Stabilization with discounted optimal control

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based on joint work with

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Setup

We consider continuous time finite dimensional control systems

\[
\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0
\]

with \( x(t) \in \mathbb{R}^n, u(t) \in U \subseteq \mathbb{R}^m, \quad u \in U = L^\infty(\mathbb{R}, U) \)
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**Goal**: Given an equilibrium \( x^e \) (i.e., \( f(x^e, u^e) = 0 \) for some \( u^e \in U \)), for any initial value \( x_0 \) find a control which steers the trajectory to \( x^e \) and keeps it there
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**Approach:** Compute this \( u \) via optimal control
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**Approach:** Compute this \( u \) via optimal control, preferably in feedback form \( u(t) = F(x(t)) \)
Special case: linear quadratic optimal control

For the special case of linear systems

\[ \dot{x}(t) = Ax(t) + Bu(t) \]
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the linear quadratic optimal control problem

\[
\text{minimize } \int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t) \, dt
\]

with matrices \( R > 0 \), \( Q > 0 \) yields such controls.
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\[ \min_{u \in U} \int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t) \, dt \]

with matrices \( R > 0, Q > 0 \) yields such controls.

This linear quadratic problem is efficiently solvable via the algebraic Riccati equation.
Nonlinear case

In the nonlinear case

\[ \dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0 \]

under standard regularity assumptions, stabilization can be achieved via the optimal control problem

\[
\min_{u \in U} \int_0^\infty \ell(x(t), u(t)) dt
\]

with \( \ell \) satisfying \( \ell(x, u) > 0 \) whenever \( x \neq x^e \).
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**Drawback:** this problem is very difficult to solve
Solution strategies

Solution strategies for

\[
\min_{u \in U} \int_0^\infty \ell(x(t), u(t)) \, dt
\]

Receding Horizon (aka Model Predictive) Control:

For \(i = 0, 1, 2, \ldots\), solve iteratively

\[
\min_{u \in U} \int_{t_i}^{t_i+T} \ell(x(t), u(t)) \, dt
\]

and apply the optimal control on \([t_i, t_i+1]\) (usually \(t_i+1 < T\)).

Advantages: yields a feedback-like control even if the problems are solved trajectorywise, very efficient for moderate \(T\).

Disadvantage: very hard to solve for large \(T\) (may be necessary to ensure stability).

Lars Grüne, Stabilization with discounted optimal control, p. 5
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Zubov's method: Transform the problem via the Kružkov transform

\[
\min_{u \in U} \int_0^\infty \ell(x(t), u(t)) \, dt = 1 - \exp\left(-\int_0^\infty \ell(x(t), u(t)) \, dt\right)
\]

Advantages: optimal value function is now bounded, dynamic programming operator is a contraction

Disadvantage: Hamilton-Jacobi-Bellman equation has a singularity at \( x \to x_e \) control only stabilizes a neighborhood of \( x_e \)
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Minimum time method:

Solve

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\minimize \int_{0}^{t(x,u)} \ell(x(t), u(t)) \, dt
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where \( t(x, u) \) is the minimum time to reach a target around \( x^e \)
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Advantage: methods for minimum time problems can be applied

Disadvantage: again, the resulting control only stabilizes a neighborhood of \( x^e \) (including the target)
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New idea:

Solve a discounted problem with \( \delta > 0 \)

\[
\min_{\delta > 0} \int_0^\infty e^{-\delta t} \ell(x(t), u(t)) \, dt
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(thanks to Maurizio we know how to solve them 😊)
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Question: will the optimal control of the discounted problem stabilize the system?
Notation and assumptions

We define the discounted functional

\[ J_\delta(x_0, u) := \int_0^\infty e^{-\delta t} \ell(x(t), u(t)) dt \]

and the optimal value function

\[ V_\delta(x_0) := \inf_{u \in U} J_\delta(x_0, u) \]
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For simplicity, we do not consider state constraints in this talk (but results can be extended provided \( V_\delta \) remains continuous)
Towards a sufficient condition

For model predictive control, consider the undiscounted optimal value function

\[ V_0(x_0) = \inf_{u \in U} \int_0^\infty \ell(x(t), u(t)) dt \]
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$$V_0(x_0) = \inf_{u \in U} \int_0^\infty \ell(x(t), u(t)) dt$$

Then, model predictive control stabilizes the equilibrium if the inequality

$$V_0(x_0) \leq \gamma \min_{u \in U} \ell(x_0, u)$$

holds for some $\gamma > 0$ and all $x_0 \in \mathbb{R}^n$.
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The larger \( \gamma \), the larger \( T \) must be for guaranteeing stability
Main theorem

Theorem: Assume that

(i) $V_\delta$ satisfies the inequality

$$\alpha_1(\|x - x^e\|) \leq V_\delta(x) \leq \alpha_2(\|x - x^e\|)$$

for functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and all $x \in \mathbb{R}^n$

(ii) there exists $K > \delta$ such that the inequality

$$KV_\delta(x) \leq \min_{u \in U} \ell(x, u)$$

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Then the discounted optimal control stabilizes the equilibrium $x^e$
Idea of proof

Idea: prove that $V_\delta$ is a Lyapunov function
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$t \mapsto V_\delta(x^*(t))$ is an absolutely continuous function
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For almost all $t \geq 0$, the dynamic programming principle and
(ii) $KV_\delta(x) \leq \min_{u \in U} \ell(x, u)$ with $K > \delta$ imply:
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$$\frac{d}{dt} V_\delta(x^*(t)) = \delta V_\delta(x^*(t)) - \ell(x^*(t), u^*(t))$$
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$$\frac{d}{dt} V_\delta(x^*(t)) = \delta V_\delta(x^*(t)) - \ell(x^*(t), u^*(t)) \leq -(K - \delta)V_\delta(x^*(t)) > 0$$
Idea of proof

**Idea:** prove that \( V_\delta \) is a Lyapunov function

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For almost all \( t \geq 0 \), the dynamic programming principle and (ii) \( KV_\delta(x) \leq \min_{u \in U} \ell(x, u) \) with \( K > \delta \) imply:

\[
\frac{d}{dt} V_\delta(x^*(t)) = \delta V_\delta(x^*(t)) - \ell(x^*(t), u^*(t)) \leq -(K - \delta) V_\delta(x^*(t)) < 0
\]

\[ \Rightarrow \quad V_\delta(x^*(t)) \leq e^{-(K-\delta)t} V_\delta(x_0) \]
Idea of proof

Idea: prove that $V_\delta$ is a Lyapunov function

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$$\frac{d}{dt} V_\delta(x^*(t)) = \delta V_\delta(x^*(t)) - \ell(x^*(t), u^*(t)) \leq -(K - \delta)V_\delta(x^*(t)) > 0$$

$$\Rightarrow \quad V_\delta(x^*(t)) \leq e^{-(K-\delta)t}V_\delta(x_0)$$

Together with the bounds (i) on $V_\delta$, this implies the claimed asymptotic stability.
Approximately optimal trajectories

The statement can be extended to approximately optimal controls $\tilde{u}^*$
Approximately optimal trajectories

The statement can be extended to approximately optimal controls $\tilde{u}^*$, provided the relative error along the trajectory $\tilde{x}^*$, i.e.,

$$\frac{|V_\delta(\tilde{x}^*(t)) - J_\delta(\tilde{x}^*(t), \tilde{u}^*(\cdot + t))|}{V_\delta(\tilde{x}^*(t))}$$

is sufficiently small for all $t$. 
Approximately optimal trajectories

The statement can be extended to approximately optimal controls \( \tilde{u}^* \), provided the relative error along the trajectory \( \tilde{x}^* \), i.e.,

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\frac{|V_\delta(\tilde{x}^*(t)) - J_\delta(\tilde{x}^*(t), \tilde{u}^*(\cdot + t))|}{V_\delta(\tilde{x}^*(t))}
\]

is sufficiently small for all \( t \).

If the absolute error

\[
|V_\delta(\tilde{x}^*(t)) - J_\delta(\tilde{x}^*(t), \tilde{u}^*(\cdot + t))|
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is small, trajectories converge to a neighborhood of \( x^e \) whose size shrinks with the error.
Discussion of the conditions

How restrictive are the conditions of the theorem?
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for functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and all $x \in \mathbb{R}^n$

(ii) there exists $K > \delta$ such that the inequality

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holds for all $x \in \mathbb{R}^n$
Discussion of the condition (i)

Assumption (i):

$$\alpha_1(\|x - x^e\|) \leq V_\delta(x) \leq \alpha_2(\|x - x^e\|)$$
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These bounds can be assured by appropriate choice of \( \ell \)
Discussion of the condition (i)

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\[ \alpha_1(\|x - x^e\|) \leq V_\delta(x) \leq \alpha_2(\|x - x^e\|) \]

These bounds can be assured by appropriate choice of \( \ell \):

\( \ell \) must be sufficiently flat near \( x^e \), sufficiently large away from \( x^e \) and fast dynamics must be penalized sufficiently strong
Discussion of condition (ii)

Assumption (ii): There exists $K > \delta$ with

$$KV_\delta(x) \leq \min_{u \in U} \ell(x, u)$$

For $\delta < 1/\gamma$, this inequality follows from the stability condition for model predictive control $V_0(x) \leq \gamma \min_{u \in U} \ell(x, u)$.

This condition, in turn, is always satisfied for suitable $\ell$ if the system is finite time or exponentially controllable to $x_e$. 
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Example

\[ \dot{x}_1 = -x_1 + x_1 x_2 \]
\[ \dot{x}_2 = x_2 - x_1 x_2 \]

The predator-prey model (\(x_1\) = predator, \(x_2\) = prey) which for \(u = 0\) has an equilibrium at \((1, 1)\)^T and periodic trajectories. The \(u\)-term models that the predators are hunted for. The goal is to stabilize \(x_e = (1, 1.26)^T\) which is an equilibrium for \(u_e = 0\). To this end we use \(U = [0, 1]\) and the running cost \(\ell(x, u) = \|x - x_e\|^2 + |u - u_e|^2\). Numerical computations were performed for different \(\delta\) using the occupational measure approach of V. Gaitsgory et al.
Example

\[
\begin{align*}
\dot{x}_1 &= -x_1 + x_1 x_2 - u x_1 \\
\dot{x}_2 &= x_2 - x_1 x_2
\end{align*}
\]

\(\Rightarrow\) predator-prey model \((x_1 = \text{predator}, x_2 = \text{prey})\) which for \(u = 0\) has an equilibrium at \((1, 1)^T\) and periodic trajectories.
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\[ \rightsquigarrow \text{ predator-prey model (} x_1 = \text{ predator, } x_2 = \text{ prey)} \text{ which for } u = 0 \text{ has an equilibrium at } (1, 1)^T \text{ and periodic trajectories.} \]

The \( u \)-term models that the predators are hunted for
Example

\[ \dot{x}_1 = -x_1 + x_1 x_2 - u x_1 \]
\[ \dot{x}_2 = x_2 - x_1 x_2 \]

\(\leadsto\) predator-prey model \((x_1 = \text{predator}, \ x_2 = \text{prey})\) which for \(u = 0\) has an equilibrium at \((1, 1)^T\) and periodic trajectories.

The \(u\)-term models that the predators are hunted for

The goal is to stabilize \(x^e = (1, 1.26)^T\) which is an equilibrium for \(u^e = 0.26\).
Example

\[ \dot{x}_1 = -x_1 + x_1 x_2 - u x_1 \]
\[ \dot{x}_2 = x_2 - x_1 x_2 \]

\text{\(\rightsquigarrow\) predator-prey model \((x_1 = \text{predator}, \ x_2 = \text{prey})\) which for \(u = 0\) has an equilibrium at \((1, 1)^T\) and periodic trajectories.}

The \(u\)-term models that the predators are hunted for

The goal is to stabilize \(x^e = (1, 1.26)^T\) which is an equilibrium for \(u^e = 0.26\). To this end we use \(U = [0, 1]\) and the running cost

\[ \ell(x, u) = \|x - x^e\|_2^2 + |u - u^e|^2 \]
Example

\[
\begin{align*}
\dot{x}_1 &= -x_1 + x_1x_2 - ux_1 \\
\dot{x}_2 &= x_2 - x_1x_2
\end{align*}
\]

\(\rightsquigarrow\) predator-prey model \((x_1 = \text{predator}, x_2 = \text{prey})\) which for \(u = 0\) has an equilibrium at \((1, 1)^T\) and periodic trajectories.

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\ell(x, u) = \|x - x^e\|^2 + |u - u^e|^2
\]

Numerical computations were performed for different \(\delta\) using the occupational measure approach of V. Gaitsgory et al.
Example

Uncontrolled
Example

Stabilized at $x^c = (1, 1.26)^T$
Conclusions

- Discounted optimal control can be used for the stabilization of nonlinear systems

Reference:
V. Gaitsgory, L. Grüne, N. Thatcher
Stabilization with discounted optimal control
Preprint available from num.math.uni-bayreuth.de
Conclusions

- **Discounted optimal control** can be used for the stabilization of nonlinear systems.
- **Sufficient stability conditions** are similar to those for model predictive control.

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Auguri, Maurizio!