Similarity solutions for the silo filling problem in granular matter theory: a numerical study

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(in collaboration with Matteo Fois)

Numerical methods for PDEs
(on the occasion of M. Falcone's 60th birthday)
Rome, December 4, 2014

Outline

Granular matter

- A shared interest with Maurizio in the last ten years: *sandpile growth on open tables or tables with walls* (finite difference and semi-Lagrangian schemes), *dunes evolution* (modeling and simulations, in progress).

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- Connections: traffic flows, crowd dynamics, mass transportation, mean field games

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- Granular matter: not only sand (cereals, sugar, rocks, pills, etc.)
- Crucial problem in the applications: its storage, that is, filling and emptying a container
- Granular materials adapt their shape to the container (like a fluid), but the free surface strictly depends on the way they are poured (intensity and dislocation of the sources)
- Moreover, the pressure on the bottom does not grow linearly with the height of the pile, since part of it discharges through arcs of grains against the walls. This has to be well considered in the construction of the container. Unpleasant consequence: silos can suddenly explode and collapse (thousands of cases in the USA every year)

A recent case in Italy (september 27, 2014)

BONEMERSE (CREMONA)

Cremona, crolla un silo
Morti due operai sepolti dal mais

Trovati i corpi di due operai di 48 e 54 anni. Erano impegnati a controllare il grosso silos che conteneva circa 700 tonnellate di mais

The HK model for granular matter in a silo

[Hadeler-Kuttler, '99]

$$\begin{cases} v_t = \beta \nabla \cdot (v \, \nabla u) - \gamma(\alpha - |\nabla u|) \, v + f & \text{in } \Omega \times (0, T] \\ u_t = \gamma(\alpha - |\nabla u|) \, v & \text{in } \Omega \times (0, T] \\ u(x,0) = u_0(x) \; , \quad v(x,0) = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \times (0, T] \end{cases} \tag{1}$$

- Ω: cross-section (base) of the silo
- f: (small) vertical source, with support $S_f \subset \Omega$
- u: standing layer; v: rolling layer;
- α: angle of repose; β, γ: mobility and collision rates of grains

BC for the silo: a motivation

Mass conservation in the absence of the source ($f \equiv 0$) implies:

$$0 = \frac{d}{dt}\int_\Omega (u+v)\, dx = \int_\Omega \beta \nabla \cdot (v\, \nabla u) = \int_{\partial\Omega} \beta v \frac{\partial u}{\partial n}\, d\sigma \ ,$$

which suggests

$$v\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \ . \tag{2}$$

On the other hand, the v layer flows in the direction of $-\nabla u$, that is $\partial\Omega$ is an *outflow region* for it, and we cannot prescribe BC on it. Then (2) reduces to a pure homogeneous Neumann condition on u.

Similarity solutions

Definition

We call a pair of functions $(U(x), V(x))$ a similarity solution of (1) if there exist a time t_0 and a constant c such that the functions

$$u(x,t) = U(x) + c(t - t_0)\ , \quad v(x,t) = V(x) \tag{3}$$

solve the system for any $t \geq t_0$.

In other words, after a certain time the free surface of the standing layer keep growing without changing its shape anymore, at a constant rate c, while the rolling layer stabilizes itself.

Theorem

If $f = f(x)$ (the source is constant in time), there exists a similarity solution $(U(x), V(x))$ of (1) (U unique up to an additive constant), with

$$c = \frac{1}{|\Omega|} \int_\Omega f\ dx \quad (\textit{average precipitation}) \tag{4}$$

Proof

Replacing (3) in (1) we get

$$\begin{cases} 0 = \beta\nabla \cdot (V\,\nabla U) - \gamma(\alpha - |\nabla U|)\,V + f \\ c = \gamma(\alpha - |\nabla U|)\,V \end{cases} \tag{5}$$

Summing up the two equations we obtain

$$c = \beta\nabla \cdot (V\,\nabla U) + f;$$

from Green formula and boundary condition, it follows by integration

$$0 = -c|\Omega| + \int_\Omega f\ dx \quad \Rightarrow \quad c = \frac{1}{|\Omega|}\int_\Omega f\ dx. \tag{6}$$

Proof (2)

If now we set $W = V\nabla U$ and $g = (f - c)/\beta$, we see that W has to solve

$$\begin{cases} \nabla \cdot W = -g & \text{in } \Omega \\ W \cdot n = 0 & \text{on } \partial\Omega \end{cases} \tag{7}$$

with $\int_\Omega g \, dx = 0$. We look for W in the form of $W = \nabla\psi$, so that the potential ψ solves a semidefinite Neumann problem for the Laplacian:

$$\begin{cases} -\Delta\psi = g & \text{in } \Omega \\ \frac{\partial\psi}{\partial n} = 0 & \text{on } \partial\Omega \end{cases} \tag{8}$$

which has a solution (unique up to an additive constant) for the zero mean property of g.

Proof (3)

From ψ we can uniquely determine $W = \nabla\psi$; then from the second equation in (5):

$$V = \frac{c}{\gamma\alpha} + \frac{1}{\alpha}|\nabla U|V = \frac{1}{\gamma\alpha|\Omega|}\int_\Omega f\ dx + \frac{1}{\alpha}|W|,$$

and

$$\nabla U = \frac{W}{V},$$

so that U is determined up to an additive constant.

(Q.E.D)

1D formulas

In one dimension one can get from that explicit integral expressions for the similarity solutions.
If for example Ω coincides with the interval $(0, L)$, one has

$$V(x) = \frac{1}{\gamma\alpha L}\int_0^L f(y)dy + \frac{1}{\alpha}|W(x)| \tag{9}$$

and

$$U_x(x) = \alpha \frac{W(x)}{\frac{1}{\gamma L}\int_0^L f(y)dy + |W(x)|}, \tag{10}$$

where

$$W(x) = \frac{x}{\beta L}\int_0^L f(y)dy - \frac{1}{\beta}\int_0^x f(y)dy.$$

These expressions give several informations about similarity solutions:

- if the source is constant ($f(x) = k$ for any $x \in (0, L)$), then $W(x) \equiv 0$, $V(x) = k/(\gamma\alpha)$ and $U_x \equiv 0$ (*flat free surface*);

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- for any nonzero source $V(x) > 0$ everywhere; at the boundary:

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- the rolling layer thickness is directly proportional to the source intensity and inversely proportional to α;

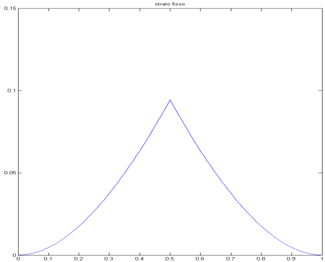
A point source at the center: the log profile

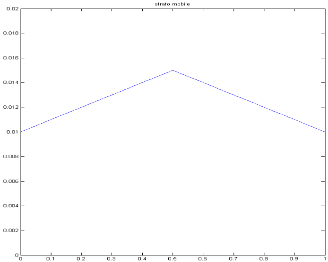
Let $\Omega = (0, L)$ and $f = k\delta_{L/2}$. Then

$$V(x) = \frac{k}{\gamma\alpha L} + \frac{k}{\alpha\beta L}\min\{x, L-x\} \tag{11}$$

and

$$U(x) = \begin{cases} \frac{\alpha\gamma}{\beta}(x - \log(1+x)) & \text{if } x \leq L/2 \\ \\ \frac{\alpha\gamma}{\beta}(L - x - \log(1+L-x)) & \text{if } x > L/2 \end{cases} \tag{12}$$



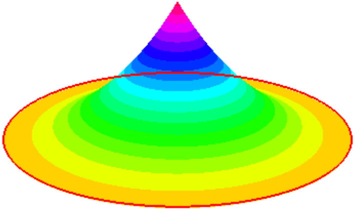


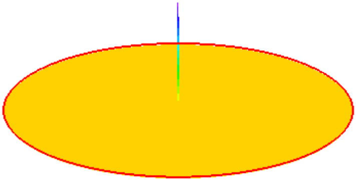
A 2D example: a point source in a cylindrical silo

Let $\Omega = B_R(0)$, and $f = \delta_{(0,0)}$; similarity solutions are radial functions :

$$V(r) = \frac{c}{\gamma\alpha}\left(1 + \frac{\gamma}{2\beta r}(R^2 - r^2)\right) \tag{13}$$

$$U_r(r) = -\alpha \frac{R^2 - r^2}{R^2 - r^2 + 2\beta r/\gamma} \qquad [U_r(0) = -\alpha]. \tag{14}$$





Numerical computation of similarity solutions

1) *Solve the semidefinite Neumann problem (8) (its weak formulation) by* P_1 *finite elements on a regular triangulation* T_h *of* Ω*.*

$$\psi_h \in V_h, \quad \int_\Omega \nabla\psi_h \cdot \nabla\phi \, dx = \int_\Omega g\phi \, dx \ , \quad \forall\phi \in V_h. \qquad (15)$$

where $V_h \subset H^1(\Omega)$ is the space of piecewise linear functions on T_h.

Numerical computation of similarity solutions

1) *Solve the semidefinite Neumann problem (8) (its weak formulation) by P_1 finite elements on a regular triangulation T_h of $\Omega_h \subset \Omega$.*

If Ω is not a polygonal domain, it has to be replaced by a suitable set $\Omega_h \subset \Omega$ (union of the triangular elements of T_h); Ω_h will be closed to Ω, but now g loses its zero mean property on Ω_h, so that the discrete problem could not be solvable. Then we have to solve:

$$\psi_h \in V_h, \quad \int_{\Omega_h} \nabla\psi_h \cdot \nabla\phi \, dx = \int_{\Omega_h} g_h \phi \, dx \ , \quad \forall \phi \in V_h. \tag{16}$$

where

$$g_h = f - c_h, \quad c_h = \frac{1}{|\Omega_h|} \int_{\Omega_h} f \, dx \ .$$

Now

$$\int_{\Omega_h} g_h \, dx = 0 \quad \text{for any } h, \quad \text{and} \quad \lim_{h\to 0} g_h = g,$$

so that (16) becomes solvable (see e.g. Capuzzo Dolcetta-FV '88).

Assume for simplicity: $\alpha = \beta = \gamma = 1$.

2) *Compute the p.w. constant vector on* T_h*:* $w_h = \nabla \psi_h$.

Then its euclidean norm $|w_h|$ belongs to the space $W_h \subset L^2(\Omega)$ of p.w. constant functions on T_h, as well as

$$v_h = c_h + |w_h| = c_h + |\nabla \psi_h| \ . \tag{17}$$

3) *From* $w_h = v_h \nabla u_h$*, compute* ∇u_h *in any triangle of* T_h*.*

$$z_k = \nabla u_h|_{\tau_k} = \frac{w_h}{v_h}|_{\tau_k}, \quad \forall \tau_k \in T_h \ . \tag{18}$$

4) *Recover* $u_h \in V_h$ *from its gradient.*

- in one dimension one can directly integrate the resulting step function on T_h, fixing for example the zero value at the left boundary node to ensure uniqueness, so that:

$$u_h(x_i) = \int_0^{x_i} (u_h)_x \, dx = \sum_{k=1}^{i} \int_{x_{k-1}}^{x_k} z_k = h \sum_{k=1}^{i} z_k$$

- in two dimensions come back to problem (16), replace $\nabla \psi_h$ with its known expression in terms of $v_h \in W_h$, and solve the discrete variational problem

$$u_h \in V_h, \quad \int_{\Omega_h} v_h \nabla u_h \cdot \nabla \phi \, dx = \int_{\Omega_h} g_h \phi \, dx \,, \quad \forall \phi \in V_h. \qquad (19)$$

A numerical scheme for the filling process

- In order to verify that similarity solutions attract the growing profiles of the heaps in the dynamic process of filling the silo, we implemented a numerical scheme for the complete system (1)-(2), by adapting the finite difference approach used in Falcone-FV '06 for the growing sandpiles on an open table.

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- We put a stopping criterium which ends the evolution when the standing layer does not change anymore its shape, that is when

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- To compare the reached solutions with the already computed similarity solutions we bring back both to the bottom of the silo, through the transformation

$$\hat{w}_i = w_i - \min_i w_i$$

1D case

Let $\Omega = (0,1)$, $h = 1/(N-1)$ the space step, Δt the time step, x_i ($i = 1, .., N$) the nodes of a h-uniform mesh. Then the scheme reads as

$$\begin{cases} v_i^{n+1} = v_i^n + \Delta t(G_i - (1 - |Du_i^n|)v_i^n + f_i) \\ u_i^{n+1} = u_i^n + \Delta t(1 - |Du_i^n|)v_i^n \\ u_i^0 = v_i^0 = 0 \quad \forall i \end{cases} \tag{20}$$

where u_i^n, v_i^n, G_i^n denote the approximate values of u, v and the flux derivative $(vu_x)_x$ at time $n\Delta t$ in x_i; the latter is the upwind approximation in the direction of the sign of Du_i, that is

$$G_i^n = \begin{cases} (v_{i+1}^n Du_{i+1}^n - v_i^n Du_i^n)/h & \text{if } Du_i^n > 0 \\ (v_i^n Du_i^n - v_{i-1}^n Du_{i-1}^n)/h & \text{if } Du_i^n < 0 \end{cases}$$

[In each node x_i, we define $Du_i = \text{maxmod}(D^- u_i, D^+ u_i)$.]

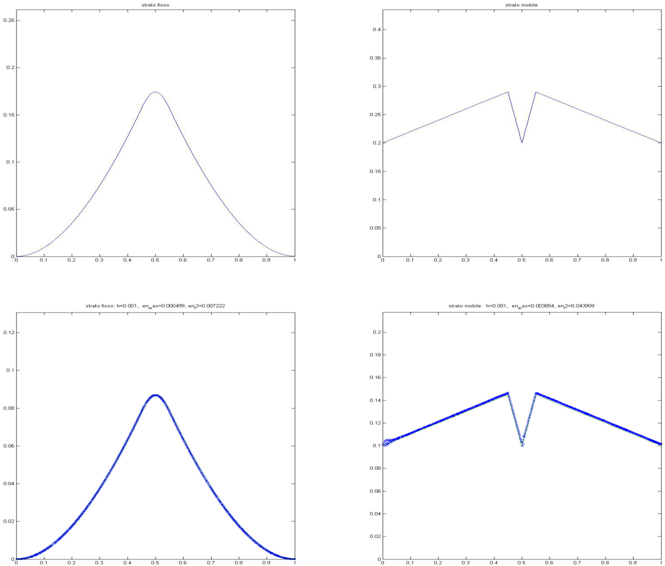
2D case: square or rectangular cross section for the silo.

- Decompose the flux term as

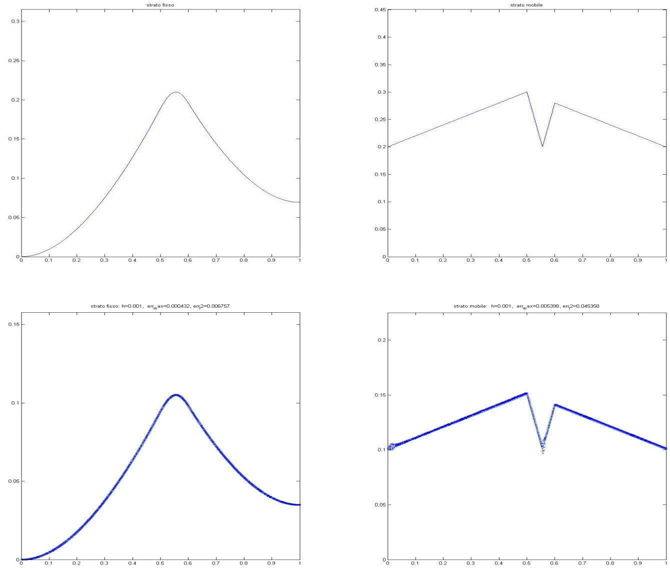
$$\nabla \cdot (v \nabla u) = (v u_x)_x + (v u_y)_y$$

- Apply the 1D approach in each direction.
- The boundary condition reduces to $u_y = 0$ on the north and south sides, to $u_x = 0$ on the east and west sides. At the four vertices a good compromise is to assume $u_\nu = 0$ where ν denotes the direction of the diagonal at that point.

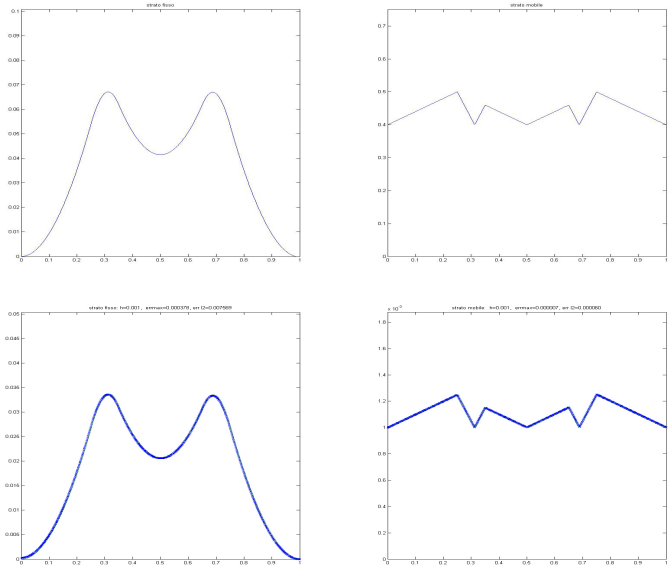
Central source: $S_f = (0.45, 0.55)$

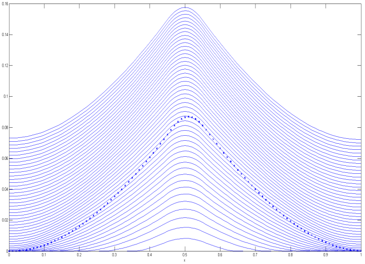


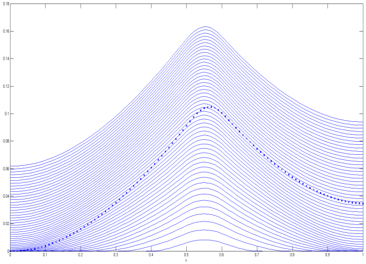
Asymmetric source: $S_f = (0.5, 0.6)$

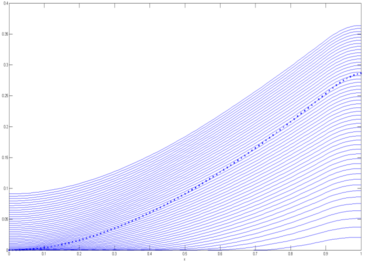


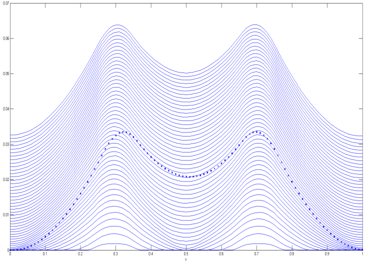
Disconnected source: $S_f = (0.25, 0.35) \cup (0.65, 0.75)$



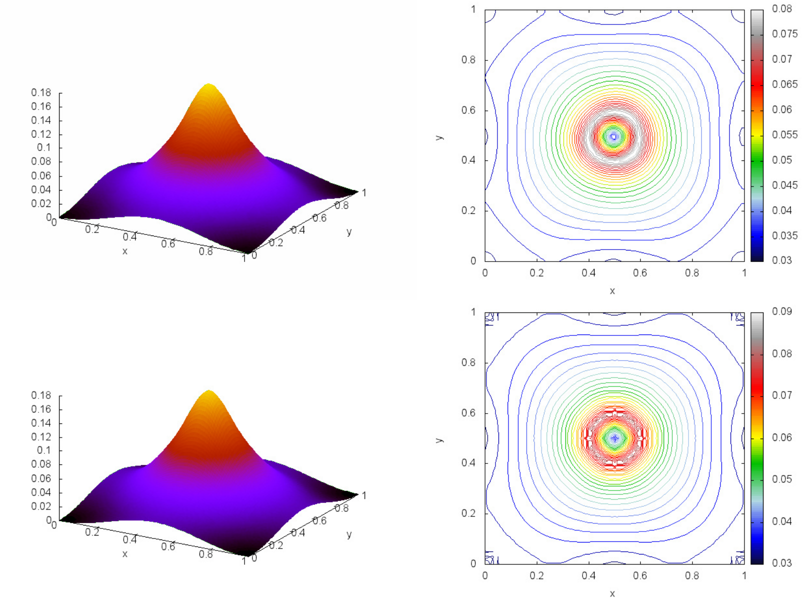




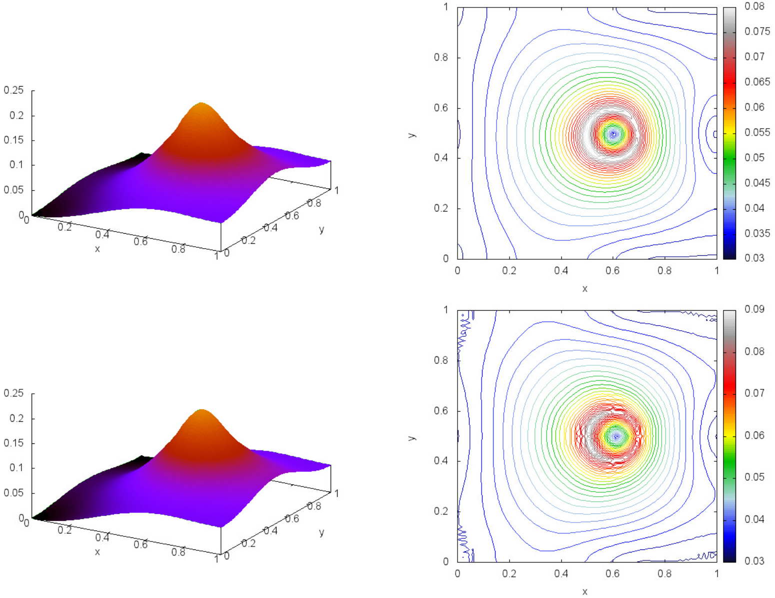




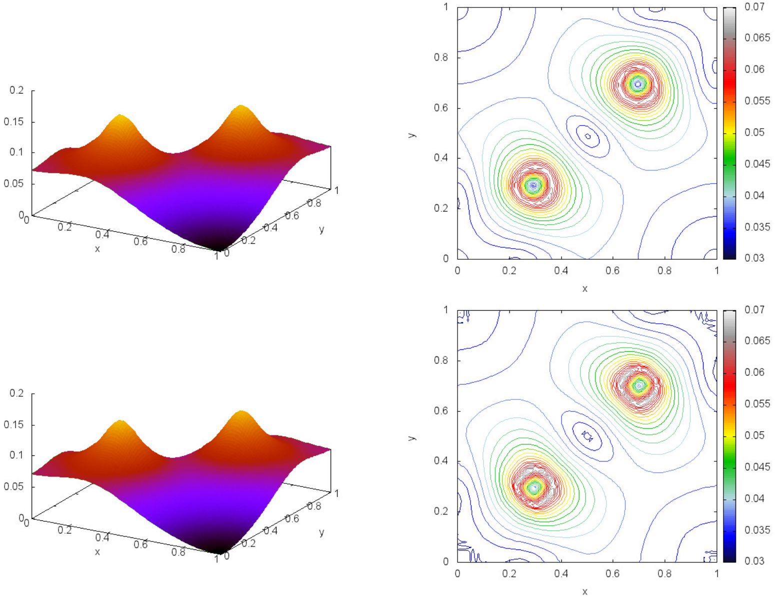
Test 1: $S_f = B_{0.1}(0.5, 0.5)$ (left: U, right: V, up: FE, down: FD)



Test 2: $S_f = B_{0.1}(0.6, 0.5)$ (left: U, right: V, up: FE, down: FD)



Test 3: $S_f = B_{0.1}(0.3, 0.3) \cup B_{0.1}(0.7, 0.7)$ (left: U, right: V, up: FE, down: FD)



Some movies ....

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