Dedicato a Maurizio Falcone, 4 Dicembre 2014

Most unstable switching laws for switched linear systems.

Nicola Guglielmi (University of L'Aquila)

jointly with:

Marino Zennaro (University of Trieste)

Linear switched dynamical systems (LSS)

We consider the linear switched system (for $n = 0, 1, \ldots$)

$$x(n+1) \;=\; A_{\sigma(n)}\, x(n), \qquad \sigma : \mathbb{N} \longrightarrow \mathcal{I} := \{1, 2, \ldots, m\}$$

where $x(0) \in \mathbb{R}^k$ and $A_{\sigma(n)} \in \mathbb{R}^{k\times k}$ is an element of the finite (this simplifies presentation) family of matrices

$$\mathcal{F} = \{A_i\}_{i\in\mathcal{I}}$$

associated to the system and σ denotes the switching law.

We are interested in the following issues:

- Stability properties of the solutions in terms of spectral characteristics of the associated family $\mathcal{F}$.
- Describing geometry of worst/best case solutions of LSS.

A few applications

(1) Discontinuous linear ODEs: switched control systems.

Liberzon: Switching in systems and control, Birkhäuser, 2003

di Bernardo, Budd, Champneys, Kowalczyk, Piecewise-smooth dynamical systems, Springer, 2008

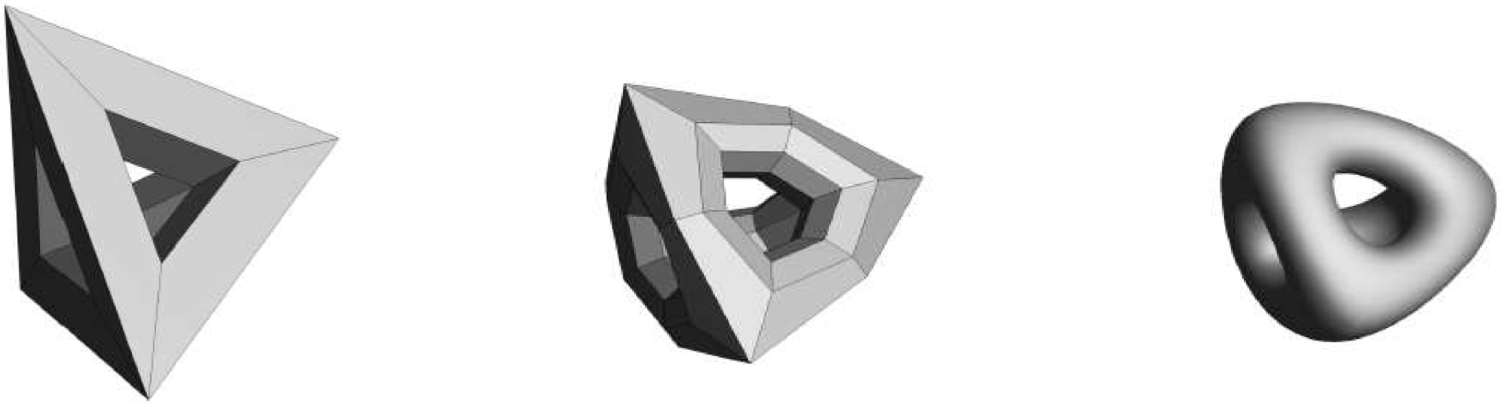
(2) Stability of numerical methods for differential equations.

e.g. G. & Zennaro: Zero stability of variable stepsize BDF formulæ, Numer. Math., 2001

(3) Wavelets, subdivision and refinement schemes.

Daubechies: Comm. Pure Appl. Math., 1988, Heil & Strang: IMA Math. Appl., 1995.

Sabin: Analysis and design of univariate subdivision schemes, Springer-Verlag, 2010



(4) Consensus problems.

Olshevsky & Tsitsiklis: Convergence speed in distributed consensus and averaging, SIAM Rev., 2011

Stability issues: worst case analysis

Aim: determining the most unstable switching law (MUSL), i.e. the law σ giving the solution with highest rate of growth ρ. Specifically we look for a law σ and a norm $\|\cdot\|$ such that

$$\|x(n)\| = \rho^n \|x(0)\| \qquad \text{for all } n.$$

The MUSL can be characterized using optimal control techniques. The variational approach leads to a Hamilton–Jacobi– Bellman equation.

Its solution is referred to as a Barabanov norm of the LSS.

"Although the Barabanov norm was studied extensively, it seems that there are only few examples where it was actually computed in closed form" (**Teichner and Margaliot, 2012**).

The multiplicative semigroup

We consider the set of products of degree n,

$$\Sigma_n(\mathcal{F}) \;=\; \{A_{i_n}\dots A_{i_1} \mid i_1,\dots,i_n \in \mathcal{I}\}$$

and define the product semigroup

$$\Sigma(\mathcal{F}) \;=\; \bigcup_{n\geq 1} \Sigma_n(\mathcal{F}).$$

Goals.

- Compute maximal asymptotic rate of growth ρ of $\Sigma(\mathcal{F})$.
- Determine a norm $\|\cdot\|$ such that for any $x(0)$ there exists a switching law σ for which the trajectory

$$x(n) = P_n x(0), \qquad P_n = A_{\sigma(n)}\dots A_{\sigma(0)}$$

fulfils $\|x(n)\| = \rho^n \|x(0)\|$ for all n.

Generalizing the spectral radius

(1) Joint spectral radius (**Rota & Strang '60**):

$$\widehat{\rho}(\mathcal{F}) = \limsup_{n\to\infty} \widehat{\rho}_n(\mathcal{F})^{1/n} \quad \text{with } \widehat{\rho}_n(\mathcal{F}) = \max_{P\in\Sigma_n(\mathcal{F})} \|P\|$$

(2) Generalized spectral radius (**Daubechies** *et al.* **'92**):

$$\overline{\rho}(\mathcal{F}) = \limsup_{n\to\infty} \overline{\rho}_n(\mathcal{F})^{1/n} \quad \text{with } \overline{\rho}_n(\mathcal{F}) = \max_{P\in\Sigma_n(\mathcal{F})} \rho(P)$$

(3) Common spectral radius (**Elsner '95**):

$$\nu(\mathcal{F}) = \inf_{\|\cdot\|\in\mathcal{N}} \|\mathcal{F}\| \quad \text{with} \quad \|\mathcal{F}\| = \max_{i\in\mathcal{I}} \|A_i\|$$

where $\mathcal{N}$ is the set of operator norms.

All 3 quantities result to be equal so we denote them as $\rho(\mathcal{F})$.

Framework

Daubechies & Lagarias proved the following inequality, where P is any product of degree d and $\| \cdot \|$ any operator norm,

$$\rho(P)^{1/d} \leq \rho(\mathcal{F}) \leq \|F\|$$

Definitions.

1. We say that $\mathcal{F}$ has the finiteness property if there exists a spectrum maximizing product, that is a product for which the left inequality is an equality.
2. We say that $\mathcal{F}$ is non defective if there exists an operator norm for which the right inequality becomes an equality. Such norm is called an extremal norm.

Both properties appear to be generic but there is no proof.

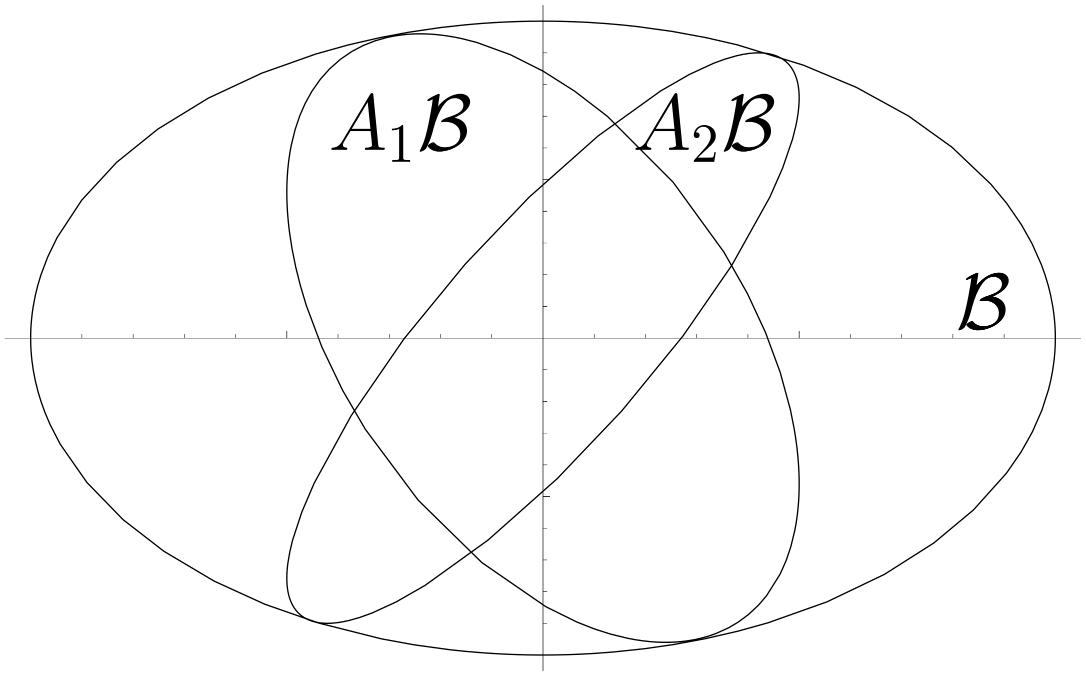
Extremal norms

Definition [extremal norm]

We say that $\|\cdot\|$ is an extremal norm for $\mathcal{F}$ if $\|\mathcal{F}\| = \rho(\mathcal{F})$, i.e.

$$\max_{i\in\mathcal{I}} \|A_i x\| \leq \rho(\mathcal{F})\|x\| \quad \forall x \in \mathbb{R}^k.$$

Assume $\rho(\mathcal{F}) = 1$ and let $\mathcal{B}$ the unit ball of $\|\cdot\|$, then $A_i x \in \mathcal{B}$ for all $x \in \mathcal{B}$ and $i \in \mathcal{I}$. Geometrically:



Extremal Barabanov norms

Definition [Barabanov norm]
We say that an extremal norm $\|\cdot\|$ for the family $\mathcal{F}$ is an (invariant) Barabanov norm if

$$\max_{i \in \mathcal{I}} \|A_i x\| = \rho(\mathcal{F}) \|x\| \quad \forall x \in \mathbb{R}^k.$$

Barabanov norms identify - for any initial vector - a most unstable solution associated to a MUSL.

Theorem (**Barabanov, 1988**)
Assume that a family of matrices $\mathcal{F}$ is irreducible. Then there exists a Barabanov operator norm for $\mathcal{F}$.

As a consequence the existence of a Barabanov norm appears generic as well as the existence of a MUSL.

Computational framework

Recent algorithms proposed in the literature start from the guess of a candidate spectrum maximizing product and attempt to obtain an extremal norm.

Assumptions.

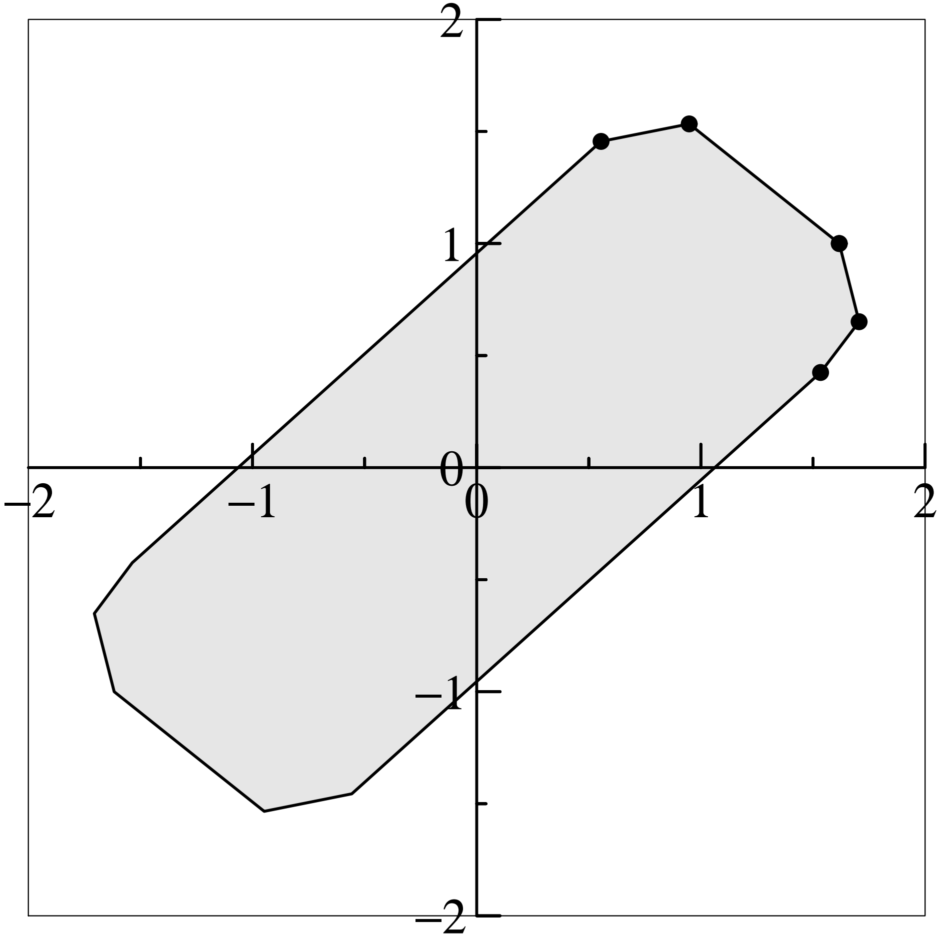
(i) Since the joint spectral radius $\rho(\mathcal{F})$ is a positively homogenous function of the set of matrices, for simplicity we assume $\rho(\mathcal{F}) = 1$.

(ii) We assume that $\mathcal{F}$ is non defective and has the finiteness property.

These assumptions imply that there exists a product P_* such that $\rho(P_*) = 1$ and a norm $\| \cdot \|_*$ such that $\|\mathcal{F}\|_* = 1$.

The polytope algorithms

These algorithms, proposed e.g. by **G., Wirth & Zennaro, 2005, G. & Protasov, 2013** attempt to compute an extremal polytope norm, that is an extremal norm whose unit ball is a centrally symmetric polytope $\mathcal{P}$.



Starting from a suitable initial vector (the leading eigenvector v_1 of the spectrum maximizing product P_*), the algorithms compute $\mathcal{P}$ recursively, i.e.

$\mathcal{P} = \mathrm{conv hull}(\pm v_1, \pm A_1 v_1, \ldots)$

Example 1

Let $\mathcal{F} = \{A_1, A_2\}$

$$A_1 = \alpha \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

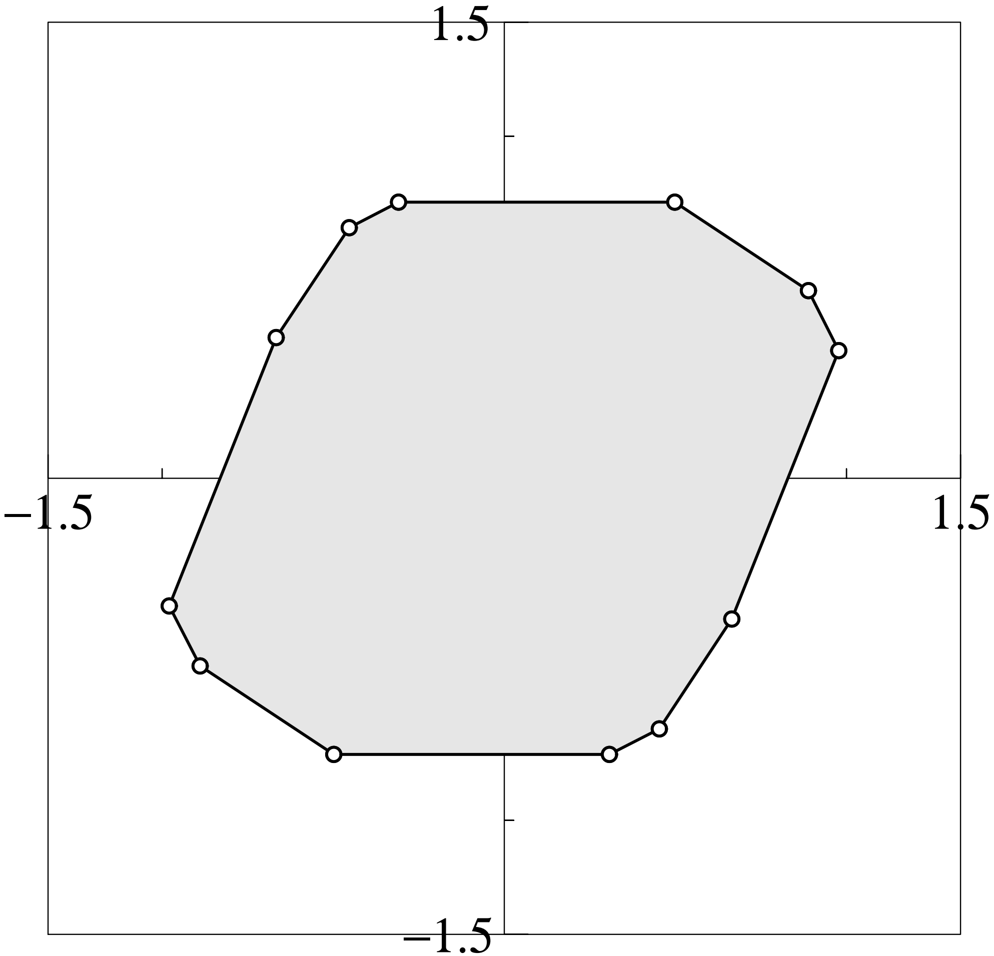
with $\alpha = \left(\frac{3+\sqrt{5}}{2}\right)^{-1/5}$, having spectral radius $\rho(\mathcal{F}) = 1$ and spectrum maximizing product $P_* = A_1 A_2 A_1^2 A_2$.

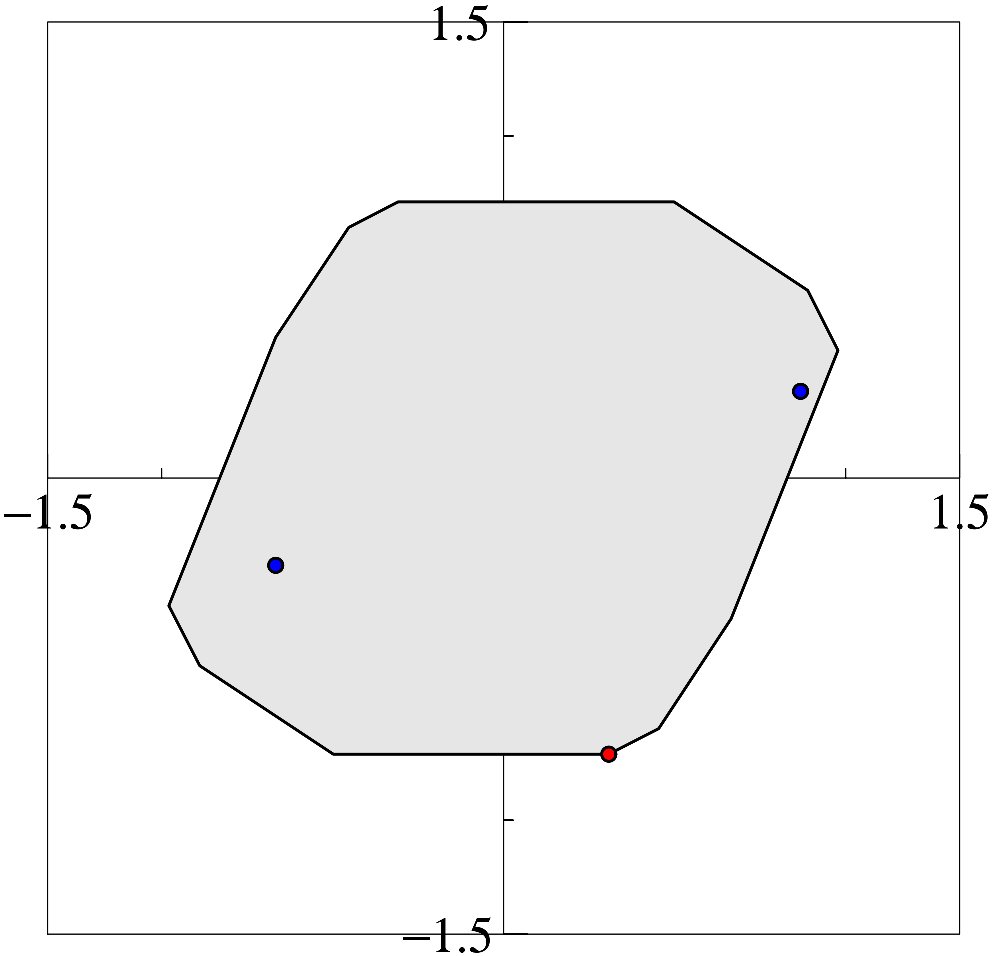
Applying the polytope algorithm

We obtain an extremal polytope norm after 5 iterations, with $\mathcal{P}$ a polytope with 6 vertices.

Is this a Barabanov norm?

Computed extremal polytope norm





In the right picture a boundary point x is drawn in red and the transformed vectors $A_1 x$ and $A_2 x$ are drawn in blue. $\Longrightarrow$

This is **not** a Barabanov norm.

Duality

Definition [adjoint polytope]
Let $\mathcal{P}$ be a real centrally symmetric polytope, that is there exists a set of vectors $V = \{v_1, \ldots, v_p\}$ such that

$$\mathcal{P} = \mathrm{convhull}\left(\pm v_1, \ldots, \pm v_p\right)$$

We define its adjoint (or dual), the polytope

$$\mathcal{P}^* = \mathrm{adj}(V) = \left\{ x \in \mathbb{R}^k \;\middle|\; \left|\langle x, v_i \rangle\right| \leq 1,\ i = 1, \ldots, p \right\}.$$

Theorem
Let $\mathcal{P}$ and $\mathcal{P}^*$ a polytope and its ajoint and $\|\cdot\|_{\mathcal{P}}$ and $\|\cdot\|_{\mathcal{P}^*}$ the associated norms. Then, for any matrix A, $\|A^{\mathrm{T}}\|_{\mathcal{P}} = \|A\|_{\mathcal{P}^*}$.

Corollary For a family $\mathcal{F}$, we have $\|\mathcal{F}\|_{\mathcal{P}^*} = \|\mathcal{F}^{\mathrm{T}}\|_{\mathcal{P}}$

How to get a Barabanov extremal norm

Key observation: the polytope algorithm determines a polytope $\mathcal{P} = \mathrm{convhull}\,(\pm v_1, \ldots, \pm v_p)$, characterized by

$$v_\ell \;=\; A_{i_\ell} v_{j_\ell} \quad \text{for some } \; j_\ell \in \{1, \ldots, p\} \;\&\; i_\ell \in \{1, \ldots, m\}.$$

This implies

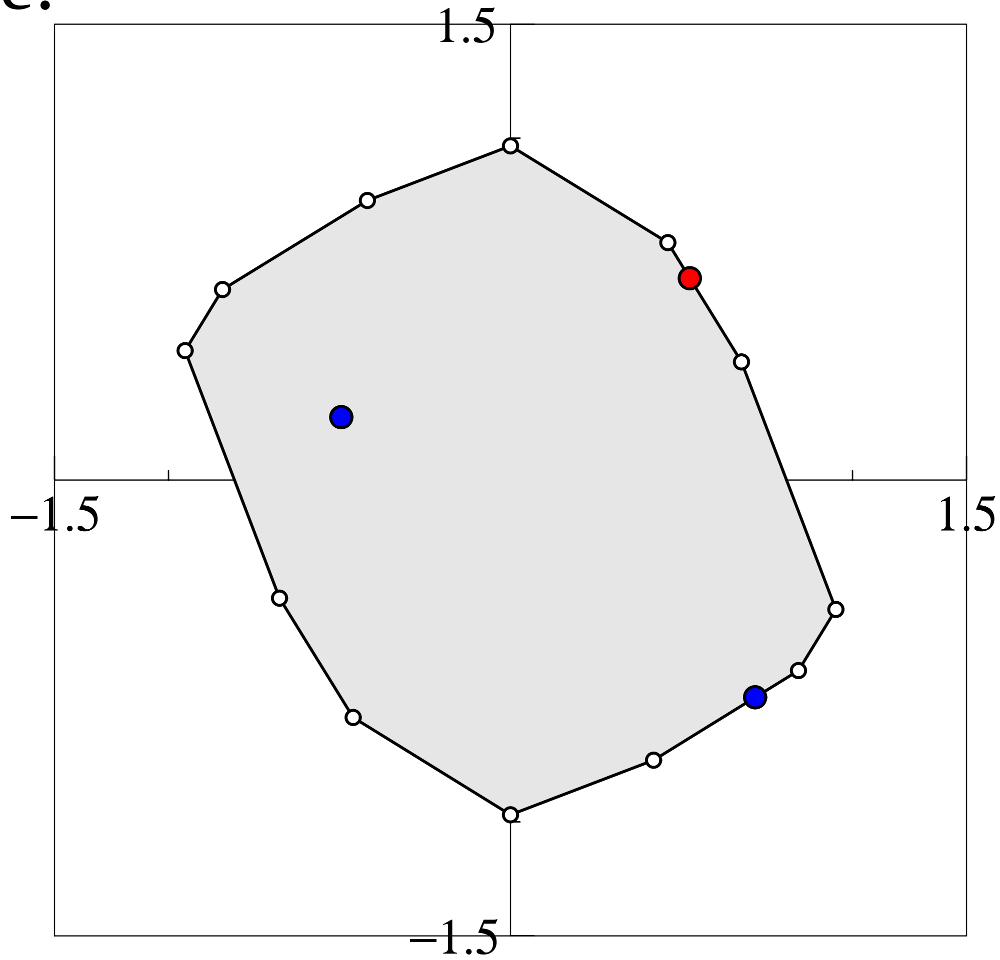
$$\mathcal{P} = \mathrm{convhull}\left(\bigcup_{i=1}^{m} A_i \mathcal{P} \right) \qquad \textbf{(H)}$$

Theorem [canonical construction of a Barabanov norm]
Let $\mathcal{P}$ define an extremal norm $\| \cdot \|_{\mathcal{P}}$ for $\mathcal{F}$ and assume that **(H)** holds. Then $\| \cdot \|_{\mathcal{P}^*}$ is a Barabanov norm for $\mathcal{F}^{\mathrm{T}}$.

Recipe: Given $\mathcal{F}$ apply the polytope algorithm to $\mathcal{F}^{\mathrm{T}}$.

Example 1 (ctd.)

Consider the family $\mathcal{F}^{\mathrm{T}} = \{A_1^{\mathrm{T}}, A_2^{\mathrm{T}}\}$ and the norm $\|\cdot\|_{\mathcal{P}^*}$.
Then we observe:



For any initial vector $x \in \partial\mathcal{P}^*$ (in red), at least one of the vectors $A_1^{\mathrm{T}}x, A_2^{\mathrm{T}}x \in \partial\mathcal{P}^*$ (in blue).

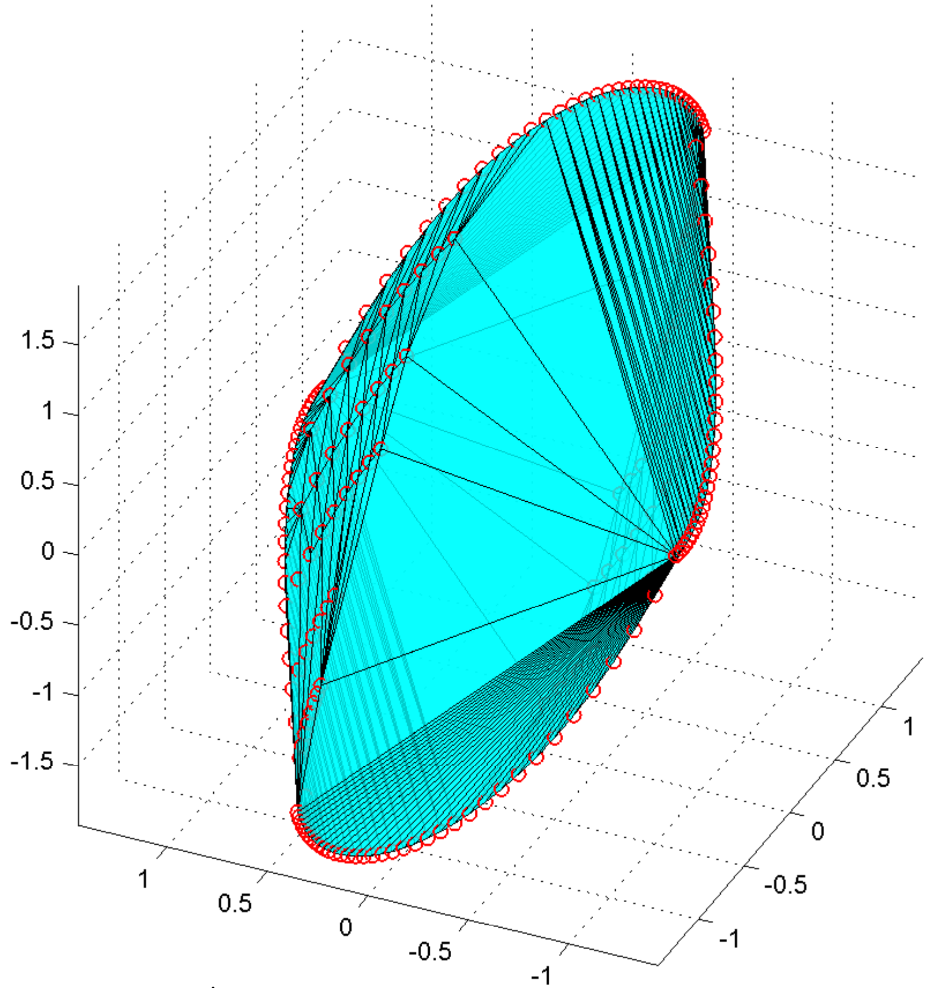
Example 2

Consider the control system (with $u : \mathbb{R}^+ \to \{1, 2\}$)

$\dot{x}(t) = B(u(t))\, x(t)$, with

$$B(1) = \begin{pmatrix} -1 & 0 & 0 \\ 10 & -1 & 0 \\ 0 & 0 & -10 \end{pmatrix}$$

$$B(2) = \begin{pmatrix} -10 & 0 & 10 \\ 0 & -10 & 0 \\ 0 & 10 & -1 \end{pmatrix}$$



and discretize it on a grid with $\Delta t = 1/256$.

A MUSL is computed through the determination of a the Barabanov norm whose unit ball $\mathcal{B}$ is shown in the figure.

Software

Matlab routines are made available at

`http://univaq.it/∼guglielm/`

TANTI AUGURI MAURIZIO