HJB approach for stochastic optimal control problems with state constraints

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ENSTA ParisTech **Univ. Paris Diderot**

Internal SADCO Meeting, Rome, 10-12 Nov. 2014

Let $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space. Consider the following controlled SDE in $\mathbb{R}^d$:

$$\begin{cases} dX(s) = b(s, X(s), u(s))ds + \sigma(s, X(s), u(s))dB_s & s \in [t, T] \\ X(t) = x \end{cases} \tag{1}$$

where, for a given compact set U:

$$u \in \mathcal{U} := \{\text{Progr. meas. processes with values in } U\}$$

$\rightsquigarrow$ $\exists\, X(\cdot) := X^u_{t,x}(\cdot)$: unique strong solution of (1) associated to the control u. (under usual regularity assumptions on b,σ).

Let $\psi : \mathbb{R}^d \to \mathbb{R}$ (terminal cost) and $\ell : [0, T] \times \mathbb{R}^d \times U \to \mathbb{R}$ (running cost).
Let $\mathcal{K} \subseteq \mathbb{R}^d$ be a non empty and closed set.

STATE-CONSTRAINED OCP:

$$v(t,x) = \inf_{u \in \mathcal{U}} \left\{ \mathbb{E}\left[\psi(X^u_{t,x}(T)) + \int_t^T \ell(s, X^u_{t,x}(s), u(s)) ds \right] : \text{such that } X^u_{t,x}(s) \in \mathcal{K}, \forall s \in [t, T] \text{ a.s.} \right\}$$

Example: No control, $\ell \equiv 0$, problem is:

$$v(t,x) = \mathbb{E}\left[\psi(X_{t,x}(T)) \right] \text{ on processes s.t. } X_{t,x}(\cdot) \subset \mathcal{K}.$$

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- Avoid if possible controllability or feasability conditions
- Framework that enables further numerical approximation

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- aircraft landing problem with wind disturbances
- energy consumption/demand problem with limited resource
- super replication problems in finance
- ... any stochastic problem with state constraints

1. Find an PDE for the value function v
2. Look for an optimal trajectory $X(.)$ in the DPP.

- Ex: NO STATE CONSTRAINTS, DETERMINISTIC CASE:

$$v(t,x) = \inf_{u \in \mathcal{U}} \psi(X^u_{t,x}(T))$$

Then the DPP reads, for $0 < h \leq T - t$,

$$v(t,x) = \inf_{u \in \mathcal{U}} v(t+h, X^u_{t,x}(t+h))$$

and the HJB equation (PDE) is

$$-v_t + \max_{u \in U}(-b(t,x,u) \cdot \nabla v) = 0 \qquad (2)$$

$$v(T,x) = \psi(x). \qquad (3)$$

- Ex: NO STATE CONSTRAINTS, STOCHASTIC CASE:

$$v(t,x) = \inf_{u\in\mathcal{U}} \mathbb{E}\big[\psi(X^u_{t,x}(T))\big]$$

Then the DPP reads

$$v(t,x) = \inf_{u\in\mathcal{U}} \mathbb{E}\big[v(t+h, X^u_{t,x}(t+h))\big]$$

and the HJB equation (PDE) is

$$-v_t + \max_{u\in U}\big(-b(t,x,u)\cdot\nabla v - \frac{1}{2}Tr(\sigma\sigma^T D^2 v)\big) = 0 \quad (4)$$

$$v(T,x) = \psi(x). \quad (5)$$

- Characterisation of viability kernels:
  Aubin- Da Prato (90,98), ...
  Buckdahn, Peng, Quincampoix, Rainer (98)
  Quincampoix-Rainer (05)
- Stochastic target problems
  Soner-Touzi (00), (02), (09)
  Bouchard-Elie-Imbert (10)

$\Rightarrow$ **may need compatibility conditions like**

- $\forall x,\ \forall p \neq 0,\ \exists a \in U,\ \langle \sigma(x,a), p\rangle = 0$
- invertibility condition on the matrix σ
- or other feasibility conditions ...

Outline of the talk

1. Deterministic case
   - Step 1: link with reachability problem
   - Step 2: level set approach

2. Stochastic case
   - Step 1: reachability problem using unbounded controls
   - Step 2: level set approach
   - Step 3: HJB equation and uniqueness result

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Deterministic case

Let us consider the following dynamics

$$\begin{cases} \dot{X}(s) = b(s, X(s), u(s)), & s \in [t, T] \\ X(t) = x \end{cases} \tag{6}$$

where

$$u \in \mathcal{U} := \left\{ u : [0, T] \to U, \text{ measurable function} \right\}.$$

The STATE-CONSTRAINED OCP is:

$$\begin{aligned} \upsilon(t,x) &= \inf_{u \in \mathcal{U}} \Big\{ \psi(X^u_{t,x}(T)) + \int_t^T \ell(s, X^u_{t,x}(s), u(s)) ds : \\ &\qquad X^u_{t,x}(s) \in \mathcal{K}, \forall s \in [t, T] \Big\} \end{aligned}$$

Deterministic case

Assumptions:

(D1) $U \subseteq \mathbb{R}^m$ is a compact set;

(D2) $b : [0, T] \times \mathbb{R}^d \times U \to \mathbb{R}^d$ continuous function, Lipschitz continuous w.r.t. x (unif. in t and u)

$\rightsquigarrow$ $X^u_{t,x}(\cdot)$ unique solution of (6).

(D3) $\ell : [0, T] \times \mathbb{R}^d \times U \to \mathbb{R}, \psi : \mathbb{R}^d \to \mathbb{R}$ continuous functions, Lipschitz continuous w.r.t. x unif. in t and u.

(D4) $\forall (t, x) \in [0, T] \times \mathbb{R}^d$, $(b, \ell)(t, x, U)$ is convex (or weaker assumption for existence of a minima)

Let us introduce

$$Z^u_{t,x,z}(\theta) := z - \int_t^\theta \ell(s, X^u_{t,x}(s), u(s))ds.$$

Then,

$$\begin{aligned} v(t,x) - z &= \inf_{u\in\mathcal{U}} \left\{ \psi(X^u_{t,x}(T)) + \int_t^T \ell(\cdot) - z, \quad X^u(\cdot) \subset \mathcal{K} \right\} \\ &= \inf_{u\in\mathcal{U}} \left\{ \psi(X^u_{t,x}(T)) - Z^u_{t,x,z}(T), \quad X^u(\cdot) \subset \mathcal{K} \right\} \end{aligned}$$

So $v(t,x) \le z \Leftrightarrow (x,z) \in \mathcal{R}^{\psi,\mathcal{K}}_t$, where

$$\begin{aligned} \mathcal{R}^{\psi,\mathcal{K}}_t \quad := \quad & \Big\{ (x,z) \in \mathbb{R}^{d+1} : \exists u \in \mathcal{U} \text{ such that} \\ & (X^u_{t,x}(T), Z^u_{t,x,z}(T)) \in \mathrm{epi}(\psi)^1 \text{ and } X^u_{t,x}(\cdot) \subset \mathcal{K} \Big\}. \end{aligned}$$

"state-constrained backward reachable set"

[1] $\mathrm{epi}(\psi) := \{(x,z) \in \mathbb{R}^{d+1} : \psi(x) \le z\}$

By using $(D4)$ and compactness of the set of trajectories, one has

$$v(t,x) = \inf \left\{ z \in \mathbb{R} : (x,z) \in \mathcal{R}_t^{\psi,\mathcal{K}} \right\}. \tag{7}$$

In order to characterize $\mathcal{R}_t^{\psi,\mathcal{K}}$ the level set approach will be applied.

Idea of the level set approach

Step 1. Interpret the set $\mathcal{R}_t^{\psi,\mathcal{K}}$ as the level set (negative region) of a continuous function.
Step 2. HJB approach

Main advantages:

- Take into account state constraints without any further assumption;
- Availability of many numerical methods for solving PDEs.

Let us introduce LEVEL SET FUNCTIONS $g_\psi : \mathbb{R}^{d+1} \to \mathbb{R}$ and $g_\mathcal{K} : \mathbb{R}^d \to \mathbb{R}$, Lipschitz continuous, such that:

$$g_\psi(x,z) \le 0 \quad \Leftrightarrow \quad (x,z) \in \mathrm{epi}(\psi),$$
$$g_\mathcal{K}(x) \le 0 \quad \Leftrightarrow \quad x \in \mathcal{K}.$$

Theorem (TWO AUXILIARY UNCONSTRAINED OCPs)

$$\text{OCP1:} \quad w_1(t,x,z) := \inf_{u \in \mathcal{U}} \underbrace{g_\psi(X^u_{t,x}(T), Z^u_{t,x,z}(T))}_{\equiv \psi(X^u_{t,x}(T)) - Z^u_{t,x,z}(T)} \bigvee \max_{s \in (t,T)} \underbrace{g_\mathcal{K}(X^u_{t,x}(s))}_{\equiv d_S(X^u_{t,x}(s),\mathcal{K})}$$

$$\text{OCP2:} \quad w_2(t,x,z) := \inf_{u \in \mathcal{U}} \underbrace{g_\psi(X^u_{t,x}(T), Z^u_{t,x,z}(T))}_{\equiv \max(\psi(X^u_{t,x}(T) - Z^u_{t,x,z}(T), 0)} + \int_t^T \underbrace{g_\mathcal{K}(X^u_{t,x}(s))}_{\equiv d(X^u_{t,x}(s),\mathcal{K})} ds$$

Then $w_1(t,x,z) \le 0 \ \Leftrightarrow w_2(t,x,z) = 0 \Leftrightarrow (x,z) \in \mathcal{R}^{\psi,\mathcal{K}}_t$.

Remarks - Reachability with state constraints: B. - Forcadel - Zidani (10')
- OCP1 approach : Altarovici-B.-Zidani (ESAIM-COCV '13)
- OCP1 and OCP2 are numericaly compared in B.-Cheng-Shu ('13)

Proposition

Let assumptions (D1)-(D4) be satisfied and $k = 1$ or $k = 2$. Then
(i)

$$\mathcal{R}_t^{\psi,\mathcal{K}} = \Big\{(x,z) \in \mathbb{R}^{d+1} : w_k(t,x,z) \le 0\Big\}$$

(ii) w_1 (resp w_2) is the unique continuous viscosity solution of the following HJB equation:

$$\begin{cases} \min\Big(-w_t + \sup_{a\in U}\big(-b(t,x,a)D_x w + \ell(t,x,a)\partial_z w\big),\ w - g_{\mathcal{K}}(x)\Big) = 0 \\ w(T,x,z) = g_\psi(x,z) & \mathbb{R}^{d+1} \end{cases}$$

resp.:

$$\begin{cases} -\partial_t w + \sup_{a\in U}\big(-b(t,x,a)D_x w + \ell(t,x,a)\partial_z w - g_{\mathcal{K}}(x)\big) = 0 & [0,T)\times\mathbb{R}^{d+1} \\ w(T,x,z) = g_\psi(x,z) & \mathbb{R}^{d+1} \end{cases}$$

Comparison of OCP1 and OCP2 (B.-Cheng-Shu '13)

- Obstacle: $$\boxed{v(t,x) := \min_{u\in\mathcal{U}} \; d^S_{\mathcal{C}}(X^u_{0,x}(t)) \bigvee \max_{s\in[0,t]} d^S_{\mathcal{K}}(X^\alpha_{0,x}(s))}$$

$$\min\Big(v_t + \max\left(0, 2\pi(-x_2, x_1)\cdot\nabla v\right),\; v - d^S_{\mathcal{K}}(x)\Big) = 0$$
$$v(0,x) = d^S_{\mathcal{C}}(x)$$

- Penalization : $$\boxed{v(t,x) := \min_{u\in\mathcal{U}} \; d_{\mathcal{C}}(X^u_{0,x}(t)) + \int_0^t d_{\mathcal{K}}(X^u_{0,x}(s))ds}$$

$$v_t + \max_\alpha\left(0, 2\pi(-x_2, x_1)\cdot\nabla v\right) - d_{\mathcal{K}}(x) = 0$$
$$v(0,x) = d_{\mathcal{C}}(x)$$

Rem 1: OCP1 (sup. costs) have already been considered by Barron-Ishii ('89), . . .
Rem 2: OCP2 is also introduced in **Kurzanski-Varaiya ('06)** for reachability problems

OCP1/obstacle (up) vs OCP2/penalization (down)

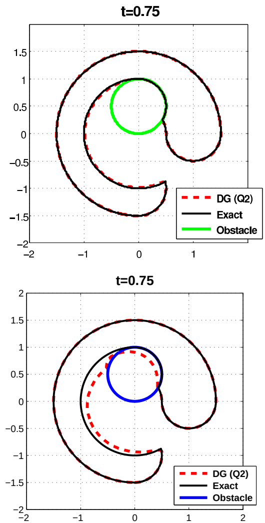

Figure: DG scheme of B.-Cheng-Shu 2011, plots at time $t = 0.75$, with Q^2 elements and 80×80 mesh cells.

The approach OCP1 is very efficient and is currently applied to various problem of average dimension (4-6):
- UAV
- space launching problem
- vehicle collision avoidance

REFS:

- Crück-Désilles-Zidani (Collision analysis for a UAV) AIAA Guidance, Nav. and Cont. conf. 2012
- B.-Bourgeois-Désilles-Zidani, Preprint (CNES climbing problem)
- Xausa-Baier-B.-Gerdts, SIAM Conference CT13, Preprint 2014 (Applications of Reachable Sets to Driver Assistance Systems)

Stochastic case

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$$\left\{ \begin{array}{l} dX(s) = b(s, X(s), u(s))ds + \sigma(s, X(s), u(s))dB_s \quad s \in [t, T] \\ X(t) = x \end{array} \right. \tag{8}$$

where

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The STATE-CONSTRAINED OCP is:

$$v(t,x) \quad = \quad \inf_{u\in\mathcal{U}} \Big\{ \mathbb{E}\Big[\psi(X^u_{t,x}(T)) + \int_t^T \ell(s, X^u_{t,x}(s), u(s))ds \Big] : X^u_{t,x}(\cdot) \subset \mathcal{K} \text{ a.s.} \Big\}$$

Stochastic case

Assumptions:

(S1) $u(\cdot) \in U$ a.s., with $U \subseteq \mathbb{R}^m$ compact set;

(S2) $b, \sigma : [0, T] \times \mathbb{R}^d \times U \to \mathbb{R}^d, \mathbb{R}^{d \times p}$ are continuous functions, Lipschitz continuous w.r.t. x (unif. in t and u).

$\rightsquigarrow$ $X^u_{t,x}(\cdot)$ unique strong solution of (8).

(S3) $\psi : \mathbb{R}^d \to \mathbb{R}$, $\ell : [0, T] \times \mathbb{R}^d \times U \to \mathbb{R}$ continuous functions, Lipschitz continuous w.r.t. x (unif. in t and u).

(S4) ψ and ℓ are bounded from below.

Link between the OCP and a reachability problem :

$$v(t,x) = \inf\Big\{ z \; : \; \exists u,\ \mathbb{E}\Big[\psi(X^u_{t,x}(T)) + \int_t^T \ell(s, X^u_{t,x}(s), u(s))ds\Big] \le z \text{ and } X^u_{t,x}(s) \in \mathcal{K}, \forall s \in [t,T] \text{ a.s.}\Big\}$$

How to link this problem with a reachability one?
Of course,

$$z \geq \mathbb{E}\left[\psi(X^u_{t,x}(T))\right] \quad \text{DO NOT IMPLY} \quad z \geq \psi(X^u_{t,x}(T)) \text{ a.s.}$$

Thanks to Ito's representation theorem this is OK up to a martingale: $\exists \alpha \in \mathcal{A} := L^2_{\mathbb{F}}$ ($\mathbb{R}^p$-valued prog. meas. process):

$$\psi(X^u_{t,x}(T)) = \mathbb{E}[\psi(X^u_{t,x}(T))] + \int_t^T \alpha_s \cdot dB_s, \quad \text{a.s.}$$

A first equivalence

So $\quad z \geq \mathbb{E}\left[\psi(X^u_{t,x}(T))\right]$

$$\iff \quad \exists \alpha \in \mathcal{A},\ z \geq \psi(X^u_{t,x}(T)) - \int_t^T \alpha_s \cdot dB_s, \quad \text{a.s.}$$

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$$\Longleftrightarrow \quad \exists \alpha \in \mathcal{A},\ z \geq \psi(X^u_{t,x}(T)) - \int_t^T \alpha_s \cdot dB_s, \quad \text{a.s.}$$

Let

$$Z^{u,\alpha}_{t,x,z}(\cdot) := z - \int_t^{\cdot} \ell(s, X^u_{t,x}(s), u(s))ds + \int_t^{\cdot} \alpha_s \cdot dB_s. \qquad (9)$$

In particular,

$$z \geq \mathbb{E}\left[\psi(X^u_{t,x}(T)) + \int_t^T \ell(s, X^u_{t,x}(s), u(s))ds\right]$$
$$\iff \quad \exists \alpha \in \mathcal{A}, \quad Z^{\alpha,u}_{t,x,z}(T) \geq \psi(X^u_{t,x}(T)) \text{ a.s.}$$

$\Rightarrow$ We consider the following stochastic state-constrained backward reachable set

$$\mathcal{R}_t^{\psi,\mathcal{K}} := \Big\{(x,z) \in \mathbb{R}^{d+1} : \exists (u,\alpha) \in \mathcal{U} \times \mathcal{A} \text{ such that}$$
$$(X_{t,x}^u(T), Z_{t,x,z}^{u,\alpha}(T)) \in \text{epi}(\psi) \text{ and } X_{t,x}^u(s) \in \mathcal{K}, \forall s \in [t,T] \text{ a.s.}\Big\}.$$

Then

$$v(t,x) = \inf\Big\{z \in \mathbb{R} : (x,z) \in \mathcal{R}_t^{\psi,\mathcal{K}}\Big\}.$$

Let us use again the level set approach. We use the OCP2 approach, with non-negative level set functions.

Let

$$g_\psi(x,z) := \max(\psi(x) - z, 0) \quad \text{and} \quad g_\mathcal{K}(x) := d(x, \mathcal{K}).$$

In particular:

$$g_\psi, g_\mathcal{K} \geq 0 \quad \text{and} \quad g_\psi(x,z) = 0 \Leftrightarrow (x,z) \in \text{epi}(\psi), \quad g_\mathcal{K}(x) = 0 \Leftrightarrow x \in \mathcal{K}.$$

AUXILIARY UNCONSTRAINED OCP:

$$w(t,x,z) = \inf_{(u,\alpha)\in\mathcal{U}\times\mathcal{A}} \mathbb{E}\Big[\underbrace{g_\psi(X^u_{t,x}(T), Z^{u,\alpha}_{t,x,z}(T))}_{\equiv \max(\psi(X^u_{t,x}(T)) - Z^{u,\alpha}_{t,x,z}(T),\ 0)} + \int_t^T \underbrace{g_\mathcal{K}(X^u_{t,x}(s))}_{\equiv d_\mathcal{K}(X^u_{t,x}(s))} ds \Big]$$

Proposition

Assume (S1)-(S3) and (I): the infimum in w is reached. Then

$$\mathcal{R}_t^{\psi,\mathcal{K}} = \left\{(x,z) \in \mathbb{R}^{d+1} : w(t,x,z) = 0\right\}$$

Application:

$$\upsilon(t,x) = \inf \left\{ z \in \mathbb{R} : w(t,x,z) = 0 \right\}.$$

• (I) can be realized under some convexity assumption on $(b, \ell, \sigma\sigma^T)$. It is also the case when b, σ depends linearly upon x and u, for $\mathcal{K}$ convex set ($\Rightarrow d_K$ convex) and Ψ, ℓ convex in (x,u).

• But α_s is unbounded ...

Some References (unbounded controls)

- Lasry - Lions (Cras '00) : "A new class of sing. sto. cont. pbs." ($b(x,u) = b_1(x) + b_2(x)u$).
- Pham (Prob. Surveys '05) : HJB equation for unbounded controls
- Brüder (Preprint HAL '05) : comparison principle for a super-replication problem in Finance
- B. - Brüder-Maroso-Zidani (SINUM '09) : convergence of SL scheme for a super-replication problem
- Debrabant-Jakobsen (Math. Comp. '13) : high order SL schemes

Recall that

$$\begin{pmatrix} dX_s \\ dZ_s \end{pmatrix} = \begin{pmatrix} b(X_s) \\ -\ell(X_s, u_s) \end{pmatrix} + \begin{pmatrix} \sigma \\ \alpha \end{pmatrix} dW_s$$

and

$$w(t,x,z) = \inf_{(u,\alpha)\in\mathcal{U}\times\mathcal{A}} \mathbb{E}\left[\max(\psi(X^u_{t,x}(T)) - Z^{u,\alpha}_{t,x,z}(T),\ 0) + \int_t^T d_{\mathcal{K}}(X^u_{t,x}(s)) \right]$$

The HJB equation associated to the AUXILIARY OCP would be, in the case $p = d = 1$:

$$\underbrace{\sup_{\substack{u\in U \\ \alpha\in\mathbb{R}}} \left\{ -\partial_t w + -b\partial_x w + \ell\partial_z w - \frac{1}{2}\sigma^2\partial_{xx} w - \alpha\sigma\partial_{xz} w - \frac{1}{2}\alpha^2\partial_{zz} w - g_{\mathcal{K}} \right\} = 0}_{=:H(t,(x,z),w_t,Dw,D^2w)}$$

$\Rightarrow$ Because of unbounded controls, H can be unbounded !

Notice that

$$\sup_{\alpha\in\mathbb{R}}\left(A - 2B\alpha + C\alpha^2\right) = 0 \quad\Leftrightarrow\quad A \le 0,\ AC \le B^2$$

Better is

$$\sup_{\alpha\in\mathbb{R}} A - 2B\alpha + C\alpha^2 = 0$$

$$\Leftrightarrow \sup_{\alpha_0,\alpha_1\in\mathbb{R}^2,\ \alpha_0\neq 0} A - 2B\frac{\alpha_1}{\alpha_0} + C(\frac{\alpha_1}{\alpha_0})^2 = 0$$

$$\Leftrightarrow \sup_{\alpha_0,\alpha_1\in\mathbb{R}^2} A\alpha_0^2 - 2B\alpha_0\alpha_1 + C\alpha_1^2 = 0$$

$$\Leftrightarrow \sup_{(\alpha_0,\alpha_1)\in S^1} \begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix}^T \begin{pmatrix}A & -B\\ -B & C\end{pmatrix}\begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix} = 0$$

$$\Leftrightarrow \Lambda_+\begin{pmatrix}A & -B\\ -B & C\end{pmatrix} = 0$$

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$$
\begin{aligned}
&\sup_{\alpha\in\mathbb{R}} A - 2B\alpha + C\alpha^2 = 0 \\
&\qquad \Leftrightarrow \quad \sup_{\alpha_0,\alpha_1\in\mathbb{R}^2,\ \alpha_0\neq 0} A - 2B\frac{\alpha_1}{\alpha_0} + C(\frac{\alpha_1}{\alpha_0})^2 = 0 \\
&\qquad \Leftrightarrow \quad \sup_{\alpha_0,\alpha_1\in\mathbb{R}^2} A\alpha_0^2 - 2B\alpha_0\alpha_1 + C\alpha_1^2 = 0 \\
&\qquad \Leftrightarrow \quad \sup_{(\alpha_0,\alpha_1)\in S^1} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}^T \begin{pmatrix} A & -B \\ -B & C \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = 0 \\
&\qquad \Leftrightarrow \quad \Lambda_+ \begin{pmatrix} A & -B \\ -B & C \end{pmatrix} = 0
\end{aligned}
$$

In the same way, if $B = (B_1, \ldots, B_p)^T \in \mathbb{R}^p$:

$$\sup_{\alpha \in \mathbb{R}^p} A - 2\langle B, \alpha \rangle + C\|\alpha\|^2 = 0$$

$$\Leftrightarrow \quad \Lambda_+ \begin{pmatrix} A & -B_1 & \ldots & -B_p \\ -B_1 & C & & 0 \\ \vdots & & \ddots & \vdots \\ & & & \\ -B_p & 0 & \ldots & C \end{pmatrix} = 0$$

Let us define the matrix of $\mathbb{R}^{(p+1)\times(p+1)}$:

$$\mathcal{L}^u(t,(x,z),Dw,D^2w) := \begin{pmatrix} A & -B_1 & \dots & -B_p \\ -B_1 & C & & 0 \\ \vdots & & \ddots & \vdots \\ -B_p & 0 & \dots & C \end{pmatrix}$$

where

$$\frac{A}{2} := -\partial_t w - b\cdot D_x w + \ell\,\partial_z w - \frac{1}{2}Tr(\sigma\sigma^T D_x^2 w) - g_{\mathcal{K}}.$$

$$B := \sigma^T D_x \partial_z w = (B_1,\dots,B_p)^T, \quad \text{and} \quad C := -\partial_{zz} w$$

By elementary calculus (for regular φ):

$$H(t,(x,z),w_t,D\varphi,D^2\varphi) = 0$$

$$\Leftrightarrow \sup_{u\in U} \Lambda^+\Big(\mathcal{L}^u(t,(x,z),D\varphi,D^2\varphi)\Big) = 0.$$

Theorem

Let assumptions (S1)-(S3) be satisfied. Then w is a viscosity solution of the following generalized HJB equation

$$\begin{cases} \sup_{u\in U} \Lambda^{+}\Big(\mathcal{L}^u(t,(x,z),Dw,D^2w)\Big) = 0, & t < T, (x,z) \in \mathbb{R}^2 \\ w(T,x,z) = g_{\psi}(x,z) & \mathbb{R}^2 \end{cases}$$

in the class of continuous function with linear growth at infinity ($|w(t,x,z)| \leq C(1+|x|+|z|)$)

Uniqueness ? With $z \in \mathbb{R}$, difficulties to get uniqueness ...

Stochastic case

Step 3: HJB equation and uniqueness result

Boundary conditions for $z \leq 0$:

To simplify, we can assume, instead of $(S4)$:

$$(S4)^* \qquad \psi \geq 0 \quad \text{and} \quad \ell \geq 0.$$

Let w_0 be the value of the following unconstrained problem:

$$w_0(t,x) := \inf_{u \in \mathcal{U}} \mathbb{E}\left[\psi(X^u_{t,x}(T)) + \int_t^T \ell(s, X^u_{t,x}(s), u(s))ds + \int_t^T g_{\mathcal{K}}(X^u_{t,x}(s)ds)\right].$$

Proposition (A Dirichlet boundary condition, under $(S4^*)$)

For any $z \leq 0$, $\quad w(t,x,z) \equiv w_0(t,x) - z.$

Proof: For any $z \in \mathbb{R}$,

$$\begin{aligned} w(t,x,z) & \\ &\geq \inf_{(u,\alpha)\in\mathcal{U}\times\mathcal{A}} \mathbb{E}\left[\psi(X^u_{t,x}(T)) - z + \int_t^T \ell(\cdot)ds - \underbrace{\int_t^T \alpha_s dB_s}_{\mathbb{E}(.)=0} + \int_t^T g_{\mathcal{K}}(X^u_{t,x}(s))ds\right] \\ &\geq w_0(t,x) - z \end{aligned}$$

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Proof (continued): On the other hand, for $z \leq 0$, choosing the particular control $\alpha_s :\equiv 0$:

$$w(t,x,z)$$

$$\leq \inf_{u \in \mathcal{U}} \mathbb{E}\Big[\max(\underbrace{\psi(X^u_{t,x}(T)) - z + \int_t^T \ell(\cdot)ds,}_{\geq 0}\ 0) + \int_t^T g_{\mathcal{K}}(.)ds \Big]$$

$$\leq w_0(t,x) - z$$

Theorem (Comparison principle, Dirichlet case (B.-Picarelli-Zidani))

Assume $(S1)-(S4^*)$. A weak comparison principle holds for (lower USC/ upper LSC) solutions of

$$\sup_{u\in U} \Lambda^+\Big(\mathcal{L}^u(t,(x,z),Dw,D^2w)\Big) = 0, \quad t<T,\ x\in\mathbb{R}^d,\ z\geq 0,$$

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$$w(t,x,0) = w_0(t,x), \quad t<T, x\in\mathbb{R}^d$$

w linearly bounded in x ($|w|\leq C(1+|x|)$)

Idea of Proof: Starts with a rescaling of the eigenvalue HJ equation, in order to construct a strict subsolution (Bruder). Then, new estimates to deal with $\mathbb{R}^p$-valued α controls instead of $\mathbb{R}$-valued controls combined with the Ishii Lemma.

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Conclusion and further work

Conclusion:

- state-constrained OCP can be recasted into a state-constrained reachability problem, adding
  - a state variable
  - an $\mathbb{R}^p$-valued unbounded control
- the state-constrained reachability problem can be modelized by a level set approach and an auxiliary unconstrained OCP
- The valule of this OCP is characterized as the unique solution of an HJB equation.

Further work:

- Numerical schemes
- Applications

Thanks for your attention!!

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