State-constrained stochastic optimal control problems via reachability approach

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Setting

For a fixed $T > 0$ let us consider the following SDE's in $\mathbb{R}^d$:

$$\begin{cases} dX(s) = b(s, X(s), u(s))ds + \sigma(s, X(s), u(s))dW(s) \\ X(t) = x \end{cases} \tag{1}$$

with

- $W(\cdot)$: p-dimensional Brownian motion;
- $u \in \mathcal{U}$: progressively measurable processes with values in a compact set U;

$\rightsquigarrow$ $X^u_{t,x}(\cdot)$: unique solution of (1) associated with the control u.

State constrained optimal control problem

Let $\mathcal{K} \subseteq \mathbb{R}^d$ be a non empty and closed set.
Let us consider (assume $\psi, \ell \geq 0$):

$$v(t,x) = \inf \left\{ \mathbb{E}\left[\psi(X^u_{t,x}(T)) + \int_t^T \ell(s, X^u_{t,x}(s), u(s))ds \right] : \right.$$
$$\left. u \in \mathcal{U} \text{ and } X^u_{t,x}(s) \in \mathcal{K}, \forall s \in [t,T] \text{ a.s.} \right\}$$

- If $\mathcal{K} = \mathbb{R}^d$: characterization of v by HJB equation;
- If $\mathcal{K} \subsetneq \mathbb{R}^d$: necessity of further assumptions on b and σ.

AIM: be able to compute the value function v also in absence of this kind of assumptions

State constrained OCP via reachability approach

APPROACH: link between optimal control problems and reachability

OCP (constrained) —(A)→ Reachability problem (constrained) —(B)→ OCP (unconstrained)

	(A)	(B)
DETERM.:	Cardaliaguet-Quincampoix-SaintPierre ('00), Aubin-Frankowska ('96), Altarovici-Bokanowski-Zidani ('13)	Osher-Sethian ('88), Bokanowski-Forcadel-Zidani ('10), Kurzhanski-Varaiya ('06)
STOCH.:	Bouchard-Dang ('12)	Bokanowski-AP-Zidani ('14)

Outline

For simplicity, let us assume $\ell \equiv 0$.

Proposition

One has

$$v(t,x) = \inf\Big\{ z \geq 0 : \exists u \in \mathcal{U} \text{ such that } z \geq \mathbb{E}\Big[\psi(X^u_{t,x}(T))\Big] \text{ and } X^u_{t,x}(s) \in \mathcal{K}, \forall s \in [t,T] \text{ a.s.} \Big\}$$

PROBLEM: How to link this problem with a reachability one?
Of course

$$\text{(I)}: \exists u \in \mathcal{U} : z \geq \mathbb{E}\left[\psi(X^u_{t,x}(T))\right] \not\Longrightarrow \text{(II)}: \exists u \in \mathcal{U} : z \geq \psi(X^u_{t,x}(T)) \text{ a.s.}$$

We extend the arguments of Bouchard-Dang ('12) (unconstrained case).

Main argument

Thanks to the Ito's representation theorem (I) and (II) are equivalent up to a martingale.

Step A: link with the reachability problem

Proposition

Let be $\alpha \in \mathcal{A}$, set square-integrable $\mathbb{R}^p-valued$ predictable processes, and

$$Z_{t,z}^{\alpha,u}(\cdot) := z + \int_t^{\cdot} \alpha_s^T dW_s.$$

Then

$$\begin{array}{c} \text{Exists } u \in \mathcal{U}: \\ z \geq \mathbb{E}[\psi(X_{t,x}^u(T))] \\ \text{and} \\ X_{t,x}^u(s) \in \mathcal{K}, \forall s \in [t,T] \ \ a.s. \end{array} \iff \begin{array}{c} \text{Exist } (u,\alpha) \in \mathcal{U} \times \mathcal{A}: \\ Z_{t,z}^{\alpha,u}(T) \geq \psi(X_{t,x}^u(T)) \\ \text{and} \\ X_{t,x}^u(s) \in \mathcal{K}, \forall s \in [t,T] \ \ a.s. \end{array}$$

Thanks to the previous result

Theorem

One has

$$v(t,x) = \inf\Big\{ z \geq 0 : \exists (u,\alpha) \in \mathcal{U} \times \mathcal{A} \text{ such that } \Big(X^u_{t,x}(s) \in \mathcal{K}, \forall s \in [t,T] \text{ and } (X^u_{t,x}(T), Z^{u,\alpha}_{t,z}(T)) \in epi(\psi) \Big) \text{ a.s.} \Big\}$$

We aim to solve a **BACKWARD REACHABILITY PROBLEM**:

$$\mathcal{R}^{\psi,\mathcal{K}}_t := \Big\{ (x,z) \in \mathbb{R}^{d+1} : \exists u \in \mathcal{U} \text{ such that } (X^u_{t,x}(T), Z^u_{t,z}(T)) \in \mathrm{epi}(\psi) \text{ and } X^u_{t,x}(s) \in \mathcal{K}, \forall s \in [t,T] \Big\}.$$

Outline

Let us introduce $g_{\mathcal{K}} : \mathbb{R}^d \to \mathbb{R}$ such that:

$$g_{\mathcal{K}} \geq 0 \quad \text{and} \quad g_{\mathcal{K}}(x) = 0 \Leftrightarrow x \in \mathcal{K}.$$

and let us consider the following **UNCONSTRAINED OCP:**

$$w(t,x,z) = \inf_{(u,\alpha)\in\mathcal{U}\times\mathcal{A}} \mathbb{E}\left[\Big(\psi(X^u_{t,x}(T)) - Z^{u,\alpha}_{t,z}(T)\Big)_+ + \int_t^T g_{\mathcal{K}}(X^u_{t,x}(s))ds\right]$$

Proposition

Let us assume that for any (t,x) the infimum in the definition of w is attained. Then

$$\mathcal{R}_t^{\psi,\mathcal{K}} = \left\{(x,z) \in \mathbb{R}^{d+1} : w(t,x,z) = 0\right\}.$$

Therefore

$$\upsilon(t,x) = \inf\left\{z \geq 0 : w(t,x,z) = 0\right\}.$$

Outline

Dealing with unbounded controls

For simplicity $p = d = 1$.

The HJB equation associated to the AUXILIARY OCP would be:

$$-w_t + \underbrace{\sup_{\substack{u \in U,\\ \alpha \in \mathbb{R}}} \left\{ -\, b\, w_x + \ell\, w_z - \frac{1}{2}\sigma^2\, w_{xx} - \alpha\sigma w_{xz} - \frac{1}{2}\alpha^2 w_{zz} - g_{\mathcal{K}} \right\}}_{=:H(t,x,Dw,D^2w)} = 0$$

⇒ The Hamiltonian can be unbounded!!

NEW ISSUE: handle unbounded controls
(Refs. Brüder ('05), Bokanowski-Brüder-Maroso-Zidani ('09)).

Dealing with unbounded controls

Let us define:

$$\mathcal{H}^u(t,x,Dw,D^2w) := \begin{pmatrix} -w_t - b\, w_x + \ell\, w_z - \frac{1}{2}\sigma^2 w_{xx} - g_{\mathcal{K}} & -\sigma w_{xz} \\ -\sigma w_{xz} & -w_{zz} \end{pmatrix}$$

and

$$w_0(t,x) := \inf_{u\in\mathcal{U}} \mathbb{E}\left[\psi(X^u_{t,x}(T)) + \int_t^T \ell(s, X^u_{t,x}(s), u(s)) + g_{\mathcal{K}}(X^u_{t,x}(s))ds\right].$$

- It is possible to prove that

$$-w_t + H(t,x,Dw,D^2w) \le 0$$
$$\Leftrightarrow \sup_{u\in U} \Lambda^+\Big(\mathcal{H}^u(t,(x,z),Dw,D^2w)\Big) \le 0;$$

- for any $z \le 0$ one has $w(t,x,z) = w_0(t,x) - z$.

Dealing with unbounded controls

Theorem

Then w is the unique viscosity solution of the following generalized HJB equation

$$\begin{cases} \sup\limits_{\substack{u\in U,\\ \xi\in\mathbb{R}^2, ||\xi||=1}} \Big\{ \xi_1^2 \Big(-w_t - bw_x + \ell w_z - \frac{1}{2}\sigma^2 w_{xx} - g_{\mathcal{K}}(x) \Big) \\ \qquad -2\xi_1\xi_2\sigma w_{xz} - \xi_2^2 w_{zz} \Big\} = 0 & t\in[0,T), x\in\mathbb{R}, z>0 \\ w(t,x,0) = w_0(t,x) & t\in[0,T), x\in\mathbb{R}^d \\ w(T,x,z) = (\psi(x)-z)_+ & x\in\mathbb{R}, z\geq 0 \end{cases}$$

in the class of continuous function with linear growth at infinity.

Conclusion:

- We translate the state-constrained OCP in a state-constrained reachability one, adding a state variable and an $\mathbb{R}^p$-valued control;
- We solved the state-constrained reachability problem by the level set method linking it with an auxiliary unconstrained OCP.
- We characterized w as the unique solution of a generalized HJB equation.

Thank you for your attention

&

Auguri

Paura, eh?
(Afraid?)