Local optimization techniques in semi-Lagrangian schemes for Hamilton-Jacobi equations

Dante Kalise
Optimization and Optimal Control Group
Radon Institute for Computational and Applied Mathematics (Linz)

(joint work with A. Kröner and K. Kunisch)

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Infinite horizon optimal control

Consider the minimization of

$$J_x(\alpha) = \int_0^\infty l(y_x(s), \alpha(s)) e^{-\lambda s}\, ds\,, \quad \lambda > 0\,,$$

subject to

$$\begin{aligned} &\dot{y}(t) = f(y, \alpha)\,, \quad y(\cdot) \in \mathbb{R}^n, \alpha(\cdot) \in A \subset \mathbb{R}^m, \\ &y(0) = x\,. \end{aligned}$$

For

$$v(x) = \inf_{\alpha \in \mathcal{U}} J_x(\alpha)\,,$$

HJB eq. for Infinite Horizon Control

$$\lambda v(x) + \sup_{a \in A}\{-f(x,a) \cdot Dv(x) - l(x,a)\} = 0\,.$$

The basic SL-FEM scheme

DPP with pseudotime parameter h, semi-discrete formulation

$$\begin{cases} v^h(x) = \min_{a \in A}\{(1-\lambda h)v^h(x+hf(x,a)) + hl(x,a)\} & \text{in } \Omega \subset \mathbb{R}^n, \\ v^h|_{\partial\Omega} = g(x). \end{cases}$$

We discretize Ω into a set of nodes x_i, $i = 1, \ldots, N$ and we consider the set of values $V := \{v^h(x_i)\}$. We define the first order interpolant

$$v^h(x) = I_1[V](x) + O(\Delta x^2), \qquad \Delta x = \max_i |x_i - x_{i-1}|.$$

Fully discrete SL-FEM

$$(V)_i = \min_{a \in A}\{(1-\lambda h)I_1[V](x_i + hf(x_i,a)) + hl(x_i,a)\}, \quad i = 1, \ldots, N.$$

The basic SL-FEM scheme

The standard way to solve this system is by a fixed point iteration of the value function.

Fully discrete SL-FEM/ Value Iteration (VI) scheme

$$\begin{aligned} V^{k+1} &= T(V^k), \quad \text{for } i = 1, \ldots, N \\ \left(T(V^k)\right)_i &\equiv \min_{a \in A} \left\{(1-\lambda h) I_1[V^k](x_i + hf(x_i, a)) + hl(x_i, a)\right\} . \end{aligned}$$

This scheme is fully-discrete except for the minimization routine.

- A standard practice in this context is to discretize the set A and compute the minimization by comparison. The computation of the optimal control

$$a_i^* = \underset{a \in A}{argmin} \left\{(1-\lambda h) I_1[V^k](x_i + hf(x_i, a)) + hl(x_i, a)\right\} ,$$

shall inherit this approach.

Local minimization in SL schemes

Previous attempts to embed other minimization techniques:

- Carlini, Cumpleañero and Ferretti (2004): Brendt algorithm.
- Grüne et al. (2005,06,09): Brendt , recursive search, Newton-Bundle.
- Bonnans et al. (2004): bilevel optimization problem HJB + parameters.

What we propose:

to replace the minimization-by-comparison approach with state of the art numerical optimization solvers, able to provide a more accurate representation of the control field at a similar computational cost.

The interpolation operator

For a triangulated 2D domain we consider local linear interpolants

$$I[V](x) = cx_1 + dx_2 + e\,,$$

with coefficients are given by

$$\begin{pmatrix} x_1^1 & x_2^1 & 1 \\ x_1^2 & x_2^2 & 1 \\ x_1^3 & x_2^3 & 1 \end{pmatrix} \begin{pmatrix} c \\ d \\ e \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}.$$

Under the assumptions

$$h \leq \|f(\cdot,\cdot)\|_\infty^{-1} \frac{\sqrt{2}}{2} k, \quad \dot{y}(t) = g(y) + Bu\,, \quad l(x,a) = w(x) + \frac{\gamma}{2}\|u\|_2^2\,,$$

Writing the local minimization problem

... and evaluating at the arrival point

$$x_a = x_d + hf(x_d, a)\,,$$

it is possible to rewrite

$$\min_{a \in A} \Big\{ (1 - \lambda h) I_1[V^k](x_i + hf(x_i, a)) + hl(x_i, a) \Big\}$$

as

$$\min_{a \in A_i \subset A} J(a)$$

with

$$J(a) := \frac{1}{2} a^t Q a + La + C\,.$$

Subdivision of the control set

Since different controls generate arrival points located at different triangles, it is necessary to establish a subdivision of the control set.

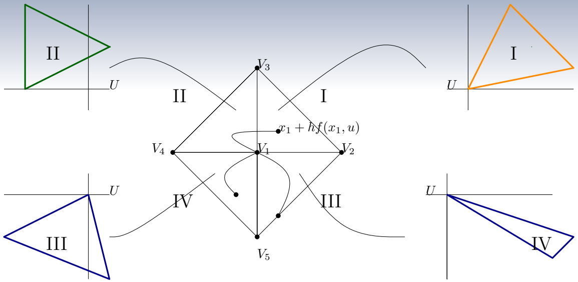


Figure: Arrival points and related control sets.

For eikonal dynamics, the relation is trivial, in other cases it requires some work.

A semismooth Newton method

We address the 2D eikonal case. We need to solve a problem of the form as

$$\min_{a \in A_1} \frac{1}{2} a^t Q a + L a + C$$

where

$$A_1 = \{(a_1, a_2) | a_1 \geq 0\,, a_2 \geq 0\,, \|a\|_2 \leq 1\}\,.$$

The first-order optimality condition can be formulated as

$$a = P(a - \vartheta \nabla J(a)) \quad \forall \vartheta > 0\,, a \in \mathbb{R}^2\,,$$

with projection

$$P : \mathbb{R}^m \to \mathbb{R}^m, \quad P(p) = \frac{\max(0,p)}{\max(1, \|max(0,p)\|_2)}\,.$$

A semismooth Newton method

We introduce $p = a - \vartheta \nabla J(a)$ and $\beta = \max(1, \| \max(0,p)\|)$. By defining the system

$$E(a,p,\beta) = \begin{pmatrix} \beta a - \max(0,p) \\ a - \vartheta \nabla J(a) - p \\ \beta - \max(1, \| \max(0,p)\|_2) \end{pmatrix},$$

the optimality condition can be formulated as $E(z) = 0$ with $z = (a,p,\beta)$, which can be solved via a Newton iteration

$$\begin{aligned} DE(z_k)\delta z &= -E(z_k)\,, \\ z_{k+1} &= z_k + \delta z\,. \end{aligned}$$

Assessing the performance of the method

method	tolerance	iterations	cpu time	error
Chambolle-Pock	1E-4	9	7.3E-5 [s]	4.31E-5
Semi-smooth Newton	1E-4	5	1.3E-2 [s]	7.74E-9
Comparison (1E4 evaluations)	–	–	1.1E-3 [s]	1.6E-2
Comparison (2E3 evaluations)	–	–	6.6E-4 [s]	4.1E-2

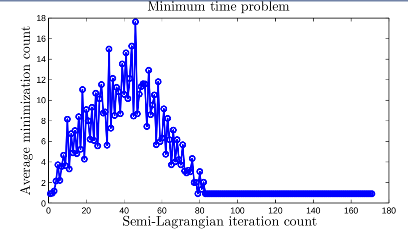
Assessing the performance of the method

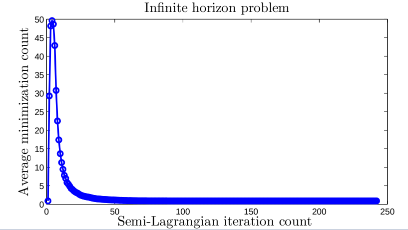
method	tolerance	iterations	cpu time	error
Semi-smooth Newton	1E-4	101	4.53E-3 [s]	1.51E-3
Comparison (4E4 evaluations)	–	–	5.18E-3 [s]	5.77E-3
Comparison (2E5 evaluations)	–	–	2.08E-2 [s]	2.80E-3

Assessing the performance of the method

method	tolerance	iterations	cpu time	error
Semi-smooth Newton	1E-4	58	1.47E-2 [s]	1.23E-3
Comparison ($2E3$ evaluations)	–	–	8.56E-4 [s]	4.18E-2
Comparison ($1E5$ evaluations)	–	–	1.20E-2 [s]	1.47E-2

Performance inside the SL scheme



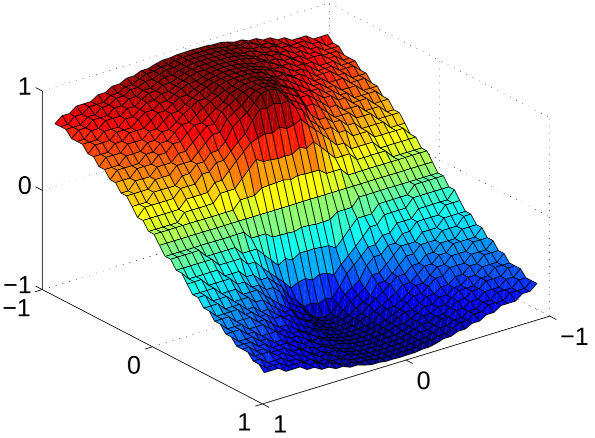


Performance inside the SL scheme

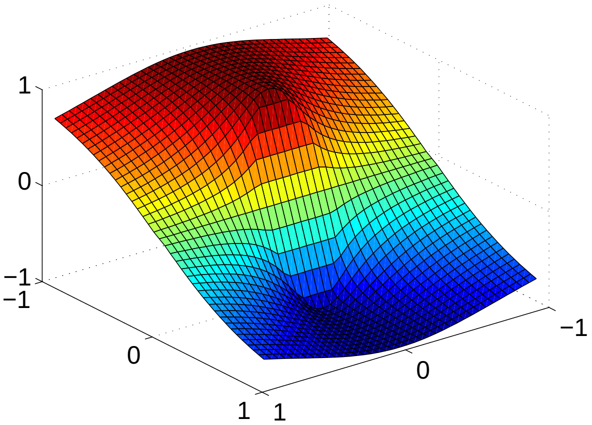
	$\Delta x = 0.05$ $(40^2$ DoF)			$\Delta x = 0.025$ $(80^2$ DoF)		
Minimization	CPU time	$\|v - V_h\|_1$	$\|u - U_h\|_1$	CPU time	$\|v - V_h\|_1$	$\|u - U_h\|_1$
Comparison	63.52 [s]	3.12E-2	3.84E-2	5.76E2 [s]	1.96E-2	1.74E-2
Semismooth	77.25 [s]	2.62E-2	1.61E-2	7.27E2 [s]	1.36E-2	7.21E-3
Chambolle-Pock	63.05 [s]	2.60E-2	1.42E-2	5.77E2 [s]	1.36E-2	6.83E-3

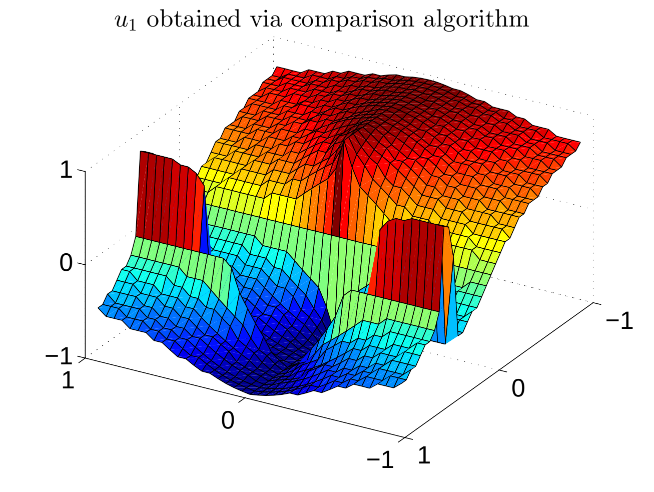
Table: Infinite horizon control of eikonal dynamics. CPU time and errors for a semi-Lagrangian scheme with different minization algorithms. The comparison algorithm was run with a discrete set of 1028 control points.

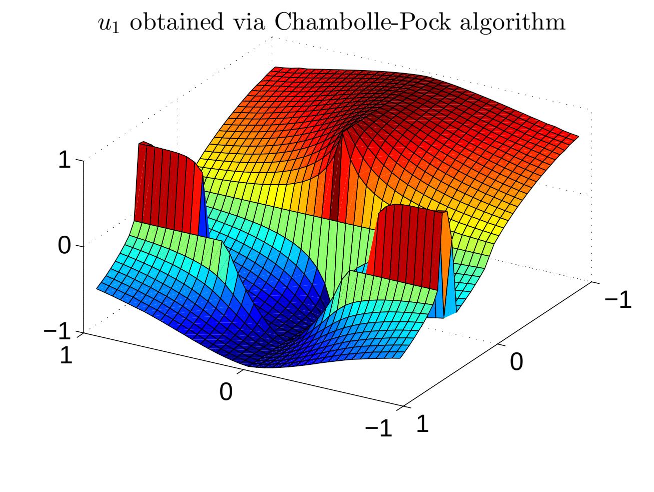
u_1 obtained via comparison algorithm



u_1 obtained via Chambolle-Pock algorithm




u_1 obtained via comparison algorithm
1
0
−1
1
0
−1
1
0
−1


u_1 obtained via Chambolle-Pock algorithm
1
0
−1
1
0
−1
1
0
−1

Concluding remarks

- We have implemented state of the art building blocks for numerical optimization in SL schemes.
- We obtain, a similar computational cost, a better resolution of the control field with different dynamics and costs.
- Similar ideas can be applied in other control problems and related numerical schemes.

Thanks for your attention.

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