{in } \Omega \\
W \cdot n &= 0 \quad \text{on } \partial \Omega
\end{align*}$$

(7)

with $\int_{\Omega} g \, dx = 0$. We look for $W$ in the form of $W = \nabla \psi$, so that the potential $\psi$ solves a semidefinite Neumann problem for the Laplacian:

$$\begin{align*}
-\Delta \psi &= g \quad \text{in } \Omega \\
\frac{\partial \psi}{\partial n} &= 0 \quad \text{on } \partial \Omega
\end{align*}$$

(8)

which has a solution (unique up to an additive constant) for the zero mean property of $g$. 

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Proof (3)

From $\psi$ we can uniquely determine $W = \nabla \psi$; then from the second equation in (5):

$$V = \frac{c}{\gamma \alpha} + \frac{1}{\alpha} |\nabla U| \ V = \frac{1}{\gamma \alpha |\Omega|} \int_{\Omega} f \ dx + \frac{1}{\alpha} |W|,$$

and

$$\nabla U = \frac{W}{V},$$

so that $U$ is determined up to an additive constant.

(Q.E.D)
1D formulas

In one dimension one can get from that explicit integral expressions for the similarity solutions. If for example $\Omega$ coincides with the interval $(0, L)$, one has

$$V(x) = \frac{1}{\gamma \alpha L} \int_{0}^{L} f(y) dy + \frac{1}{\alpha} |W(x)|$$  \hspace{1cm} (9)

and

$$U_x(x) = \alpha \frac{W(x)}{\frac{1}{\gamma L} \int_{0}^{L} f(y) dy + |W(x)|},$$  \hspace{1cm} (10)

where

$$W(x) = \frac{x}{\beta L} \int_{0}^{L} f(y) dy - \frac{1}{\beta} \int_{0}^{x} f(y) dy.$$
These expressions give several informations about similarity solutions:

- if the source is constant \( f(x) = k \) for any \( x \in (0, L) \), then \( W(x) \equiv 0 \), \( V(x) = k/(\gamma \alpha) \) and \( U_x \equiv 0 \) (flat free surface);
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- for any nonzero source \(V(x) > 0\) everywhere; at the boundary:

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- $|U_x| \leq \alpha$ everywhere, that is the slope of the standing layer never exceeds the angle of repose, and $U_x = 0$ at the boundary;

- the rolling layer thickness is directly proportional to the source intensity and inversely proportional to $\alpha$;
A point source at the center: the log profile

Let \( \Omega = (0, L) \) and \( f = k \delta_{L/2} \). Then

\[
V(x) = \frac{k}{\gamma \alpha L} + \frac{k}{\alpha \beta L} \min\{x, L - x\}
\]

and

\[
U(x) = \begin{cases} 
\frac{\alpha \gamma}{\beta} (x - \log(1 + x)) & \text{if } x \leq L/2 \\
\frac{\alpha \gamma}{\beta} (L - x - \log(1 + L - x)) & \text{if } x > L/2
\end{cases}
\]
A 2D example: a point source in a cylindrical silo

Let $\Omega = B_R(0)$, and $f = \delta_{(0,0)}$; similarity solutions are radial functions:

$$V(r) = \frac{c}{\gamma \alpha}(1 + \frac{\gamma}{2\beta r}(R^2 - r^2))$$

$$U_r(r) = -\alpha \frac{R^2 - r^2}{R^2 - r^2 + 2\beta r / \gamma} \quad [U_r(0) = -\alpha].$$
1) Solve the semidefinite Neumann problem (8) (its weak formulation) by $P_1$ finite elements on a regular triangulation $T_h$ of $\Omega$.

\[ \psi_h \in V_h, \quad \int_{\Omega} \nabla \psi_h \cdot \nabla \phi \; dx = \int_{\Omega} g \phi \; dx, \quad \forall \phi \in V_h. \quad (15) \]

where $V_h \subset H^1(\Omega)$ is the space of piecewise linear functions on $T_h$. 

Numerical computation of similarity solutions
Numerical computation of similarity solutions

1) Solve the semidefinite Neumann problem (8) (its weak formulation) by $P_1$ finite elements on a regular triangulation $T_h$ of $\Omega_h \subset \Omega$.

If $\Omega$ is not a polygonal domain, it has to be replaced by a suitable set $\Omega_h \subset \Omega$ (union of the triangular elements of $T_h$); $\Omega_h$ will be closed to $\Omega$, but now $g$ loses its zero mean property on $\Omega_h$, so that the discrete problem could not be solvable. Then we have to solve:

$$
\psi_h \in V_h, \quad \int_{\Omega_h} \nabla \psi_h \cdot \nabla \phi \, dx = \int_{\Omega_h} g_h \phi \, dx, \quad \forall \phi \in V_h. \quad (16)
$$

where

$$
g_h = f - c_h, \quad c_h = \frac{1}{|\Omega_h|} \int_{\Omega_h} f \, dx.
$$

Now

$$
\int_{\Omega_h} g_h \, dx = 0 \quad \text{for any } h, \quad \text{and} \quad \lim_{h \to 0} g_h = g,
$$

so that (16) becomes solvable (see e.g. Capuzzo Dolcetta-FV ’88).
Assume for simplicity: \( \alpha = \beta = \gamma = 1 \).

2) Compute the p.w. constant vector on \( T_h \): \( w_h = \nabla \psi_h \).

Then its euclidean norm \( |w_h| \) belongs to the space \( W_h \subset L^2(\Omega) \) of p.w. constant functions on \( T_h \), as well as

\[
\nu_h = c_h + |w_h| = c_h + |\nabla \psi_h|. \tag{17}
\]

3) From \( w_h = \nu_h \nabla u_h \), compute \( \nabla u_h \) in any triangle of \( T_h \).

\[
z_k = \nabla u_h \big|_{\tau_k} = \frac{w_h}{\nu_h} \big|_{\tau_k}, \quad \forall \tau_k \in T_h. \tag{18}
\]
4) Recover $u_h \in V_h$ from its gradient.

- in one dimension one can directly integrate the resulting step function on $T_h$, fixing for example the zero value at the left boundary node to ensure uniqueness, so that:

$$u_h(x_i) = \int_0^{x_i} (u_h)_x \, dx = \sum_{k=1}^{i} \int_{x_{k-1}}^{x_k} z_k = h \sum_{k=1}^{i} z_k$$

- in two dimensions come back to problem (16), replace $\nabla \psi_h$ with its known expression in terms of $v_h \in W_h$, and solve the discrete variational problem

$$u_h \in V_h, \quad \int_{\Omega_h} v_h \nabla u_h \cdot \nabla \phi \, dx = \int_{\Omega_h} g_h \phi \, dx, \quad \forall \phi \in V_h. \quad (19)$$
A numerical scheme for the filling process

- In order to verify that similarity solutions attract the growing profiles of the heaps in the dynamic process of filling the silo, we implemented a numerical scheme for the complete system (1)-(2), by adapting the finite difference approach used in Falcone-FV ’06 for the growing sandpiles on an open table.
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- We put a stopping criterium which ends the evolution when the standing layer does not change anymore its shape, that is when

\[(u_i^{n+1} - u_i^n) \approx c_h \Delta t \quad \forall i\]
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- We put a stopping criterium which ends the evolution when the standing layer does not change anymore its shape, that is when

  \[(u_i^{n+1} - u_i^n) \approx c_h \Delta t \quad \forall i\]

- To compare the reached solutions with the already computed similarity solutions we bring back both to the bottom of the silo, through the transformation

  \[\hat{w}_i = w_i - \min w_i\]
1D case

Let $\Omega = (0, 1)$, $h = 1/(N-1)$ the space step, $\Delta t$ the time step, $x_i$ ($i = 1, \ldots, N$) the nodes of a $h$-uniform mesh. Then the scheme reads as

\begin{align*}
  v_i^{n+1} &= v_i^n + \Delta t (G_i - (1 - |Du_i^n|)v_i^n + f_i) \\
  u_i^{n+1} &= u_i^n + \Delta t (1 - |Du_i^n|)v_i^n \\
  u_i^0 &= v_i^0 = 0 \quad \forall i
\end{align*}

(20)

where $u_i^n, v_i^n, G_i^n$ denote the approximate values of $u, v$ and the flux derivative $(vv_x)_x$ at time $n\Delta t$ in $x_i$; the latter is the upwind approximation in the direction of the sign of $Du_i$, that is

\begin{align*}
  G_i^n &= \begin{cases} 
    (v_{i+1}^n Du_{i+1}^n - v_i^n Du_i^n)/h & \text{if } Du_i^n > 0 \\
    (v_i^n Du_i^n - v_{i-1}^n Du_{i-1}^n)/h & \text{if } Du_i^n < 0
  \end{cases}
\end{align*}

[In each node $x_i$, we define $Du_i = \max\text{mod}(D^- u_i, D^+ u_i)$.]
2D case: square or rectangular cross section for the silo.

- Decompose the flux term as
  \[
  \nabla \cdot (v \nabla u) = (vu_x)_x + (vu_y)_y
  \]

- Apply the 1D approach in each direction.

- The boundary condition reduces to \( u_y = 0 \) on the north and south sides, to \( u_x = 0 \) on the east and west sides. At the four vertices a good compromise is to assume \( u_\nu = 0 \) where \( \nu \) denotes the direction of the diagonal at that point.
Central source: $S_f = (0.45, 0.55)$
Asymmetric source: $S_f = (0.5, 0.6)$
Disconnected source: $S_f = (0.25, 0.35) \cup (0.65, 0.75)$
Test 1: $S_f = B_{0.1}(0.5, 0.5)$ (left: U, right: V, up: FE, down: FD)
Test 2: $S_f = B_{0.1}(0.6, 0.5)$ (left: U, right: V, up: FE, down: FD)
Test 3: $S_f = B_{0.1}(0.3, 0.3) \cup B_{0.1}(0.7, 0.7)$ (left: U, right: V, up: FE, down: FD)
Some movies ....
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