From discrete microscopic models to macroscopic models and applications to traffic flow.

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Plan

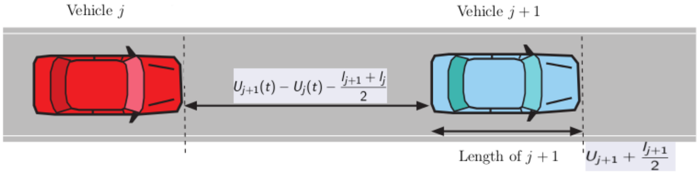
1. Motivations
2. Homogenization result
3. Idea of the proof

Plan

Microscopic traffic flow model

- Discrete model of traffic :

$$\dot{U}_j(t) = V\left(U_{j+1}(t) - U_j(t) - \frac{l_{j+1} + l_j}{2}\right). \quad (1)$$



- U_j: position of the vehicle j.
- V: Optimal velocity function (OVF) of the driver.

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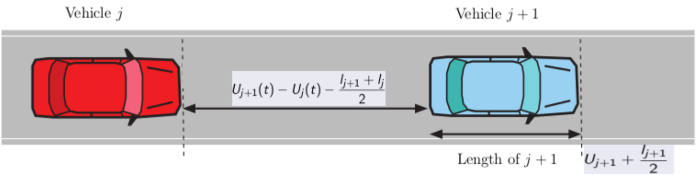

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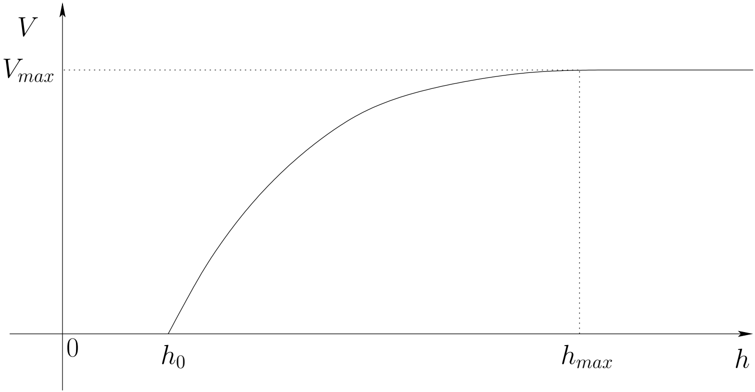
Optimal velocity function



Passing from micro to macro

- Goal : Describe the traffic in term of density of vehicles, i.e. passing from the microscopic model to a macroscopic one.
- LWR macroscopic model:

$$\rho_t + (\rho v(\rho))_x = 0$$

where v is the average speed of vehicles.

Some existing results

- 1 single road, first order model: [NF, Imbert, Monneau]
- 1 single road, second order model, different type of drivers: [NF, Salazar]
- Perturbation at macroscopic level: [Galise, Imbert, Monneau]

A model with a perturbation

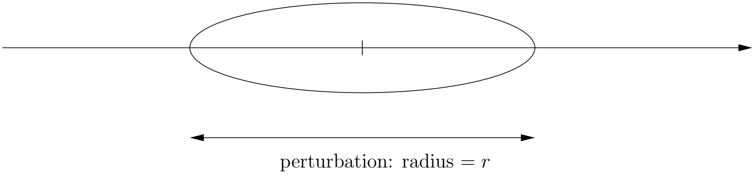
$$\dot{U}_j(t) = V\Big(U_{j+1}(t) - U_j(t)\Big)\phi(U_j(t)). \tag{2}$$

with

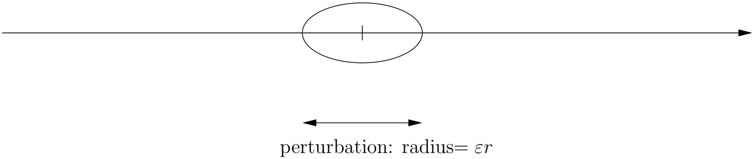
$$\phi(x) = \begin{cases} 1 & \text{if } x \in \mathbb{R}\backslash B(0,r) \\ \mu(x) & \text{if } x \in B(0,r), \end{cases}$$

and $\phi(x) \geq 0$.

A model with a perturbation



Rescalling



Passing to the limit : a model with junction

Some references: [Achou, Camilli, Cutri, Tchou], [Imbert, Monneau, Zidani], [Imbert, Monneau],....

Problem of junction : [Imbert, Monneau]

Given $H : \mathbb{R} \to \mathbb{R}$ decreasing on $]-\infty, p_0]$ and increasing on $[p_0, +\infty[$, $A \in \mathbb{R}$ and $F_A : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, we consider the problem

$$\begin{cases} u_t + H(u_x) = 0 & \text{on} \quad (0, +\infty) \times \mathbb{R}\backslash\{0\} \\ u_t + F_A(u_x(t, 0^-), u_x(t, 0^+)) = 0 & \text{on} \quad (0, +\infty) \times \{0\} \end{cases} \quad (3)$$

with

$$F_A(p^-, p^+) = \max(A, H^+(p^-), H^-(p^+)).$$

Problem of junction : [Imbert, Monneau]

Definition (Definition of the solution on the junction)

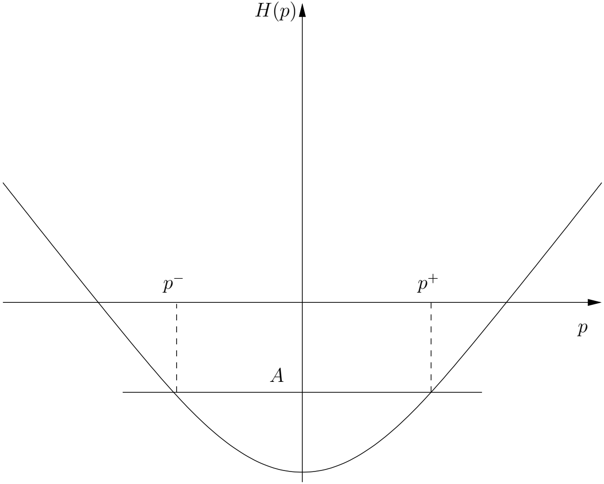
We denote $J := (0, +\infty) \times \mathbb{R}$, $J^+ := (0, +\infty) \times (0, +\infty)$ and $J^- := (0, +\infty) \times (-\infty, 0)$ and

$$\mathcal{C}^2(J) = \left\{\varphi \in C(J), \text{ the restriction of } \varphi \text{ to } J^+ \text{ and to } J^- \text{ are } C^2\right\}.$$

An usc (resp. lsc) function $u : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ is a viscosity sub-solution (resp. super-solution) of (3) if for all $(t, x) \in J$ and for all $\varphi \in \mathcal{C}^2(J)$ such that $u - \varphi$ reaches a local maximum (resp. minimum) at (t, x), we have

$$\begin{aligned} &\varphi_t(t,x) + H(\varphi_x(t,x)) \le 0 \quad \text{(resp. } \ge 0) && \text{if } x \neq 0, \\ &\varphi_t(t,x) + F_A(\varphi_x(t,0^-), \varphi_x(t,0^+)) \le 0 \quad \text{(resp. } \ge 0) && \text{if } x = 0. \end{aligned} \tag{4}$$

Another definition at the junction



Problem of junction : [Imbert, Monneau]

Proposition (Equivalent definition of the solution at the junction)

In the previous definition, if $x = 0$, we get an equivalent definition with test functions φ satisfying

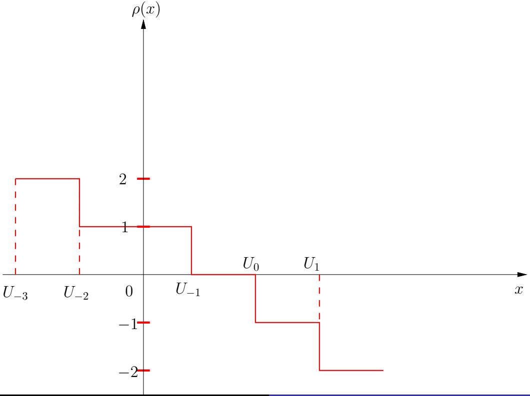
$$\varphi(t,x) = \psi(t) + p^- x 1_{\{x \le 0\}} + p^+ x 1_{\{x \ge 0\}},$$

with $\psi \in C^1(0,+\infty)$.

Plan

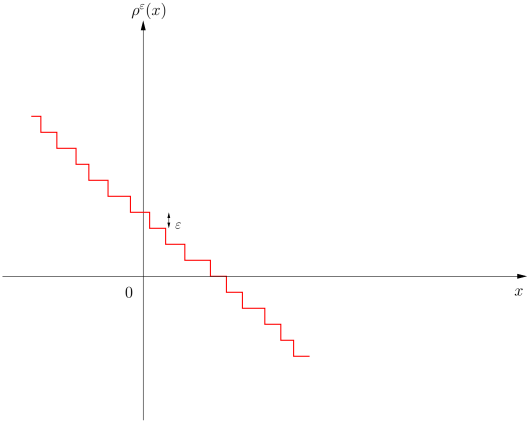
Injecting the system of ODE in a PDE

$$\rho(t,y) = -\left(\sum_{i\geq 0} H\left(y - U_i(t)\right) + \sum_{i<0}\left(-1 + H\left(y - U_i(t)\right)\right)\right)$$



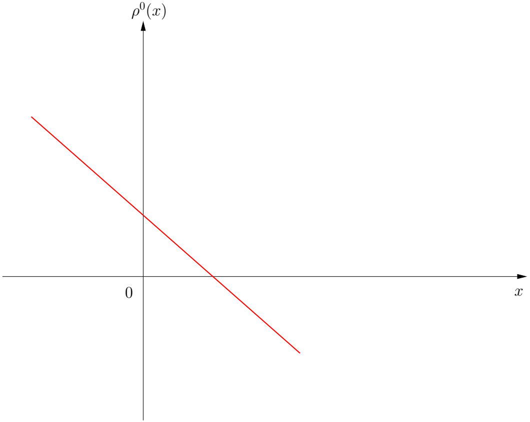
Rescalling

$$\rho^{\varepsilon}(t,y) = \varepsilon\rho\left(t/\varepsilon, x/\varepsilon\right)$$



Passing to the limit

$$\rho^{\varepsilon} \to \rho^{0}$$



Convergence result

Theorem (NF, Salazar)

Assume that

$$U_i(0) + h_0 \leq U_{i+1}(0).$$

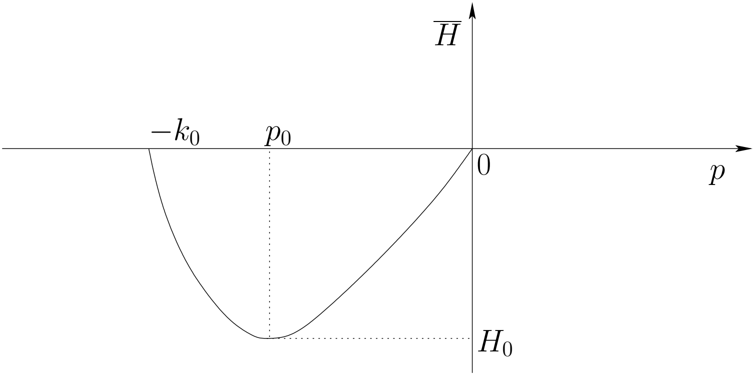
Then, there exists $\overline{A}$ *and* $\overline{H}$ *such that* $\rho^\varepsilon \to u^0$ *with* u^0 *solution of*

$$\begin{cases} u_t^0 + \overline{H}(u_x^0) = 0 & \text{for } (t,x) \in (0,+\infty) \times \mathbb{R}\backslash\{0\} \\ u_t^0 + F_{\overline{A}}\left(u_x^0(t,0^-), u_x^0(t,0^+)\right) = 0 & \text{for } (t,x) \in (0,+\infty) \times \{0\} \\ u^0(0,x) = u_0(x) & \text{for } x \in \mathbb{R}, \end{cases}$$

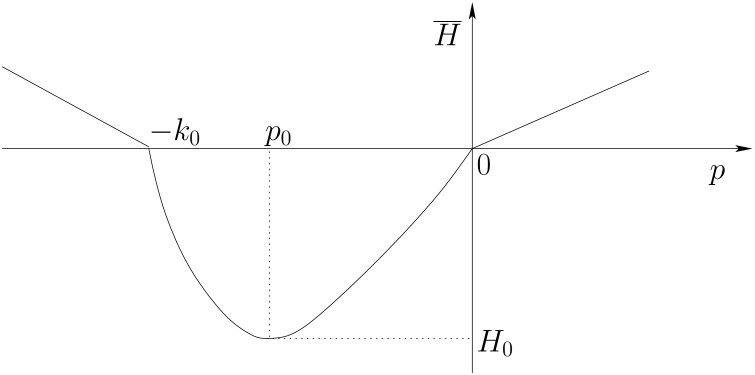
Moreover $-1/h_0 =: -k_0 \leq u_x^0 \leq 0$ *and for* $p \in [-k_0, 0]$, *we have*

$$\overline{H}(p) = -V\left(\frac{-1}{p}\right)|p|.$$

Effective hamiltonian



Extended effectif hamiltonian



Plan

Injection of the system of ODE in a PDE

The function ρ^ε satisfies

$$\begin{cases} u_t^\varepsilon + M^\varepsilon \left[\dfrac{u^\varepsilon(t,.)}{\varepsilon}\right](x).\phi\left(\dfrac{x}{\varepsilon}\right).|u_x^\varepsilon| = 0 \\ \\ u^\varepsilon(0,x) = u_0(x). \end{cases}$$

where M^ε is a non-local operator defined by

$$M^\varepsilon[U](x) = \int_{-\infty}^{+\infty} J(z) E\left(U(x+\varepsilon z) - U(x)\right) dz - \frac{3}{2} V_{max},$$

and with

$$E(z) = \begin{cases} 0 & \text{if } z > 0 \\ \dfrac{1}{2} & \text{if } -1 < z \leq 0 \\ \dfrac{3}{2} & \text{if } z \leq -1, \end{cases} \quad \text{and} \quad J = V' \text{on } \mathbb{R}.$$

Proof of convergence

We want to show that $\overline{\rho} = \limsup^* \rho^\varepsilon$ is a sub solution of the limit problem. Let φ such that $\overline{\rho} - \varphi$ reaches a maximum at $(\bar{t}, \bar{x})$

- If $\bar{x} \neq 0$ the proof is rather classical (see [NF, Imbert, Monneau])
- If $\bar{x} = 0$, then $\varphi(t,x) = \psi(t) + p^- x 1_{\{x \leq 0\}} + p^+ x 1_{\{x \geq 0\}}$.
  We set
  $$\varphi^\varepsilon = \psi(t) + w^\varepsilon(x)$$
  with $w^\varepsilon(x) = \varepsilon w\left(\frac{x}{\varepsilon}\right)$ and w solution of
  $$M[w](x).\phi(x).|w_x| = \lambda \quad \text{for } x \in \mathbb{R}$$
  such that $w^\varepsilon \to p^- x 1_{\{x \leq 0\}} + p^+ x 1_{\{x \geq 0\}}$.
  Classically, φ^ε is a super-solution of the same problem as ρ^ε and we get the result using the comparison principle.

Difficulty

How to construct w solution of

$$M[w](x).\phi(x).|w_x| = \lambda \quad \text{for } x \in \mathbb{R}$$

such that

$$w^\varepsilon \to p^- x 1_{\{x\leq 0\}} + p^+ x 1_{\{x \geq 0\}}.$$

Truncated cell problem

- Idea of [Achdou, Tchou] and [Galise, Imbert, Monneau]: construct a corrector on a bounded domain with appropriate boundary condition and pass to the limit.

Truncated cell problem

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- For $r \leq R << l$, we consider the truncated cell problem

$$\begin{cases} G_R\left(x, [w^{l,R}], w_x^{l,R}\right) = \lambda_{l,R} & \text{if } x \in (-l, l) \\ \overline{H}^-(w_x^{l,R}) = \lambda_{l,R} & \text{if } x \in \{-l\} \\ \overline{H}^+(w_x^{l,R}) = \lambda_{l,R} & \text{if } \in \{l\}, \end{cases}$$

with

$$G_R(x, U, q) = \psi_R(x).\phi(x).M[U](x).|q| + (1 - \psi_R(x)).\overline{H}(q),$$

and $\psi_R \in C^\infty(\mathbb{R}, [0, 1])$, such that

$$\psi_R \equiv \begin{cases} 1 & \text{on } [-R, R] \\ 0 & \text{outside } [-R-1, R+1], \end{cases}$$

Approximated truncated cell problem

- For $\delta > 0$, we consider

$$\begin{cases} \delta v^\delta + G_R\left(x, [w^{l,R}], w_x^{l,R}\right) = 0 & \text{for } x \in (-l, l) \\ \delta v^\delta + \overline{H}^-(v_x^\delta) = 0 & \text{for } x \in \{-l\} \\ \delta v^\delta + \overline{H}^+(v_x^\delta) = 0 & \text{for } x \in \{l\} \end{cases}$$

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- v^δ is not Lipschitz continuous BUT

$$-k_0(x-y) - 1 \leq v^\delta(x) - v^\delta(y) \leq 0 \quad \text{for } x \geq y.$$

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- This implies that there exists m^δ uniformly Lipschitz continuous such that

$$|v^\delta(x) - m^\delta(x)| \le C \quad \text{for all } x \in [-l, l].$$

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- This allows us to pass to the limit as $\delta \to 0$ (the limit $l \to +\infty$ and $R \to +\infty$ are easier).

Characterization of the effective flux limiter

Theorem

We denote by $\mathcal{S}$ the set of functions w such that there exists a Lipschitz continuous function such that $|w - m| \leq C$. Then

$$\overline{A} = \inf\{\lambda, \textit{ there exists a corrector } w \in \mathcal{S}\}.$$

Moreover

$$0 \geq \bar{A} \geq \min_{p \in \mathbb{R}} \overline{H}(p).$$

Conclusions and Perspectives

- Conclusions :
  - Homogenization results for discrete traffic flow models
  - This allows to model microscopic phenomena.
- Perspectives :
  - Homogenization for second order models
  - Microscopic perturbation depending on time (red light for example)
  - Homogenization on networks
  - Numerical computation of $\overline{A}$
  - Homogenization in random media