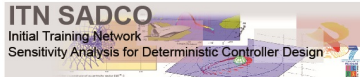


# State-constrained stochastic optimal control problems via reachability approach

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For a fixed  $T > 0$  let us consider the following SDE's in  $\mathbb{R}^d$ :

$$\begin{cases} dX(s) = b(s, X(s), u(s))ds + \sigma(s, X(s), u(s))dW(s) \\ X(t) = x \end{cases} \quad (1)$$

with

- $W(\cdot)$ :  $p$ -dimensional Brownian motion;
  - $u \in \mathcal{U}$ : progressively measurable processes with values in a compact set  $U$ ;
- ↪  $X_{t,x}^u(\cdot)$ : unique solution of (1) associated with the control  $u$ .

# State constrained optimal control problem

Let  $\mathcal{K} \subseteq \mathbb{R}^d$  be a non empty and closed set.

Let us consider (assume  $\psi, \ell \geq 0$ ):

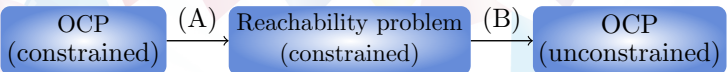
$$v(t, x) = \inf \left\{ \mathbb{E} \left[ \psi(X_{t,x}^u(T)) + \int_t^T \ell(s, X_{t,x}^u(s), u(s)) ds \right] : \right. \\ \left. u \in \mathcal{U} \text{ and } X_{t,x}^u(s) \in \mathcal{K}, \forall s \in [t, T] \text{ a.s.} \right\}$$

- If  $\mathcal{K} = \mathbb{R}^d$ : characterization of  $v$  by HJB equation;
- If  $\mathcal{K} \subsetneq \mathbb{R}^d$ : necessity of **further assumptions** on  $b$  and  $\sigma$ .

**AIM:** be able to compute the value function  $v$  also in **absence of this kind of assumptions**

# State constrained OCP via reachability approach

**APPROACH:** link between optimal control problems and reachability



**DETERM.:**

Cardaliaguet-Quincampoix-SaintPierre ('00),  
Aubin-Frankowska ('96),  
Altarovici-Bokanowski-Zidani ('13)

Osher-Sethian ('88),  
Bokanowski-Forcadet-Zidani ('10),  
Kurzanski-Varaiya ('06)

**STOCH.:**

Bouchard-Dang ('12)

Bokanowski-AP-Zidani ('14)

- 1 STEP A: link with the reachability problem
- 2 STEP B: the level set approach
- 3 The HJB characterization

# Step A: link with the reachability problem

For simplicity, let us assume  $\ell \equiv 0$ .

## Proposition

One has

$$v(t, x) = \inf \left\{ z \geq 0 : \exists u \in \mathcal{U} \text{ such that } z \geq \mathbb{E} \left[ \psi(X_{t,x}^u(T)) \right] \right. \\ \left. \text{and } X_{t,x}^u(s) \in \mathcal{K}, \forall s \in [t, T] \text{ a.s.} \right\}$$

# Step A: link with the reachability problem

**PROBLEM:** How to link this problem with a reachability one?

Of course

$$(I): \exists u \in \mathcal{U} : z \geq \mathbb{E} \left[ \psi(X_{t,x}^u(T)) \right] \not\Rightarrow (II): \exists u \in \mathcal{U} : z \geq \psi(X_{t,x}^u(T)) \text{ a.s.}$$

We extend the arguments of [Bouchard-Dang \('12\)](#) (unconstrained case).

## Main argument

Thanks to the Ito's representation theorem (I) and (II) are equivalent up to a martingale.

# Step A: link with the reachability problem

## Proposition

Let be  $\alpha \in \mathcal{A}$ , set square-integrable  $\mathbb{R}^P$  – valued predictable processes, and

$$Z_{t,z}^{\alpha,u}(\cdot) := z + \int_t^\cdot \alpha_s^T dW_s.$$

Then

$$\begin{array}{ccc} \begin{array}{l} \text{Exists } u \in \mathcal{U} : \\ z \geq \mathbb{E}[\psi(X_{t,x}^u(T))] \\ \text{and} \\ X_{t,x}^u(s) \in \mathcal{K}, \forall s \in [t, T] \text{ a.s.} \end{array} & \iff & \begin{array}{l} \text{Exist } (u, \alpha) \in \mathcal{U} \times \mathcal{A} : \\ Z_{t,z}^{\alpha,u}(T) \geq \psi(X_{t,x}^u(T)) \\ \text{and} \\ X_{t,x}^u(s) \in \mathcal{K}, \forall s \in [t, T] \text{ a.s.} \end{array} \end{array}$$



# Step A: link with the reachability problem

Thanks to the previous result

## Theorem

*One has*

$$v(t, x) = \inf \left\{ z \geq 0 : \exists (u, \alpha) \in \mathcal{U} \times \mathcal{A} \text{ such that} \right. \\ \left. \left( X_{t,x}^u(s) \in \mathcal{K}, \forall s \in [t, T] \text{ and } (X_{t,x}^u(T), Z_{t,z}^{u,\alpha}(T)) \in \text{epi}(\psi) \right) \text{ a.s.} \right\}$$

We aim to solve a **BACKWARD REACHABILITY PROBLEM**:

$$\mathcal{R}_t^{\psi, \mathcal{K}} := \left\{ (x, z) \in \mathbb{R}^{d+1} : \exists u \in \mathcal{U} \text{ such that} \right. \\ \left. \left( X_{t,x}^u(T), Z_{t,z}^u(T) \right) \in \text{epi}(\psi) \text{ and } X_{t,x}^u(s) \in \mathcal{K}, \forall s \in [t, T] \right\}.$$

1 STEP A: link with the reachability problem

2 STEP B: the level set approach

3 The HJB characterization

# Step B: the level set approach

Let us introduce  $g_{\mathcal{K}} : \mathbb{R}^d \rightarrow \mathbb{R}$  such that:

$$g_{\mathcal{K}} \geq 0 \quad \text{and} \quad g_{\mathcal{K}}(x) = 0 \Leftrightarrow x \in \mathcal{K}.$$

and let us consider the following **UNCONSTRAINED OCP**:

$$w(t, x, z) = \inf_{(u, \alpha) \in \mathcal{U} \times \mathcal{A}} \mathbb{E} \left[ \left( \psi(X_{t,x}^u(T)) - Z_{t,z}^{u,\alpha}(T) \right)_+ + \int_t^T g_{\mathcal{K}}(X_{t,x}^u(s)) ds \right]$$

## Proposition

*Let us assume that for any  $(t, x)$  the infimum in the definition of  $w$  is attained. Then*

$$\mathcal{R}_t^{\psi, \mathcal{K}} = \left\{ (x, z) \in \mathbb{R}^{d+1} : w(t, x, z) = 0 \right\}.$$

*Therefore*

$$v(t, x) = \inf \left\{ z \geq 0 : w(t, x, z) = 0 \right\}.$$

- 1 STEP A: link with the reachability problem
- 2 STEP B: the level set approach
- 3 The HJB characterization

For simplicity  $p = d = 1$ .

The HJB equation associated to the AUXILIARY OCP would be:

$$-w_t + \sup_{\substack{u \in U, \\ \alpha \in \mathbb{R}}} \left\{ -b w_x + \ell w_z - \frac{1}{2} \sigma^2 w_{xx} - \alpha \sigma w_{xz} - \frac{1}{2} \alpha^2 w_{zz} - g_{\mathcal{K}} \right\} = 0$$

$\underbrace{\hspace{15em}}_{=: H(t, x, Dw, D^2w)}$

⇒ The Hamiltonian can be unbounded!!

**NEW ISSUE:** handle unbounded controls

(Refs. Brüder ('05), Bokanowski-Brüder-Maroso-Zidani ('09)).

Let us define:

$$\mathcal{H}^u(t, x, Dw, D^2w) := \begin{pmatrix} -w_t - b w_x + \ell w_z - \frac{1}{2}\sigma^2 w_{xx} - g_{\mathcal{K}} & -\sigma w_{xz} \\ -\sigma w_{xz} & -w_{zz} \end{pmatrix}$$

and

$$w_0(t, x) := \inf_{u \in \mathcal{U}} \mathbb{E} \left[ \psi(X_{t,x}^u(T)) + \int_t^T \ell(s, X_{t,x}^u(s), u(s)) + g_{\mathcal{K}}(X_{t,x}^u(s)) ds \right].$$

- It is possible to prove that

$$-w_t + H(t, x, Dw, D^2w) \leq 0$$

$$\Leftrightarrow \sup_{u \in \mathcal{U}} \Lambda^+ \left( \mathcal{H}^u(t, (x, z), Dw, D^2w) \right) \leq 0;$$

- for any  $z \leq 0$  one has  $w(t, x, z) = w_0(t, x) - z$ .

## Theorem

Then  $w$  is the unique viscosity solution of the following generalized HJB equation

$$\left\{ \begin{array}{l} \sup_{u \in U, \xi \in \mathbb{R}^2, \|\xi\|=1} \left\{ \xi_1^2 \left( -w_t - bw_x + \ell w_z - \frac{1}{2} \sigma^2 w_{xx} - g_K(x) \right) \right. \\ \left. - 2\xi_1 \xi_2 \sigma w_{xz} - \xi_2^2 w_{zz} \right\} = 0 \quad t \in [0, T), x \in \mathbb{R}, z > 0 \\ w(t, x, 0) = w_0(t, x) \quad t \in [0, T), x \in \mathbb{R}^d \\ w(T, x, z) = (\psi(x) - z)_+ \quad x \in \mathbb{R}, z \geq 0 \end{array} \right.$$

in the class of continuous function with linear growth at infinity.



## Conclusion:

- We translate the state-constrained OCP in a state-constrained reachability one, adding a state variable and an  $\mathbb{R}^p$ -valued control;
- We solved the state-constrained reachability problem by the level set method linking it with an auxiliary unconstrained OCP.
- We characterized  $w$  as the unique solution of a generalized HJB equation.

\*Thank you for your attention  
&  
Auguri



Paura, eh?  
(Afraid?)