A unified approach to Shape-from-Shading models for non-Lambertian surfaces

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Numerical methods for PDEs: optimal control, games and image processing

(On the Occasion of Maurizio Falcone's 60th birthday)

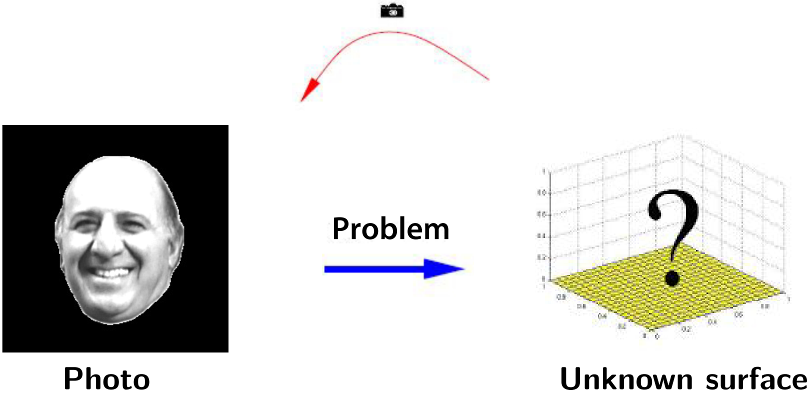
December 5, 2014, Rome

Outline

- Introduction
- Some Reflectance Models in a unified approach
  a. Lambertian Model
  b. Oren-Nayar Model
  c. Phong Model
- Semi-Lagrangian Approximation
- Numerical Tests
- Conclusions and Perspectives

Problem:

We want to obtain the 3D shape of an object starting from its image



The SfS problem is described by the following irradiance equation:

$$R(\boldsymbol{N}(\mathbf{x})) = I(\mathbf{x}) \tag{1}$$

where

- $R(\boldsymbol{N}(\mathbf{x}))$ is the reflectance function;
- $\boldsymbol{N}(\mathbf{x})$ is the unit normal to the surface at point $(\mathbf{x}, u(\mathbf{x}))$;
- $I(\mathbf{x})$ is the greylevel measured in the image at point $\mathbf{x}$.

$I : \overline{\Omega} \to [0, 1]$, with $\overline{\Omega}$ compact domain ($\Omega \subset \mathbb{R}^2$ open subset).

Assumptions:

1. One light source located at infinity in the direction of $\boldsymbol{\omega}$;
2. no self-reflections on the surface;
3. the light source is sufficiently far from the surface so perspective deformations are neglected;
4. the diffuse and specular albedos $\gamma_D(\mathbf{x})$ and $\gamma_S(\mathbf{x})$ are known (for simplicity we put $\gamma_D(\mathbf{x}) = \gamma_S(\mathbf{x}) = 1$);

SfS Problem: general unique formulation

As proposed in [T., 2014], it is useful to rewrite (1) as

$$I(\mathbf{x}) = k_A I_A + k_D I_D(\mathbf{x}) + k_S I_S(\mathbf{x})$$

where

- k_A, k_D, and k_S (with $k_A + k_D + k_S = 1$): ratio of ambient, diffuse, and specular reflection;

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In the whole talk we neglect the contribution of the ambient component ($k_A = 0$).

Idea: The surface is Lambertian, i.e. the intensity reflected by a point of the surface is equal from all points of view.

Remark: This is a purely diffuse model $\rightarrow I_S$ doesn't exist
$\Rightarrow I(\mathbf{x}) \equiv I_D(\mathbf{x})$ $(k_D \equiv 1)$

Goal: Finding $u : \overline{\Omega} \rightarrow \mathbb{R}$ s. t. satisfy the following equation:

$$I(\mathbf{x}) = \boldsymbol{N}(\mathbf{x}) \cdot \boldsymbol{\omega}, \quad \forall\, \mathbf{x} \in \Omega \tag{2}$$

where

- $\boldsymbol{N}(\mathbf{x}) = \frac{\boldsymbol{n}(\mathbf{x})}{|\boldsymbol{n}(\mathbf{x})|} = \frac{1}{\sqrt{1+|\nabla u(\mathbf{x})|^2}}(-\nabla u(\mathbf{x}), 1)$
- $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3) = (\tilde{\boldsymbol{\omega}}, \omega_3)$ (general light direction)

Lambertian PDE [Falcone-Sagona-Seghini, 2003]

Hamilton-Jacobi equation (HJE) associated to (2):

$$I(\mathbf{x})\sqrt{1+|\nabla u(\mathbf{x})|^2}+\tilde{\omega}\cdot\nabla u(\mathbf{x})-\omega_3=0, \text{ in } \Omega.$$

By using the exponential transform $\mu v(\mathbf{x})=1-e^{-\mu u(\mathbf{x})}$ we arrive to the following problem in new variable v

Fixed point form

$$\begin{cases}\mu v(\mathbf{x})=\min\limits_{a\in\partial B_3}\{b^L(\mathbf{x},a)\cdot\nabla v(\mathbf{x})+f^L(\mathbf{x},a,v(\mathbf{x}))\}, & \text{for } \mathbf{x}\in\Omega,\\ v(\mathbf{x})=0, & \text{for } \mathbf{x}\in\partial\Omega,\end{cases}$$

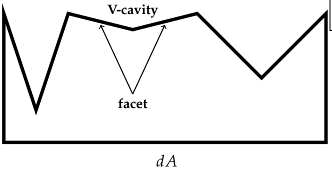
where

$$(b^L,f^L)=\left(\frac{I(\mathbf{x})\mathbf{a}_{1,2}-\tilde{\omega}}{\omega_3},\frac{-I(\mathbf{x})a_3}{\omega_3}(1-\mu v(\mathbf{x}))+1\right),$$

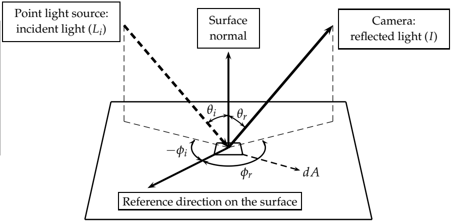
and B_3 is the unit ball in $\mathbb{R}^3$.

Oren-Nayar reflectance model (ON–model)

Idea: Representing a rough surface as an aggregation of V-shaped cavities, each with Lambertian reflectance properties.



(a) Facet model for surface patch dA consisting of many V-shaped Lambertian cavities.



(b) Diffuse reflectance for SfS with Oren-Nayar.

Figure: Sketch of the Oren-Nayar surface reflection model.

Remark:

This is a purely diffuse model $\rightarrow$ I_S doesn't exist
$\Rightarrow I(\mathbf{x}) \equiv I_D(\mathbf{x})$ $(k_D \equiv 1)$

Oren-Nayar reflectance model

General Brightness equation [Oren-Nayar, 1995]:

$$I(\mathbf{x}) = \cos(\theta_i)\ \ (A + B\sin(\alpha)\tan(\beta)\max[0, \cos(\varphi_i - \varphi_r)])$$

where

- $A = 1 - 0.5\,\sigma^2\,(\sigma^2 + 0.33)^{-1}$; $B = 0.45\sigma^2\,(\sigma^2 + 0.09)^{-1}$;
- σ: roughtness parameter of the surface;
- θ_i: angle between $\mathbf{N}$ and $\boldsymbol{\omega}$;
- θ_r: angle between $\mathbf{N}$ and viewer direction $\mathbf{V}$;
- $\alpha = \max\,[\theta_i, \theta_r]$; $\qquad \beta = \min\,[\theta_i, \theta_r]$;
- φ_i: angle between the projection of $\boldsymbol{\omega}$ and the x_1 axis onto the (x_1, x_2)-plane;
- φ_{r}: angle between the projection of $\mathbf{V}$ and the x_1 axis.

Oren-Nayar reflectance model

Brightness equation in the case $\omega \equiv V$

$$I(\mathbf{x}) = \cos(\theta) \ \left(A + B\sin(\theta)^2\cos(\theta)^{-1}\right)$$

where $\theta := \theta_i = \theta_r = \alpha = \beta$.

Dirichlet problem associated to the brightness equation:

$$\begin{cases} (I(\mathbf{x}) - B)(\sqrt{1 + |\nabla u|^2}) + A(\widetilde{\omega} \cdot \nabla u - \omega_3) \\ \qquad + B\frac{(-\widetilde{\omega}\cdot\nabla u + \omega_3)^2}{\sqrt{1+|\nabla u|^2}} = 0, & \mathbf{x} \in \Omega, \\ u(\mathbf{x}) = 0, & \mathbf{x} \in \partial\Omega, \end{cases} \tag{3}$$

Remark:

When $\sigma = 0$ the ON–model brings back to the L–model.

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Remark:

When $\sigma = 0$ the ON–model brings back to the L–model.

Oren-Nayar PDE [T.-Falcone, 2014]

Exponential transform $\mu v(\mathbf{x}) = 1 - e^{-\mu u(\mathbf{x})}$ to write (3) as

$$\begin{cases} \mu v(\mathbf{x}) + \max_{a \in \partial B_3} \{ -b^{ON}(\mathbf{x}, a) \cdot \nabla v(\mathbf{x}) + f^{ON}(\mathbf{x}, z, a, v(\mathbf{x})) \} = 1, & \mathbf{x} \in \Omega, \\ v(\mathbf{x}) = 0, & \mathbf{x} \in \partial \Omega, \end{cases}$$

where

$$b^{ON}(\mathbf{x}, a) = \frac{1}{A\omega_3} \left(c(\mathbf{x}, z) a_1 - A\omega_1, c(\mathbf{x}, z) a_2 - A\omega_2 \right),$$

$$f^{ON}(\mathbf{x}, z, a, v(\mathbf{x})) = \frac{c(\mathbf{x}, z) a_3}{A\omega_3} (1 - \mu v(\mathbf{x})),$$

$$c(\mathbf{x}, z) = I(\mathbf{x}) - B + B \left(\frac{\nabla S(\mathbf{x}, z)}{|\nabla S(\mathbf{x}, z)|} \cdot \boldsymbol{\omega} \right)^2$$

with

$$\nabla S(\mathbf{x}, z) = (-\nabla u(\mathbf{x}), 1).$$

Phong reflectance model (PH–model)

General Brightness equation [B.T. Phong, 1975]:

$$I(\mathbf{x}) = k_D(\cos(\theta_i)) + k_S(\cos(\theta_s))^\alpha$$

where

- θ_i: angle between $\mathbf{N}$ and $\boldsymbol{\omega}$.
- θ_s: angle between reflected light direction $\mathbf{R}$ and $\mathbf{V}$. $0 \leq \theta_s \leq \frac{\pi}{2}$ because for greater angles the viewer does not perceive the light reflected specularly;
- α: models the specular reflected light for each material;
- $\mathbf{N}$ and $\mathbf{R}$ are unitary and coplanar.

Phong reflectance model

Fixing $\alpha = 1$, the PH–brightness equation becomes

HJE in case $\mathbf{V} = (0, 0, 1)$ and $\alpha = 1$:

$$I(\mathbf{x})(1 + |\nabla u(\mathbf{x})|^2) - k_D(-\nabla u(\mathbf{x}) \cdot \omega + \omega_3)(\sqrt{1 + |\nabla u(\mathbf{x})|^2}) - k_S\,(-2\widetilde{\omega} \cdot \nabla u(\mathbf{x}) + \omega_3(1 - |\nabla u(\mathbf{x})|^2)) = 0, \tag{4}$$

Remark:

The cosine in the specular term is usually replaced by zero if $\mathbf{R}(\mathbf{x}) \cdot \mathbf{V} < 0$ (and in that case we get back to the L–model).

Phong reflectance model

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Remark:

The cosine in the specular term is usually replaced by zero if $\mathbf{R}(\mathbf{x}) \cdot \mathbf{V} < 0$ (and in that case we get back to the L–model).

Phong PDE [T.-Falcone, 2014 submitted]

Exponential transform $\mu v(\mathbf{x}) = 1 - e^{-\mu u(\mathbf{x})}$ to write (4) as

$$\begin{cases} \mu v(\mathbf{x}) + \max_{a \in \partial B_3} \{-b^{PH}(\mathbf{x}, a) \cdot \nabla v(\mathbf{x}) + f^{PH}(\mathbf{x}, z, a, v(\mathbf{x}))\} = 1, & \mathbf{x} \in \Omega, \\ v(\mathbf{x}) = 0, & \mathbf{x} \in \partial\Omega, \end{cases}$$

where

$$b^{PH}(\mathbf{x}, a) = \frac{1}{Q(\mathbf{x}, z)} \left(c(\mathbf{x})a_1 - k_D\omega_1, c(\mathbf{x})a_2 - k_D\omega_2\right),$$

$$\begin{aligned} f^{PH}(\mathbf{x}, z, a, v(\mathbf{x})) &= \frac{c(\mathbf{x})a_3}{Q(\mathbf{x}, z)}(1 - \mu v(\mathbf{x})), \\ Q(\mathbf{x}, z) &= 2k_S \left(\frac{\nabla S(\mathbf{x}, z)}{|\nabla S(\mathbf{x}, z)|} \cdot \omega\right) + k_D\omega_3, \\ c(\mathbf{x}) &= I(\mathbf{x}) + \omega_3 k_S, \end{aligned}$$

Semi-Lagrangian Approximation

Fixed point algorithm

Given an initial guess $W^{(0)}$ iterate on the grid G

$$W^{(n)} = T[W^{(n-1)}] \qquad n = 1, 2, 3, \ldots$$

until $$\max_{x_i \in G} |W^{(n)}(x_i) - W^{(n-1)}(x_i)| < \eta$$

We can write in a unique way the three different operators as

$$T_i^M(W) = \min_{a \in \partial B_3} \{e^{-\mu h} w(x_i + h b^M(x_i, a)) - \tau P^M a_3 (1 - \mu w(x_i))\} + \tau$$

where $M = L$, ON or PH and P^M is, respectively,

$$P^L = \frac{I(x_i)}{\omega_3}, \qquad P^{ON} = \frac{c(x_i, z)}{A\omega_3}, \qquad P^{PH} = \frac{c(x_i)}{Q(x_i, z)}$$

Operators' properties [T., 2014]

The following properties are true:

1. Let $P^M \overline{a}_3 \leq 1$, with $\overline{a}_3 \equiv$

   $$arg \min_{a \in \partial B_3} \{ e^{-\mu h} w(x_i + h b^M(x_i, a)) - \tau P^M a_3 (1 - \mu w(x_i)) \}.$$

   Then $0 \leq W \leq \frac{1}{\mu}$ implies $0 \leq T^M(W) \leq \frac{1}{\mu}$

2. $v \leq u$ implies $T^M(v) \leq T^M(u)$

3. T^M is a contraction mapping in $[0, 1/\mu)^G$ if $P^M \overline{a}_3 < \mu$

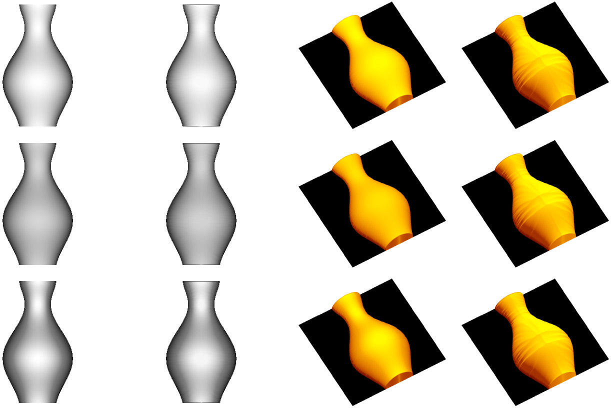
Test 1: Synthetic Vase

in | **out** | **vertical view** | **oblique view**

Lambertian

ON ($\sigma = 0.4$)

Phong ($k_S = 0.3$ $k_D = 0.7$)



Test 1: Synthetic Vase

Model	σ	k_S	$L_1(I)$	$L_2(I)$	$L_\infty(I)$	$L_1(S)$	$L_2(S)$	$L_\infty(S)$
LAM			0.0063	0.0380	0.7333	0.0267	0.0286	0.0569
ON	0		0.0063	0.0380	0.7333	0.0267	0.0286	0.0569
ON	0.4		0.0054	0.0316	0.6118	0.0263	0.0282	0.0562
ON	0.6		0.0049	0.0277	0.5373	0.0259	0.0277	0.0553
ON	1		0.0044	0.0229	0.4510	0.0254	0.0274	0.0547
PHO		0	0.0063	0.0380	0.7333	0.0267	0.0286	0.0569
PHO		0.3	0.0068	0.0396	0.8078	0.0264	0.0283	0.0561
PHO		0.6	0.0073	0.0411	0.8824	0.0247	0.0265	0.0526
PHO		0.9	0.0077	0.0373	0.9569	0.0141	0.0164	0.0432

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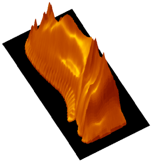
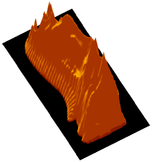
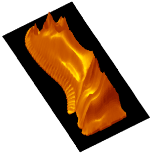
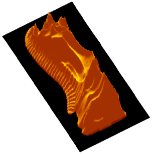
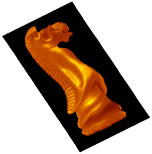
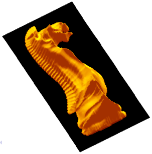
Model	σ	k_S	$L_1(I)$	$L_2(I)$	$L_\infty(I)$	$L_1(S)$	$L_2(S)$	$L_\infty(S)$
LAM			0.0063	0.0380	0.7333	0.0267	0.0286	0.0569
ON	0		0.0063	0.0380	0.7333	0.0267	0.0286	0.0569
ON	0.4		0.0054	0.0316	0.6118	0.0263	0.0282	0.0562
ON	0.6		0.0049	0.0277	0.5373	0.0259	0.0277	0.0553
ON	1		0.0044	0.0229	0.4510	0.0254	0.0274	0.0547
PHO		0	0.0063	0.0380	0.7333	0.0267	0.0286	0.0569
PHO		0.3	0.0068	0.0396	0.8078	0.0264	0.0283	0.0561
PHO		0.6	0.0073	0.0411	0.8824	0.0247	0.0265	0.0526
PHO		0.9	0.0077	0.0373	0.9569	0.0141	0.0164	0.0432

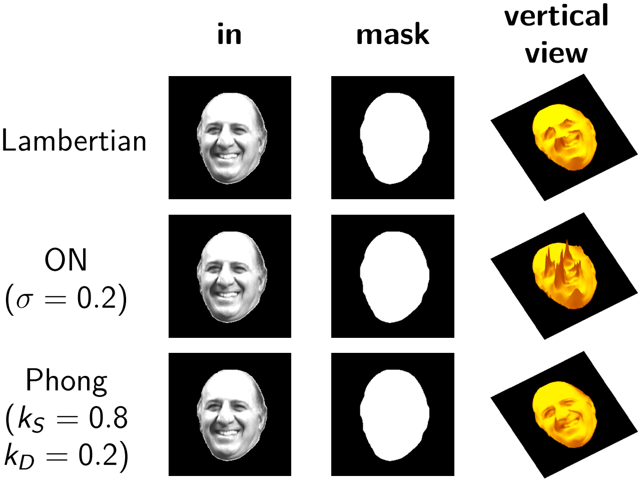
Test 2: Real Horse

	in	out	vertical view	oblique view
Lambertian				
ON ($\sigma = 1$)				
Phong ($k_S = 0.7$ $k_D = 0.3$)				

Test 2: Real Horse

Model	σ	k_S	$L_1(I)$	$L_2(I)$	$L_\infty(I)$
LAM			0.0333	0.0580	0.6941
ON	0		0.0333	0.0580	0.6941
ON	0.4		0.0338	0.0587	0.6980
ON	0.8		0.0345	0.0598	0.6941
ON	1		0.0347	0.0600	0.6941
PHO		0	0.0334	0.0584	0.6941
PHO		0.4	0.0345	0.0599	0.6902
PHO		0.7	0.0359	0.0638	0.6941
PHO		1	0.0807	0.1057	0.8235

Test 3: Who is he?



Test 3: Who is he?

Model	σ	k_S	$L_1(I)$	$L_2(I)$	$L_\infty(I)$
LAM			0.0333	0.0539	0.5608
ON	0		0.0333	0.0539	0.5608
ON	0.2		0.0727	0.0841	0.5765
ON	0.4		0.1534	0.1615	0.6196
ON	0.8		0.2675	0.2836	0.5804
ON	1		0.2924	0.3131	0.5647
PHO		0	0.0333	0.0539	0.5608
PHO		0.2	0.0368	0.0557	0.5529
PHO		0.4	0.0401	0.0581	0.5569
PHO		0.8	0.0457	0.0635	0.5843
PHO		1	0.0498	0.0681	0.6000

Conclusions

- A new unique mathematical formulation for different reflectance models
- The ON–model is more general and incorporates the L–model
- The PH–model recognizes better the silhouette so it seems to be a more realistic model;
- The choice of parameters is crucial for accuracy;
- The choice of the subject is crucial too! (See Test 3)

Conclusions

- A new unique mathematical formulation for different reflectance models
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- The choice of the subject is crucial too! (See Test 3)

Work in progress/Future Perspective

1. Combining specular-reflection effects with the more complex and general Oren-Nayar diffuse model in order to arrive to the "best" and the most general model;

2. Photometric stereo: using more than one input image (as already done for the L–model [Mecca-T., 2013]);

3. Parallel algorithms

4. Acceleration methods

References

- M. Oren, S.K. Nayar, Generalization of the Lambertian Model and Implications for Machine Vision, *Int. J. of Comp. Vis.*, 14(3):227-251,1995.
- B.T. Phong, Illumination for computer generated pictures, In: *Communications of the ACM*, 18(6):311–317, 1975.
- S. Tozza and M. Falcone, *A semi-Lagrangian Approximation of the Oren–Nayar PDE for the Orthographic Shape–from–Shading Problem*, Proc. 9th International Conference on Computer Vision Theory and Applications (VISAPP), vol.3, pp. 711-716, SCITEPRESS, 2014.
- S. Tozza and M. Falcone, *A comparison of non-Lambertian models for the Shape–from–Shading problem*, Submitted to Michael Breuss, Alfred Bruckstein, Petros Maragos, Stefanie Wuhrer (Editors), New Perspectives in Shape Analysis, Springer Edition.
- S. Tozza, *Analysis and Approximation of Non-Lambertian Shape-from-Shading Models*, PhD thesis, Dipartimento di Matematica della “Sapienza - Universit´a di Roma”, Roma, Italy, November 2014.
- R. Mecca and S. Tozza, Shape Reconstruction of Symmetric Surfaces using Photometric Stereo, In: *Innovations for Shape Analysis: Models and Algorithms*, pp. 219-243, Springer Edition, 2013.