A unified approach to Shape-from-Shading models for non-Lambertian surfaces

S. Tozza
Joint work with M. Falcone

Dipartimento di Matematica, SAPIENZA - Università di Roma

Numerical methods for PDEs: optimal control, games and image processing
(On the Occasion of Maurizio Falcone’s 60th birthday)
December 5, 2014, Rome
Outline

- Introduction

- Some Reflectance Models in a unified approach
  - Lambertian Model
  - Oren-Nayar Model
  - Phong Model

- Semi-Lagrangian Approximation

- Numerical Tests

- Conclusions and Perspectives
Introduction - Shape from Shading (SfS) Problem

Problem:
We want to obtain the 3D shape of an object starting from its image.
The SfS problem is described by the following irradiance equation:

\[ R(\mathbf{N}(x)) = I(x) \]  

(1)

where

- \( R(\mathbf{N}(x)) \) is the reflectance function;
- \( \mathbf{N}(x) \) is the unit normal to the surface at point \((x, u(x))\);
- \( I(x) \) is the greylevel measured in the image at point \( x \).

\[ I : \overline{\Omega} \rightarrow [0, 1], \text{ with } \overline{\Omega} \text{ compact domain } (\Omega \subset \mathbb{R}^2 \text{ open subset}). \]
Assumptions:

1. One light source located at infinity in the direction of $\omega$;
2. no self-reflections on the surface;
3. the light source is sufficiently far from the surface so perspective deformations are neglected;
4. the diffuse and specular albedos $\gamma_D(x)$ and $\gamma_S(x)$ are known (for simplicity we put $\gamma_D(x) = \gamma_S(x) = 1$);
As proposed in [T., 2014], it is useful to rewrite (1) as

\[ l(x) = k_A l_A + k_D l_D(x) + k_S l_S(x) \]

where

- \( k_A, k_D, \) and \( k_S \) (with \( k_A + k_D + k_S = 1 \)): ratio of ambient, diffuse, and specular reflection;
As proposed in [T., 2014], it is useful to rewrite (1) as

\[ l(x) = k_A I_A + k_D I_D(x) + k_S I_S(x) \]

where

- \( k_A, k_D, \) and \( k_S \) (with \( k_A + k_D + k_S = 1 \)): ratio of ambient, diffuse, and specular reflection;

In the whole talk we neglect the contribution of the ambient component (\( k_A = 0 \)).
Lambertian reflectance model (L–model)

**Idea:** The surface is Lambertian, i.e. the intensity reflected by a point of the surface is equal from all points of view.

**Remark:** This is a purely diffuse model $\Rightarrow I_S$ doesn’t exist $\Rightarrow I(x) \equiv I_D(x) \ (k_D \equiv 1)$

**Goal:** Finding $u : \Omega \rightarrow \mathbb{R}$ s. t. satisfy the following equation:

$$I(x) = N(x) \cdot \omega, \quad \forall x \in \Omega \quad (2)$$

where

- $N(x) = \frac{n(x)}{|n(x)|} = \frac{1}{\sqrt{1+|\nabla u(x)|^2}}(-\nabla u(x), 1)$
- $\omega = (\omega_1, \omega_2, \omega_3) = (\tilde{\omega}, \omega_3) \ (\text{general light direction})$
Lambertian PDE [Falcone-Sagona-Seghini, 2003]

Hamilton-Jacobi equation (HJE) associated to (2):

\[ I(x)\sqrt{1 + |\nabla u(x)|^2} + \tilde{\omega} \cdot \nabla u(x) - \omega_3 = 0, \text{ in } \Omega. \]

By using the exponential transform \( \mu v(x) = 1 - e^{-\mu u(x)} \) we arrive to the following problem in new variable \( v \)

**Fixed point form**

\[
\begin{aligned}
\mu v(x) &= \min_{a \in \partial B_3} \left\{ b^L(x, a) \cdot \nabla v(x) + f^L(x, a, v(x)) \right\}, & \text{for } x \in \Omega, \\
v(x) &= 0, & \text{for } x \in \partial \Omega,
\end{aligned}
\]

where

\[
(b^L, f^L) = \left( \frac{I(x)a_{1,2} - \tilde{\omega}}{\omega_3}, -\frac{I(x)a_3}{\omega_3}(1 - \mu v(x)) + 1 \right),
\]

and \( B_3 \) is the unit ball in \( \mathbb{R}^3 \).
Oren-Nayar reflectance model (ON–model)

Idea: Representing a rough surface as an aggregation of V-shaped cavities, each with Lambertian reflectance properties.

(a) Facet model for surface patch dA consisting of many V-shaped Lambertian cavities.

(b) Diffuse reflectance for SfS with Oren-Nayar.

Figure: Sketch of the Oren-Nayar surface reflection model.

Remark:
This is a purely diffuse model \( \Rightarrow I_S \) doesn’t exist
\( \Rightarrow I(x) \equiv I_D(x) \) \( (k_D \equiv 1) \)
Oren-Nayar reflectance model

**General Brightness equation [Oren-Nayar, 1995]:**

\[
I(x) = \cos(\theta_i) \left( A + B \sin(\alpha) \tan(\beta) \max[0, \cos(\varphi_i - \varphi_r)] \right)
\]

where

- \( A = 1 - 0.5 \sigma^2 (\sigma^2 + 0.33)^{-1} \); \( B = 0.45\sigma^2 (\sigma^2 + 0.09)^{-1} \);
- \( \sigma \): roughness parameter of the surface;
- \( \theta_i \): angle between \( \mathbf{N} \) and \( \omega \);
- \( \theta_r \): angle between \( \mathbf{N} \) and viewer direction \( \mathbf{V} \);
- \( \alpha = \max[\theta_i, \theta_r] \); \( \beta = \min[\theta_i, \theta_r] \);
- \( \varphi_i \): angle between the projection of \( \omega \) and the \( x_1 \) axis onto the \( (x_1, x_2) \)-plane;
- \( \varphi_r \): angle between the projection of \( \mathbf{V} \) and the \( x_1 \) axis.
Oren-Nayar reflectance model

**Brightness equation in the case** $\omega \equiv V$

$$I(x) = \cos(\theta) \left( A + B \sin(\theta)^2 \cos(\theta)^{-1} \right)$$

where $\theta := \theta_i = \theta_r = \alpha = \beta$.

**Dirichlet problem associated to the brightness equation:**

$$\begin{cases} 
(I(x) - B)(\sqrt{1 + |\nabla u|^2}) + A(\tilde{\omega} \cdot \nabla u - \omega_3) \\
+ B \frac{(-\tilde{\omega} \cdot \nabla u + \omega_3)^2}{\sqrt{1 + |\nabla u|^2}} = 0, & x \in \Omega, \\
u(x) = 0, & x \in \partial\Omega,
\end{cases} \quad (3)$$

**Remark:**

When $\sigma = 0$ the ON–model brings back to the L–model.
Oren-Nayar reflectance model

**Brightness equation in the case** $\omega \equiv V$

$$I(x) = \cos(\theta) \left( A + B \sin(\theta)^2 \cos(\theta)^{-1} \right)$$

where $\theta := \theta_i = \theta_r = \alpha = \beta$.

**Dirichlet problem associated to the brightness equation:**

$$\begin{cases}
(l(x) - B)(\sqrt{1 + |\nabla u|^2}) + A(\tilde{\omega} \cdot \nabla u - \omega_3) \\
+ B \frac{(-\tilde{\omega} \cdot \nabla u + \omega_3)^2}{\sqrt{1 + |\nabla u|^2}} = 0, & x \in \Omega, \\

u(x) = 0, & x \in \partial\Omega,
\end{cases}$$

(3)

**Remark:**

When $\sigma = 0$ the ON–model brings back to the L–model.
Exponential transform \( \mu v(x) = 1 - e^{-\mu u(x)} \) to write (3) as

\[
\begin{align*}
\mu v(x) + \max_{a \in \partial B_3} \left\{ -b^{ON}(x, a) \cdot \nabla v(x) + f^{ON}(x, z, a, v(x)) \right\} &= 1, \\
v(x) &= 0,
\end{align*}
\]

where

\[
b^{ON}(x, a) = \frac{1}{A \omega_3} \left( c(x, z)a_1 - A \omega_1, c(x, z)a_2 - A \omega_2 \right),
\]

\[
f^{ON}(x, z, a, v(x)) = \frac{c(x, z)a_3}{A \omega_3}(1 - \mu v(x)),
\]

\[
c(x, z) = I(x) - B + B \left( \frac{\nabla S(x, z)}{||\nabla S(x, z)||} \cdot \omega \right)^2
\]

with

\[
\nabla S(x, z) = (-\nabla u(x), 1).
\]
Phong reflectance model (PH–model)

General Brightness equation [B.T. Phong, 1975]:

\[ I(x) = k_D(\cos(\theta_i)) + k_S(\cos(\theta_s))^\alpha \]

where

- \( \theta_i \): angle between \( \mathbf{N} \) and \( \omega \).
- \( \theta_s \): angle between reflected light direction \( \mathbf{R} \) and \( \mathbf{V} \).
  \( 0 \leq \theta_s \leq \frac{\pi}{2} \) because for greater angles the viewer does not perceive the light reflected specularly;
- \( \alpha \): models the specular reflected light for each material;
- \( \mathbf{N} \) and \( \mathbf{R} \) are unitary and coplanar.
Phong reflectance model

Fixing $\alpha = 1$, the PH–brightness equation becomes

$$HJE \text{ in case } \mathbf{V} = (0, 0, 1) \text{ and } \alpha = 1:$$

$$I(x)(1 + |\nabla u(x)|^2) - k_D(-\nabla u(x) \cdot \omega + \omega_3)(\sqrt{1 + |\nabla u(x)|^2})$$

$$- k_S(-2\tilde{\omega} \cdot \nabla u(x) + \omega_3(1 - |\nabla u(x)|^2)) = 0,$$

(4)

Remark:
The cosine in the specular term is usually replaced by zero if $R(x) \cdot V < 0$ (and in that case we get back to the L–model).
Fixing $\alpha = 1$, the PH–brightness equation becomes

**HJE in case $\mathbf{V} = (0, 0, 1)$ and $\alpha = 1$:**

$$I(\mathbf{x})(1 + |\nabla u(\mathbf{x})|^2) - k_D(-\nabla u(\mathbf{x}) \cdot \omega + \omega_3)(\sqrt{1 + |\nabla u(\mathbf{x})|^2})$$

$$- k_S(-2\tilde{\omega} \cdot \nabla u(\mathbf{x}) + \omega_3(1 - |\nabla u(\mathbf{x})|^2)) = 0,$$

(4)

**Remark:**

The cosine in the specular term is usually replaced by zero if $\mathbf{R}(\mathbf{x}) \cdot \mathbf{V} < 0$ (and in that case we get back to the L–model).
Exponential transform $\mu v(x) = 1 - e^{-\mu u(x)}$ to write (4) as

$$
\begin{cases}
\mu v(x) + \max_{a \in \partial B_3} \{-b^{PH}(x, a) \cdot \nabla v(x) + f^{PH}(x, z, a, v(x))\} = 1, \\
v(x) = 0,
\end{cases}
$$

where

$$
b^{PH}(x, a) = \frac{1}{Q(x, z)} \left(c(x)a_1 - k_D\omega_1, c(x)a_2 - k_D\omega_2\right),$$

$$
f^{PH}(x, z, a, v(x)) = \frac{c(x)a_3}{Q(x, z)} (1 - \mu v(x)),
$$

$$
Q(x, z) = 2k_S \left(\frac{\nabla S(x, z)}{|\nabla S(x, z)|} \cdot \omega\right) + k_D\omega_3,
$$

$$
c(x) = I(x) + \omega_3 k_S,
$$
Semi-Lagrangian Approximation

Fixed point algorithm

Given an initial guess \( W^{(0)} \) iterate on the grid \( G \)

\[
W^{(n)} = T[W^{(n-1)}] \quad n = 1, 2, 3, ...
\]

until

\[
\max_{x_i \in G} |W^{(n)}(x_i) - W^{(n-1)}(x_i)| < \eta
\]

We can write in a unique way the three different operators as

\[
T_i^M(W) = \min_{a \in \partial B_3} \left\{ e^{-\mu h} w(x_i + h b^M(x_i, a)) - \tau P^M a_3 (1 - \mu w(x_i)) \right\} + \tau
\]

where \( M = L, ON \) or \( PH \) and \( P^M \) is, respectively,

\[
P^L = \frac{I(x_i)}{\omega_3}, \quad P^{ON} = \frac{c(x_i, z)}{A \omega_3}, \quad P^{PH} = \frac{c(x_i)}{Q(x_i, z)}
\]

S. Tozza - SAPIENZA, Università di Roma

A unified approach to SfS models for non-Lambertian surfaces
The following properties are true:

1. Let $P^M \bar{a}_3 \leq 1$, with $\bar{a}_3 \equiv \arg \min_{a \in \partial B_3} \{ e^{-\mu h} w(x_i + h b^M(x_i, a)) - \tau P^M a_3 (1 - \mu w(x_i)) \}$. Then $0 \leq W \leq \frac{1}{\mu}$ implies $0 \leq T^M(W) \leq \frac{1}{\mu}$.

2. $v \leq u$ implies $T^M(v) \leq T^M(u)$.

3. $T^M$ is a contraction mapping in $[0, 1/\mu)^G$ if $P^M \bar{a}_3 < \mu$.
Test 1: Synthetic Vase

Lambertian

ON
$(\sigma = 0.4)$

Phong
$(k_S = 0.3, k_D = 0.7)$

S. Tozza - SAPIENZA, Università di Roma
A unified approach to SfS models for non-Lambertian surfaces
Test 1: Synthetic Vase

<table>
<thead>
<tr>
<th>Model</th>
<th>$\sigma$</th>
<th>$k_S$</th>
<th>$L_1(I)$</th>
<th>$L_2(I)$</th>
<th>$L_\infty(I)$</th>
<th>$L_1(S)$</th>
<th>$L_2(S)$</th>
<th>$L_\infty(S)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>LAM</td>
<td></td>
<td>0.0063</td>
<td>0.0380</td>
<td>0.7333</td>
<td>0.0267</td>
<td>0.0286</td>
<td>0.0569</td>
<td></td>
</tr>
<tr>
<td>ON</td>
<td>0</td>
<td>0.0063</td>
<td>0.0380</td>
<td>0.7333</td>
<td>0.0267</td>
<td>0.0286</td>
<td>0.0569</td>
<td></td>
</tr>
<tr>
<td>ON</td>
<td>0.4</td>
<td>0.0054</td>
<td>0.0316</td>
<td>0.6118</td>
<td>0.0263</td>
<td>0.0282</td>
<td>0.0562</td>
<td></td>
</tr>
<tr>
<td>ON</td>
<td>0.6</td>
<td>0.0049</td>
<td>0.0277</td>
<td>0.5373</td>
<td>0.0259</td>
<td>0.0277</td>
<td>0.0553</td>
<td></td>
</tr>
<tr>
<td>ON</td>
<td>1</td>
<td>0.0044</td>
<td>0.0229</td>
<td>0.4510</td>
<td>0.0254</td>
<td>0.0274</td>
<td>0.0547</td>
<td></td>
</tr>
<tr>
<td>PHO</td>
<td>0</td>
<td>0.0063</td>
<td>0.0380</td>
<td>0.7333</td>
<td>0.0267</td>
<td>0.0286</td>
<td>0.0569</td>
<td></td>
</tr>
<tr>
<td>PHO</td>
<td>0.3</td>
<td>0.0068</td>
<td>0.0396</td>
<td>0.8078</td>
<td>0.0264</td>
<td>0.0283</td>
<td>0.0561</td>
<td></td>
</tr>
<tr>
<td>PHO</td>
<td>0.6</td>
<td>0.0073</td>
<td>0.0411</td>
<td>0.8824</td>
<td>0.0247</td>
<td>0.0265</td>
<td>0.0526</td>
<td></td>
</tr>
<tr>
<td>PHO</td>
<td>0.9</td>
<td>0.0077</td>
<td>0.0373</td>
<td>0.9569</td>
<td>0.0141</td>
<td>0.0164</td>
<td>0.0432</td>
<td></td>
</tr>
</tbody>
</table>
Test 1: Synthetic Vase

<table>
<thead>
<tr>
<th>Model</th>
<th>$\sigma$</th>
<th>$k_S$</th>
<th>$L_1(I)$</th>
<th>$L_2(I)$</th>
<th>$L_\infty(I)$</th>
<th>$L_1(S)$</th>
<th>$L_2(S)$</th>
<th>$L_\infty(S)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>LAM</td>
<td>0.0063</td>
<td>0.0380</td>
<td>0.7333</td>
<td>0.0267</td>
<td>0.0286</td>
<td>0.0569</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ON</td>
<td>0</td>
<td>0.0063</td>
<td>0.0380</td>
<td>0.7333</td>
<td>0.0267</td>
<td>0.0286</td>
<td>0.0569</td>
<td></td>
</tr>
<tr>
<td>ON</td>
<td>0.4</td>
<td>0.0054</td>
<td>0.0316</td>
<td>0.6118</td>
<td>0.0263</td>
<td>0.0282</td>
<td>0.0562</td>
<td></td>
</tr>
<tr>
<td>ON</td>
<td>0.6</td>
<td>0.0049</td>
<td>0.0277</td>
<td>0.5373</td>
<td>0.0259</td>
<td>0.0277</td>
<td>0.0553</td>
<td></td>
</tr>
<tr>
<td>ON</td>
<td>1</td>
<td>0.0044</td>
<td>0.0229</td>
<td>0.4510</td>
<td>0.0254</td>
<td>0.0274</td>
<td>0.0547</td>
<td></td>
</tr>
<tr>
<td>PHO</td>
<td>0</td>
<td>0.0063</td>
<td>0.0380</td>
<td>0.7333</td>
<td>0.0267</td>
<td>0.0286</td>
<td>0.0569</td>
<td></td>
</tr>
<tr>
<td>PHO</td>
<td>0.3</td>
<td>0.0068</td>
<td>0.0396</td>
<td>0.8078</td>
<td>0.0264</td>
<td>0.0283</td>
<td>0.0561</td>
<td></td>
</tr>
<tr>
<td>PHO</td>
<td>0.6</td>
<td>0.0073</td>
<td>0.0411</td>
<td>0.8824</td>
<td>0.0247</td>
<td>0.0265</td>
<td>0.0526</td>
<td></td>
</tr>
<tr>
<td>PHO</td>
<td>0.9</td>
<td>0.0077</td>
<td>0.0373</td>
<td>0.9569</td>
<td>0.0141</td>
<td>0.0164</td>
<td>0.0432</td>
<td></td>
</tr>
</tbody>
</table>
Test 2: Real Horse

Lambertian

ON
($\sigma = 1$)

Phong
($k_S = 0.7$
$k_D = 0.3$)
### Test 2: Real Horse

<table>
<thead>
<tr>
<th>Model</th>
<th>$\sigma$</th>
<th>$k_S$</th>
<th>$L_1(I)$</th>
<th>$L_2(I)$</th>
<th>$L_\infty(I)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>LAM</td>
<td></td>
<td>0.0333</td>
<td>0.0580</td>
<td>0.6941</td>
<td></td>
</tr>
<tr>
<td>ON</td>
<td>0</td>
<td>0.0333</td>
<td>0.0580</td>
<td>0.6941</td>
<td></td>
</tr>
<tr>
<td>ON</td>
<td>0.4</td>
<td>0.0338</td>
<td>0.0587</td>
<td>0.6980</td>
<td></td>
</tr>
<tr>
<td>ON</td>
<td>0.8</td>
<td>0.0345</td>
<td>0.0598</td>
<td>0.6941</td>
<td></td>
</tr>
<tr>
<td>ON</td>
<td>1</td>
<td>0.0347</td>
<td>0.0600</td>
<td>0.6941</td>
<td></td>
</tr>
<tr>
<td>PHO</td>
<td>0</td>
<td>0.0334</td>
<td>0.0584</td>
<td>0.6941</td>
<td></td>
</tr>
<tr>
<td>PHO</td>
<td>0.4</td>
<td>0.0345</td>
<td>0.0599</td>
<td>0.6902</td>
<td></td>
</tr>
<tr>
<td>PHO</td>
<td>0.7</td>
<td>0.0359</td>
<td>0.0638</td>
<td>0.6941</td>
<td></td>
</tr>
<tr>
<td>PHO</td>
<td>1</td>
<td>0.0807</td>
<td>0.1057</td>
<td>0.8235</td>
<td></td>
</tr>
</tbody>
</table>
Test 3: Who is he?

Lambertian

ON
$(\sigma = 0.2)$

Phong
$(k_S = 0.8 \quad k_D = 0.2)$

S. Tozza - SAPIENZA, Università di Roma
A unified approach to SfS models for non-Lambertian surfaces
Test 3: Who is he?

A unified approach to SfS models for non-Lambertian surfaces

<table>
<thead>
<tr>
<th>Model</th>
<th>$\sigma$</th>
<th>$k_S$</th>
<th>$L_1(I)$</th>
<th>$L_2(I)$</th>
<th>$L_\infty(I)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>LAM</td>
<td></td>
<td>0.0333</td>
<td>0.0539</td>
<td>0.5608</td>
<td></td>
</tr>
<tr>
<td>ON</td>
<td>0</td>
<td>0.0333</td>
<td>0.0539</td>
<td>0.5608</td>
<td></td>
</tr>
<tr>
<td>ON</td>
<td>0.2</td>
<td>0.0727</td>
<td>0.0841</td>
<td>0.5765</td>
<td></td>
</tr>
<tr>
<td>ON</td>
<td>0.4</td>
<td>0.1534</td>
<td>0.1615</td>
<td>0.6196</td>
<td></td>
</tr>
<tr>
<td>ON</td>
<td>0.8</td>
<td>0.2675</td>
<td>0.2836</td>
<td>0.5804</td>
<td></td>
</tr>
<tr>
<td>ON</td>
<td>1</td>
<td>0.2924</td>
<td>0.3131</td>
<td>0.5647</td>
<td></td>
</tr>
<tr>
<td>PHO</td>
<td>0</td>
<td>0.0333</td>
<td>0.0539</td>
<td>0.5608</td>
<td></td>
</tr>
<tr>
<td>PHO</td>
<td>0.2</td>
<td>0.0368</td>
<td>0.0557</td>
<td>0.5529</td>
<td></td>
</tr>
<tr>
<td>PHO</td>
<td>0.4</td>
<td>0.0401</td>
<td>0.0581</td>
<td>0.5569</td>
<td></td>
</tr>
<tr>
<td>PHO</td>
<td>0.8</td>
<td>0.0457</td>
<td>0.0635</td>
<td>0.5843</td>
<td></td>
</tr>
<tr>
<td>PHO</td>
<td>1</td>
<td>0.0498</td>
<td>0.0681</td>
<td>0.6000</td>
<td></td>
</tr>
</tbody>
</table>
Conclusions

- A new unique mathematical formulation for different reflectance models
- The ON–model is more general and incorporates the L–model
- The PH–model recognizes better the silhouette so it seems to be a more realistic model;
- The choice of parameters is crucial for accuracy;
- The choice of the subject is crucial too! (See Test 3)
Conclusions

- A new unique mathematical formulation for different reflectance models

- The ON–model is more general and incorporates the L–model

- The PH–model recognizes better the silhouette so it seems to be a more realistic model;

- The choice of parameters is crucial for accuracy;

- The choice of the subject is crucial too! (See Test 3)
Combining specular-reflection effects with the more complex and general Oren-Nayar diffuse model in order to arrive to the “best” and the most general model;

Photometric stereo: using more than one input image (as already done for the L–model [Mecca-T., 2013]);

Parallel algorithms

Acceleration methods


