Optimal Control with Heterogeneous Agents in Continuous Time

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The aim of this paper

Optimal control problems in which there is a continuum of ex-ante identical agents

- Individual state follow controllable stochastic (Itô) processes
- Continuous time $\rightarrow$ the state distribution is the solution of the Kolmogorov forward (KF) equation

$\rightarrow$ **Main result**: the solution is characterized by a system of 2 coupled PDEs: a Hamilton-Jacobi-Bellman (HJB) equation and a KF.

We provide a new numerical algorithm to solve it.

Contribution to the Literature

1. A characterization of the planner's problem as a system of coupled PDEs: HJB + KF
   1.1 Mean-field games in math: Lasry and Lions (2007), Guéant, Lasry and Lions (2011) or Carmona, Delarue and Lachapelle (2013).
   1.2 Optimal control in MFG: Bensoussan, Frehse and Yam (2013), Lasry (2013) and Lucas and Moll (2013).
2. A numerical method to compute it.
   2.1 Here we extend Achdou et al. (2013).

Individual agents

- Continuous-time infinite-horizon economy. There is a continuum of unit mass of agents indexed by $j \in [0, 1]$.
- Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ be a filtered probability space.
- Let B_t^j is a n-dimensional $\mathcal{F}_t$-Brownian motion. The state of an agent j at time t is

$$dX_t^j = b\left(X_t^j, \mu(t, X_t^j), \Gamma_t\right) dt + \sigma\left(X_t^j\right) dB_t^j.$$

Aggregate distribution and aggregate variables

- The dynamics of the distribution of agents $f(t,x)$ are given by the KF equation

$$
\begin{aligned}
\frac{\partial f}{\partial t} &= -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[b_i\left(x,\mu,\Gamma\right)f\right] + \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\left[\sigma_{ii}^2(x)f\right], \\
\int_{\mathbb{R}^n} f(t,x)dx &= 1.
\end{aligned}
$$

- The vector of aggregate variables ("prices") is determined by a system of p equations:

$$
G_k^{-1}\left(\Gamma_k^*(t)\right) = H[f(t,\cdot),\mu^*] = \int_{\mathbb{R}^n} h(x,\mu^*)f(t,x)dx, \quad k = 1,..,p.
$$

Planner's problem

- The planner chooses the controls $\mu(t, X_t^j)$ and the prices $\Gamma_k(t)$ to maximize
$$W[f, \mu] = \int_{\mathbb{R}^n} g(x, \mu) f(t, x) dx,$$
- We have an optimal value **functional** $V[f]$
$$V[f(t, \cdot)] \equiv \sup_{\Gamma,\ \mu \in A,} \int_t^\infty \int_{\mathbb{R}^n} e^{-\rho(s-t)} g(x, \mu) f(t, x) dx ds,$$
where A is the space of all admissible controls contained in the set of all Markov controls.

Marginal social value

- If $g(x, \mu^*)$ is bounded, we may apply the Riesz representation theorem

$$V[f(t,\cdot)] = \int_{\mathbb{R}^n} v(t,x;\mu^*,\Gamma^*) f(t,x) dx,$$

conversely

$$v(t,x) = \frac{\delta V[f(t,\cdot)]}{\delta f(t,x)},$$

that is, $v(t,\cdot;\mu,\Gamma) \in L^\infty(\mathbb{R}^n)$ is the **functional derivative** of the value functional with respect to the state distribution.

Proposition (Necessary conditions)

Assume that a marginal social value v, an optimal admissible Markov control μ^* and an optimal price vector Γ^* exist. Then, they satisfy

$$\begin{aligned}\rho v &= \sup_{\mu\in A} g(x,\mu) + \sum_{k=1}^{p} \lambda_k(t) h(x,\mu) + \frac{\partial v}{\partial t} \\ &+ \sum_{i=1}^{n} b_i\left(x,\mu,\Gamma^*\right)\frac{\partial v}{\partial x_i} + \sum_{i=1}^{n} \frac{\sigma_{ii}^2(x)}{2}\frac{\partial^2 v}{\partial x_i^2},\end{aligned}$$

with $\lambda_k(t) : [0,\infty) \to \mathbb{R}$, $k = 1, .., p$ *:*

$$\lambda_k(t) = -G_k' \int_{\mathbb{R}^n} v(t,y)\left(\sum_{i=1}^{n}\left[\begin{array}{c} \frac{\partial^2 b_i(y,\mu^*,\Gamma^*)}{\partial\Gamma_k\partial x_i} f(t,y) \\ +\sum_{j=1}^{m}\frac{\partial^2 b_i}{\partial\Gamma_k\partial\mu_j}\frac{\partial\mu_j^*}{\partial x_i} f + \frac{\partial b_i}{\partial\Gamma_k}\frac{\partial f}{\partial x_i}\end{array}\right]\right) dy.$$

and $G_k' = G_k'\left(\int_{\mathbb{R}^n} h(x,\mu) f(t,x)dx\right)$, *together with the KF equation, price equations and the transversality condition*

$$\lim_{t\uparrow\infty} e^{-rt} v(t,x) = 0.$$

Example: Constrained efficiency in economics

- **Representative-agent framework** $\longrightarrow$ Benevolent social planner maximizing the utility of the agent.
- **Heterogeneous agents** $\longrightarrow$ Necessary to include some interpersonal utility comparison
    - Typically embodied in a social welfare function (SWF)
- Ex. Davila et al. (2012) $\longrightarrow$ "**Constrained efficiency**"
    - A social planner who maximizes a SWF s.t. the same equilibrium budget constraints and competitive price setting as the individual agents.

Households

Households' utility

$$\mathbb{E}_t\left[\int_t^{\infty} e^{-\rho(s-t)}u(c_t)ds\right].$$

The budget constraint

$$da_t = (wz_t + ra_t - c_t)\,dt,$$

with a borrowing constraint,

$$a_t \geq \bar{a},$$

and productivity

$$dz_t = \eta(z_t)dt + \sigma_z(z_t)dB_t, \quad z_t \in [\underline{z}, \bar{z}] \text{ with } \underline{z} \geq 0.$$

Firms

Production function $Y_t = F(K_t, L_t)$.

Prices:

$$r_t = \frac{\partial}{\partial K} F\left(K_t, 1\right) - \delta, \qquad w_t = \frac{\partial}{\partial L} F\left(K_t, 1\right),$$

State distribution and market clearing

There is a continuum of households of unit mass.

The KF equation is

$$\begin{aligned} \frac{\partial f(t,a,z)}{\partial t} &= -\frac{\partial}{\partial a}\left[(wz+ra-c)\,f\right] - \frac{\partial}{\partial z}\left[\eta(z)f\right] + \frac{1}{2}\frac{\partial^2}{\partial z^2}\left[\sigma_z^2(z)f\right], \\ \int f(t,a,z)dadz &= 1, \end{aligned}$$

and the market clearing

$$K_t = \int af(t,a,z)dadz.$$

Constrained efficiency

The problem of the social planner is to maximize

$$\sup_{w,r,\ c\geq 0} \int_t^\infty e^{-\rho(s-t)} \left[\int u(c) f(s,a,z) dadz \right] ds,$$

subject to the KF equation and to the price equations.

Numerical method

For the stationary problem

Given $\theta \in (0,1)$, begin with an initial guess of the aggregate capital K^0 and the Lagrange multipliers $\lambda_1^0 = \lambda_2^0 = 0$, set $n = m = 0$. Then:

1. Compute $r^n = \frac{\partial}{\partial K} F\left(K^n, 1\right) - \delta$ and $w^n = \frac{\partial}{\partial L} F\left(K^n, 1\right)$.
2. Given r^n and w^n, solve the planner's HJB equation to obtain an estimate of the value function v^n and of the consumption c^n.
3. Given c^n, solve the KF equation and compute the aggregate distribution f^n.
4. Compute the aggregate capital stock $\hat{K}^n = \int a f^n da dz$.
5. Compute $K^{n+1} = \theta K^n + (1-\theta)\hat{K}^n$. If K^{n+1} is close enough to K^n, stop. If not set $n := n+1$ and go to step 1.
6. Compute the Lagrange multipliers $\hat{\lambda}_1^m$ and $\hat{\lambda}_2^m$.
7. Compute $\lambda_i^{m+1} = \theta \lambda_i^m + (1-\theta)\hat{\lambda}_i^m$, $i = 1, 2$. If λ_1^{m+1} and λ_2^{m+1} are close enough to λ_1^m and λ_2^m, stop. If not set $m := m+1$ and go to step 1.

Numerical method (II)

- To solve the HJB and the KF equations, we employ an upwind **finite difference** method.
    - It approximates $V(a,z)$ and $f(a,z)$ on a finite grid with steps Δa and Δz : $a \in \{a_1, ..., a_I\}$, $z \in \{z_1, ..., z_J\}$.
    - In this case $\hat{K} = \sum_{i=1}^{I} \sum_{j=1}^{J} a_i f_{i,j} \Delta a \Delta z$.

- The proposed finite difference method converges to the unique viscosity solution of this problem (Barles and Souganidis, 1991).

Results

Constrained solution for the calibration in Aiyagari (1994)

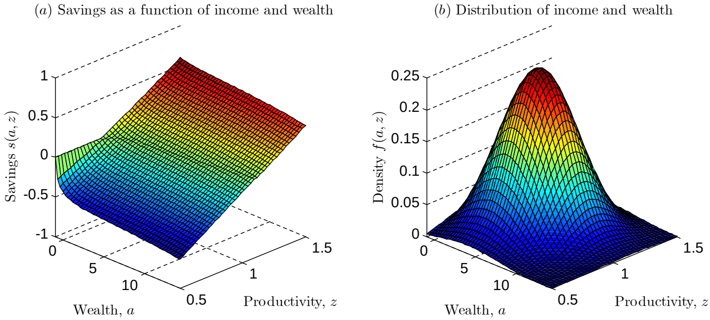

Figure: Savings policy and distribution of income and wealth: constrained optimum.

Conclusions

- We provide both a theoretical and a numerical approach to solve optimal control problems in heterogeneous-agents economies.
- Looking for interesting examples to apply it.