Necessary Conditions for Delayed Optimal Control Problems

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Outline

Problem Formulation:
Optimal Control Problems with Time-Delay & Free End-Time

Motivation:
The Failure of Standard Techniques

Necessary Optimality Conditions:
A new Transversality Condition

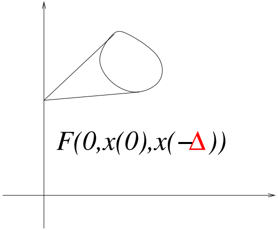
Sensitivity Analysis

Computational Methods and Numerical Examples

Concluding Remarks

An Optimal Control Problem with Constant Time-Delay & Free End-Time

$$\begin{cases} \text{Minimize } g(T, x(T)) \\ \text{over pairs } (T, x(.)) \text{ satisfying} \\ \dot{x}(t) \in F(t, x(t), x(t-\Delta)), \quad t \in [0, T] \\ x(s) = x_0(s), \quad s \in [-\Delta, 0] \end{cases}$$



Here $\Delta > 0$ is a fix constant and $T > 0$ is a choice variable!

$F(.)$ is a given set-valued map. We could adopt the standard identification

$$F(t, x, y) = \{f(t, x, y, u) : u \in U\}.$$

Quick Overview on Classical Techniques

Free End-Time (Time Transformation)

Consider a classical delay-free problem

$$\begin{cases} \text{Min } g(T, x(T)) : \\ \dot{x}(t) \in F(t, x(t)), \text{ a.e.} \\ x(0) = x_0 \end{cases}$$

Free time	$\longrightarrow$	Fixed time
$x(t)$	$\longrightarrow$	$y(s)$,

Apply the time transformation

$$t = sT$$

$$\begin{cases} \text{Min } g(t(1), y(1)) : \\ \dot{y}(s) \in T \cdot F(t(s), y(s)), \; s \in [0, 1] \\ x(0) = x_0 \end{cases}$$

Where
$y(s) := x(sT)$
T is a new "control"
t(s) is a new state

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Standard Optimal Control Problem!

Reduction for Time-Delay Systems

Consider the following problem

$$\begin{cases} \text{Min } g(T, x(T)) : \\ \dot{x}(t) \in F(t, x(t), x(t-\Delta)), \text{ a.e.} \\ x(0) = x_0 \end{cases}$$

Apply the time transformation

$$t = sT$$

Now, how do we rewrite the delay bit?

$$x(t-\Delta) = x(sT - \Delta) = x(T(s - T^{-1}\Delta)) = y(s - T^{-1}\Delta)$$

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The new delay depend on the control parameter T!

- **C. Liu, R. Loxton, K.L. Teo**, *A computational method for solving time-delay optimal control problems with free terminal time* , 2014

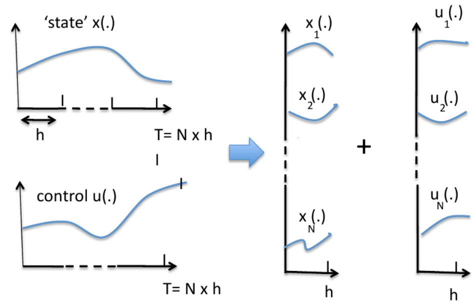
Reduction to a Delay-Free case

If the final time $T > 0$ is fixed, then

$$(P_T)\begin{cases} \text{Min } g(T, x(T)) : \\ \dot{x}(t) \in F(t, x(t), x(t-\Delta), \\ x(0) = x_0 \end{cases} \Rightarrow$$

$$\begin{array}{lllll} t \in [0,\Delta] & \rightarrow & F(t, x(t), x_0(t-\Delta)) & \rightarrow & x_1(.) \\ t \in [\Delta, 2\Delta] & \rightarrow & F(t, x(t), x_1(t-\Delta)) & \rightarrow & x_2(.) \quad \ldots \end{array}$$

Guinn Transformation



Some References

Techniques not based on a time transformation:

- **J. Warga**, *Controllability, extremality, and abnormality in nonsmooth optimal control*, 1983
- **G. L. Kharatishvili and T. A. Tadumadze**, *Formulas for variations of solutions to a differential equation with retarded arguments and a discontinuous initial condition*, 2005
- **A. Boccia, P. Falugi, H. Maurer, R.B. Vinter**, *Free time optimal control problems with time delays*, 2014

Numerical Approach to deal with parameter dependent time-delay

- **C. Liu, R. Loxton, K.L. Teo**, *A computational method for solving time-delay optimal control problems with free terminal time* , 2014

Can we neglect delays?

Metal Cutting

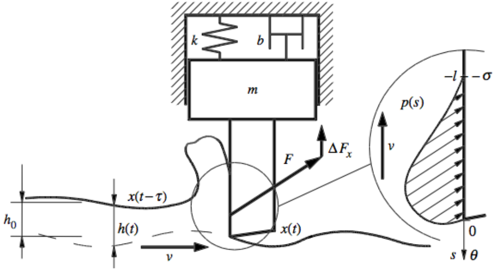


Figure 1. Mechanical model of regenerative chatter for orthogonal cutting.

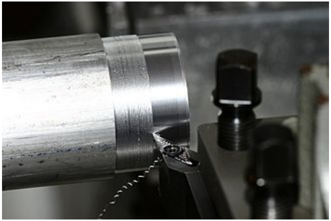
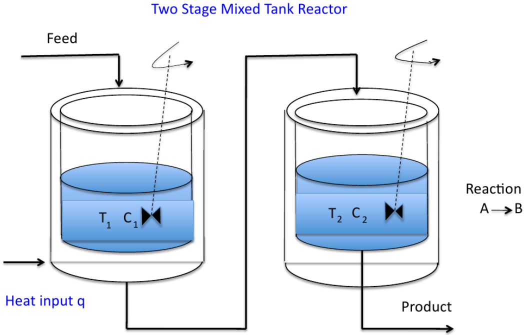


Figure: From Wikipedia

- **Gabor Stepan**, *Modelling nonlinear regenerative effects in metal cutting*

Chemical Engineering Transportation Delay



Time Delays caused by transport between two tanks

- **Göllmann, Kern, Maurer**, *Optimal control problems with delays in state and control variables subject to mixed controlstate constraints*

Necessay conditions for

$$\begin{cases} \text{Min } g(T, x(T)) : \\ \dot{x}(t) = f(t, x(t), x(t-\Delta), u(t)), \text{ a.e.} \\ u(t) \in U \\ x(s) = x_0(s), \ s \in [-\Delta, 0] \end{cases}$$

We used techniques developed in

- **F. H. Clarke and R. B. Vinter**, *Optimal multiprocesses*, 1989

Delayed Maximum Principle

$\min\{g(T, x(T)) : \dot{x}(t) = f(t, x(t), x(t-h), u(t)), \dots\}$

Let $(\bar{x}(.), \bar{u}(.))$ be a minimizer. When we freeze

$$T = \bar{T}$$

a standard delayed maximum principle for fixed end-time must be satisfied. Then there exists $p(.) \in AC([0,1]; \mathbb{R}^n)$

- Adjoint Equation: for a.e. $t \in [0, \bar{T}]$

$$-\dot{p}(t) = p(t) \cdot \nabla_x f(t, \bar{x}(t), \bar{x}(t-h), \bar{u}(t)) + p(t+h) \cdot \nabla_y f(t+h, \bar{x}(t+h), \bar{x}(t), \bar{u}(t+h)) \cdot \chi_{[0,1-h]}(t)$$

(costate satisfies delay equation in reverse time)

- Transversality Condition: $-p(\bar{T}) = \nabla_x g(\bar{T}, \bar{x}(\bar{T}))$.
- Weierstrass Condition: for a.e. $t \in [0, \bar{T}]$

$$p(t) \cdot f(t, \bar{x}(t), \bar{x}(t-h), \bar{u}(t)) = \max_{u \in U} \{p(t) \cdot f(t, \bar{x}(t), \bar{x}(t-h), u)\}.$$

Transversality Condition

We need to derive an extra condition to take account of the extra degree of freedom. Defining

$$\mathcal{H}(t, x, y, p) := \max_{u \in U} \{p \cdot f(t, x, y, u)\}$$

We can prove the following

$$\nabla_T g(\bar{T}, \bar{x}(\bar{T})) = \mathcal{H}(\bar{T}, \bar{x}(\bar{T}), \bar{u}(\bar{T}), p(\bar{T}))$$

IDEA:

$$g(\bar{T}, \bar{x}(\bar{T})) \le g(\bar{T} - \epsilon, \bar{x}(\bar{T} - \epsilon))$$
$$= g(\bar{T}, \bar{x}(\bar{T})) - \epsilon \nabla_T g(\bar{T}, \bar{x}(\bar{T})) - \int_{\bar{T}-\epsilon}^{\bar{T}} \nabla_x g(\bar{T}, \bar{x}(\bar{T})) \cdot \dot{x}(t)\, dt$$

$$\Rightarrow \nabla_T g(\bar{T}, \bar{x}(\bar{T})) \le \frac{1}{\epsilon} \int_{\bar{T}-\epsilon}^{\bar{T}} p(\bar{T}) \cdot \dot{x}(t)\, dt$$

Sensitivity Analysis

Sensitivity Information

Fix the end-time T. How does the minimum cost change with T?

$$(P_T)\begin{cases} \text{Minimize } g(T, x(T)) \text{ s.t.} \\ dx(t)/dt = f(t, x(t), x(t-h), u(t)) \quad a.e. \\ u(t) \in U \\ x(t) = x_0(t), \quad t \in [-h, 0] \\ x(T) \in C \ . \end{cases}$$

Define $V(.) : \mathbb{R} \to \mathbb{R}$

$$V(T) := \min\{P_T\} \ .$$

Sensitivity Information (cont.)

Notice that, if $(\bar{x}(.), \bar{u}(.))$ solves $(P_{\bar{T}})$ then

$$\begin{array}{c} g(\bar{T}, \bar{x}(\bar{T})) = V(\bar{T}) \\ g(T, x(T)) \geq V(T) \end{array}$$

for any $(x(.), u(.))$ on $[0, T]$. Hence $(\bar{T}, \bar{x}(.), \bar{u}(.))$ solves

$$\left\{ \begin{array}{l} \text{Minimize } g(T, x(T)) - V(T) \text{ s.t.} \\ dx(t)/dt = f(t, x(t), x(t-h), u(t)) \quad a.e. \\ u(t) \in U \\ x(t) = x_0(t), \quad t \in [-h, 0] \\ x(T) \in C \ . \end{array} \right.$$

PMP gives

$$\nabla_T V(\bar{T}) \ = \ \nabla_T g(\bar{T}, \bar{x}(\bar{T})) - \mathcal{H}(\bar{T}, \bar{x}(\bar{T}), \bar{u}(\bar{T}), p(\bar{T}))$$

Computational Aspects

Consider the fixed time problem

$$
(P_T)\begin{cases}
\text{Minimize } g(T, x(T)) \text{ s.t.} \\
dx(t)/dt = f(t, x(t), x(t-h), u(t)) \quad a.e. \\
u(t) \in U \\
x(t) = x_0(t), \quad t \in [-h, 0] \\
x(T) \in C\ .
\end{cases}
$$

Solution Technique

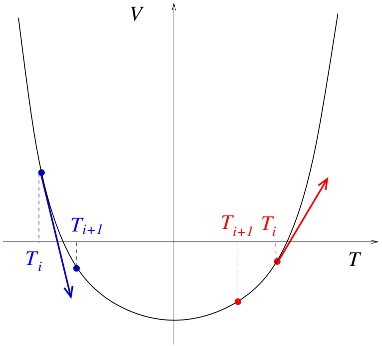
- Apply Guinn transformation to eliminate delay
- Reduce to NLP by time discretization
- Solve and generate costate trajectory $p(.)$, using IPOPT, or other optimization software.

Computational Aspects (cont.)

Solution of free-time problems is based on

- For fixed T_i, we can compute solution $(x_i(.), u_i(.))$ to P_{T_i} and costate $p_i(.)$ and also
- Formulae of sensitivity to change of end-time:

$$\frac{dV}{dT}(T_i) = \nabla_T g(T_i, x_i(T_i)) - \mathcal{H}(T_i, x_i(T_i), u_i(T_i), p_i(T_i))$$



Example: Optimal fishing

$$\begin{cases} \text{Minimize } \int_0^T e^{-\beta t}(C_E x(t)^{-1} u(t)^3 - p u(t))\, dt + 0.1\, T^2 \\ \text{over } T > 0, x(.) \text{ and } u(.) \text{ satisfying} \\ \dot{x}(t) = a x(t) \left(1 - \frac{x(t-h)}{b}\right) - u(t) \\ x(t) = 2, \quad t \in [-h, 0] \\ u(t) \geq 0, \quad t \in [0, T]\,. \end{cases}$$

$x(t)$: biomass of population.
$u(t)$: harvesting effort.
$C_E = 0.2$ (harvesting cost) , $a = 3$ and $b = 5$ (growth rates),
$\beta = 0.05$ (discount rate) and $p = 2$ (market price), $h = 0.5$.

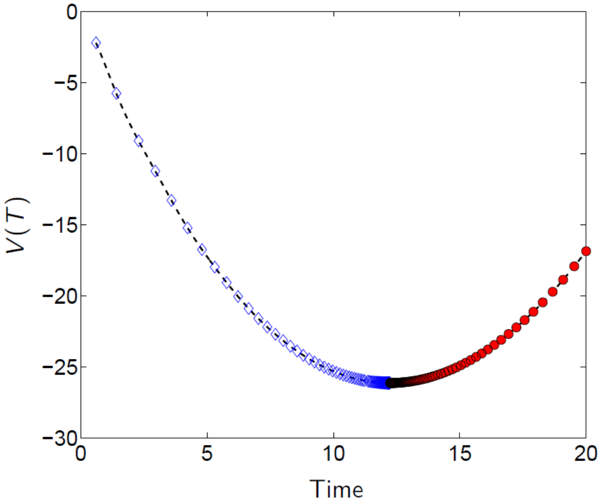


Figure: End-time value function and performance of algorithm based on sensitivity formulae, for various starting times: $T_0 = 0.5(\circ)$, $T_0 = 3.5(\Diamond)$.

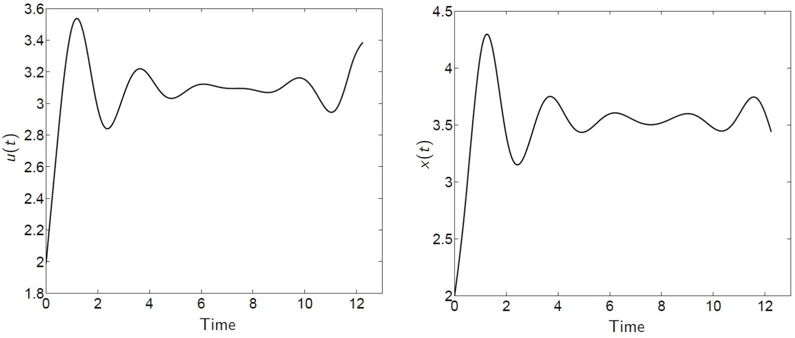


Figure: Optimal input (left) and respective fluctuation of the fish population (right)

Summing up

- We developed an analysis to address Delayed & Free End-Time Optimal Control Problems
- We derived numerical schemes (better convergence)

Future work

- State constraints
- Time dependent delays
- Input delays . . .

Grazie...

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