# Large deviations for some fast stochastic volatility models by viscosity methods

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# Plan

1. Stochastic volatility models
2. Large deviations by viscosity methods
3. As applications, asymptotic estimates for European out-of-the-money option prices near maturity and asymptotic formula for implied volatility.
4. Extension to the non-compact case (i.e. when the coefficients of the stochastic system are not periodic), work in progress.

# Stochastic system with fast oscillating random parameter

We consider a stochastic system in $\mathbb{R}^n$ with random coefficients, in particular with coefficients dependent on random parameter $Y_t$.

$$dX_t = \phi(X_t, Y_t)dt + \sqrt{2}\sigma(X_t, Y_t)dW_t, \quad X_0 = x_0 \in \mathbb{R}^n.$$

Assumption: we model this new parameter as a markov process evolving on a faster time scale $\tau = \frac{t}{\delta}$:

$$dY_t = \frac{1}{\delta}b(Y_t)dt + \sqrt{\frac{2}{\delta}}\tau(Y_t)dW_t, \quad Y_0 = y_0 \in \mathbb{R}^m.$$

Notation: $X_t$ are the slow components of the system, and $Y_t$ are the fast components.

# Assumptions

Tyical Assumptions: The fast variable are constrained in a compact set, say: the coefficients of the processes are $\mathbb{Z}^m$-periodic with respect to the variable $y$.

More generally: $Y_t$ is a recurrent process.
In particular we can generalize the results presented under the hypothesis that $Y_t$ is ergodic, this means that $Y$ forgets the initial condition for large time (i.e. as $\delta \to 0$) and its distribution becomes stationary.

**For technical simplicity from now on we assume the condition of periodicity on the coefficients.**
This condition can be relaxed to ergodicity and will be treated in an article in preparation.

Further assumption: the diffusion matrix $\tau$ is non-degenerate.

# Motivation:: analysis of financial models with stochastic volatility

Black-Scholes model: the evolution of the price of a stock S is described by

$$dlogS_t = \gamma dt + \sigma dW_t, \quad \text{t=time}, W_t = \text{Wiener proc.},$$

and the classical Black-Scholes formula for option pricing is derived assuming parameters are constants.
In reality the parameters of such models are not constants. In particular, the volatility $\sigma$, a measure for variation of price over time, is not constant but exhibits random behaviour.
Therefore it has been modeled as a positive function $\sigma = \sigma(Y_t)$ of a stochastic process $Y_t$ with

1. negative correlation (prices go up when volatility goes down)
2. mean reversion (the time it takes for agents to adjust their thresholds to current market conditions)

Refs.: Hull-White 87, Heston 93, Fouque-Papanicolaou-Sircar 2000,...

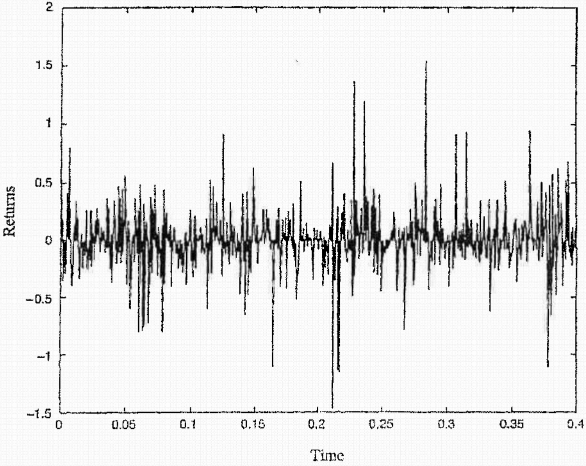


Figure 3.7. 1996 S&P 500 returns computed from half-hourly data.

# Fast stochastic volatility

It is argued in the book
**Fouque, Papanicolaou, Sircar:** Derivatives in financial markets with stochastic volatility, 2000,
that $Y_t$ also evolves on a faster time scale than the stock prices, modelling better the typical bursty behavior of volatility, see previous picture.

For this reason we put ourselves into the framework of multiple time scale systems and sigular perturbation and we model $Y_t$ with the fast stochastic process for $\delta > 0$

$$dY_t = \frac{1}{\delta} b(Y_t)dt + \sqrt{\frac{2}{\delta}} \tau(Y_t) dW_t \quad Y_0 = y_0 \in \mathbb{R}^m.$$

Passing to the limit as $\delta \to 0$ is a classical singular perturbation problem, its solution leads to the elimination of the state variable $Y_t$ and to the definition of an averaged system defined in $\mathbb{R}^n$ only. There is a large literature on the subject (**Bensoussan, Kushner, Hasminskii, Pardoux, Borkar, Galtsgory, Alvarex, Bardi...**)

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# Small time asymptotics for the system

We study the small time behaviour of the system, so we rescale time as

$$t \to \varepsilon t.$$

We study the asymptotics when both parameters go to 0 and we expect different limit behaviors depending on the rate $\varepsilon/\delta$. Therefore we put

$$\delta = \varepsilon^\alpha, \text{ with } \alpha > 1.$$

We consider the limit of the system for $\varepsilon \to 0$

$$\begin{cases} dX_t^\varepsilon = \varepsilon\phi(X_t^\varepsilon, Y_t^\varepsilon)dt + \sqrt{2\varepsilon}\sigma(X_t^\varepsilon, Y_t^\varepsilon)dW_t, & X_0^\varepsilon = x_0 \in \mathbb{R}^n \\ dY_t^\varepsilon = \frac{1}{\varepsilon^{\alpha-1}}b(Y_t^\varepsilon)dt + \sqrt{\frac{2}{\varepsilon^{\alpha-1}}}\tau(Y_t^\varepsilon)dW_t, & Y_0^\varepsilon = y_0 \in \mathbb{R}^m. \end{cases} \quad (1)$$

# Motivation: asymptotic estimates for volatility of option prices near maturity

**Avellaneda and collaborators** (2002, 2003) used the theory of large deviations to give asymptotic estimates for the Black-Scholes implied volatility of option prices near maturity (small time) in models with constant (local) volatility.

We carry on the same type of analysis in models with stochastic volatility. In this case finding explicit estimates happens to be more difficult and we need to assume condition of periodicity/ergodicity on the fast process.

**Remark**

*In this model:*

- $\varepsilon$ : *short maturity of the option*
- $\delta = \varepsilon^{\alpha}$ : *rate of mean reversion of the volatility.*

# Further references

1. **J. Feng, J.-P. Fouque, R. Kumar** (2012) studied large deviations for systems of the form that we defined for $\alpha = 2, 4$ in the one-dimensional case $n = m = 1$, assuming that $Y_t$ is an Ornstein-Uhlenbeck process and the coefficients in the equation for $X_t$ do not depend on $X_t$. The methods are based on the monograph by Feng and Kurtz, **Large deviations for stochastic processes** 2006.

2. Related works by **P. Dupuis, K. Spiliopoulos, K. Spiliopoulos** (2012, 2013) deal with different scaling and use different methods based on weak convergence

# Large Deviation Principle

Let $\{\mu_\varepsilon\}$ be a family of probability measures. A large deviation principle (LDP) characterizes the limiting behavior, as $\epsilon \to 0$, of $\{\mu_\epsilon\}$ in terms of a rate function through asymptotic upper and lower exponential bounds on the values that $\mu_\epsilon$ assigns to measurable subsets of $\mathbb{R}^n$.

Roughly speaking, large deviation theory concerns itself with the exponential decline of the probability measures of certain kinds of extreme or tail events.

In the context of financial mathematics, large deviations theory arises in the computations of small maturity out-of-the-money option prices.

# Main results-Large Deviation principle

We prove a Large Deviation Principle (LDP) for the process $X_t^\varepsilon$ (i.e. for probability measures generated by the laws of $X_t^\varepsilon$).

In other words we prove that then for every $t > 0$ and for any open set $B \subseteq \mathbb{R}^n$

$$P(X_t^\varepsilon \in B) = e^{-\inf_{x\in B} \frac{I(x;x_0,t)}{\varepsilon} + o(\frac{1}{\varepsilon})}, \quad \text{as } \varepsilon \to 0.$$

for some (good) rate function $I$, non-negative and continuous, which we will define in the next slides.

# The Large Deviation Principle

## Bryc's inverse Varadhan lemma

Assume that for all $t > 0$

1. $X_t^\varepsilon$ is exponentially tight.
2. for every $h$ bounded and continuous the limit

$$\lim_{\varepsilon \to 0} \varepsilon \log E\left[ e^{\varepsilon^{-1} h(X_t^\varepsilon)} \,|\, X_0 = x_0, Y_0 = y_0 \right] := L_h(x_0, t)$$

   exists finite.

Then $X_t^\varepsilon$ satisfies a large deviation principle with good rate function

$$I(x, x_0, t) = \sup_{h \in BC(\mathbb{R}^n)} \{h(x) - L_h(x_0, t)\}.$$

# PDE viscosity methods

We define the following logarithmic payoff

$$v^{\varepsilon}(t, x_0, y_0) := \varepsilon \log E\left[e^{\varepsilon^{-1} h(X_t^{\varepsilon})} \,|\, X_0 = x_0, Y_0 = y_0\right], x_0 \in \mathbb{R}^n, y_0 \in \mathbb{R}^m, t \geq 0,$$

where $h$ is a bounded continuous function defined on $\mathbb{R}^n$.

Then, to obtain the LDP we have:

1. prove that $v^{\varepsilon}$ converges to some function $v(t, x)$ and characterize $v$;
2. compute the rate function $I$ in term of the limit of $v^{\varepsilon}$.

# Convergence by viscosity methods

The associated HJB equation to $v_\varepsilon$ is the following parabolic pde with quadratic nonlinearity in the gradient ($b, \tau$ computed in $y$, $\phi, \sigma$ in $(x, y)$).

$$v_t^\varepsilon = |\sigma^T D_x v^\varepsilon|^2 + \varepsilon \left(\mathrm{tr}(\sigma\sigma^T D_{xx}^2 v^\varepsilon) + \phi \cdot D_x v^\varepsilon\right) + 2\varepsilon^{-\frac{\alpha}{2}}(\tau\sigma^T D_x v^\varepsilon) \cdot D_y v^\varepsilon +$$
$$+ 2\varepsilon^{1-\frac{\alpha}{2}}\mathrm{tr}(\sigma\tau^T D_{xy}^2 v^\varepsilon) + \varepsilon^{1-\alpha}\left(b \cdot D_y v^\varepsilon + \mathrm{tr}(\tau\tau^T D_{yy}^2 v^\varepsilon)\right) + \varepsilon^{-\alpha}|\tau^T D_y v^\varepsilon|^2.$$

## Remark

*This problem falls in the class of averaging/homogenization problems for nonlinear HJB type equations where the fast variable lives in a compact space*

## Theorem-Convergence by viscosity methods

Let $h$ be continuous and bounded.
Then

$$v^{\varepsilon}(x,y,t) = \varepsilon \log E e^{\frac{h(X_t^{\varepsilon})}{\varepsilon}} \to v(x,t)$$

locally uniformly in $y$ where $v$ is the unique viscosity solution to the effective equation

$$\begin{cases} v_t - \bar{H}(x, Dv) = 0 & \text{in } ]0,T[\times\mathbb{R}^n, \\ v(0,x) = h(x) & \text{in } \mathbb{R}^n. \end{cases}$$

where $\bar{H}$ is the limit or effective Hamiltonian.

## Remark

- $v^{\varepsilon}$ is uniformly bounded in $\varepsilon$.
- To prove the convergence we use relaxed semilimits Barles-Parthame procedure and the techniques stem form Evans' perturbed test function method for homogenization and its extension to singular perturbations Alvarez, Bardi (2003) and regular perturbations of singular perturbation Alvarez, Bardi, Marchi (2007).

# The effective Hamiltonian

We identify the limit or effective Hamiltonian, by solving three different cell problems depending on $\alpha$. We point out three regimes depending on how fast the volatility oscillates relative to the horizon length:

$$\begin{cases} \alpha > 2 & \text{supercritical case,} \\ \alpha = 2 & \text{critical case,} \\ \alpha < 2 & \text{subcritical case.} \end{cases}$$

1. In all the cases the limit Hamiltonian $\bar{H}$ is continuous on $\mathbb{R}^n \times \mathbb{R}^n$ and convex in the second variable.
2. In all the cases we provide some representation formulas for the limit Hamiltonian $\bar{H}$.

More interesting case: $\alpha = 2$.

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2. In all the cases we provide some representation formulas for the limit Hamiltonian $\bar{H}$.

More interesting case: $\alpha = 2$.

# The effective Hamiltonian, $\alpha = 2$

$\bar{H}$ is continuous, convex,

$$\inf_y |\sigma^T(\bar{x}, y)\bar{p}|^2 \leq \bar{H}(\bar{x}, \bar{p}) \leq \sup_y |\sigma^T(\bar{x}, y)\bar{p}|^2.$$

More precisely

$$\bar{H}(\bar{x}, \bar{p}) = \lim_{t \to \infty} \frac{1}{t} \log E\left[ e^{\int_0^t |\sigma^T(\bar{x}, Y_s)\bar{p}|^2 \, ds} \,|\, Y_0 = y \right],$$

where $Y_t$ is the stochastic process defined by

$$dY_t = \left( b(Y_t) + 2\tau(Y_t)\sigma^T(\bar{x}, Y_t)\bar{p} \right) dt + \sqrt{2}\tau(Y_t) dW_t.$$

# The Large Deviation Principle

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exists finite.

Then $X_t^\varepsilon$ satisfies a large deviation principle with good rate function

$$I(x, x_0, t) = \sup_{h \in BC(\mathbb{R}^n)} \{h(x) - L_h(x_0, t)\}.$$

# The rate function

Throughout the section we suppose that $\sigma$ is uniformly non degenerate, that is, for some $\nu > 0$ and for all $x, p \in \mathbb{R}^n$

$$|\sigma^T(x,y)p|^2 > \nu |p|^2. \tag{2}$$

Note that under the previous assumption, the effective Hamiltonian is coercive. Let $\bar{L}$ be the effective Lagrangian associated to the effective Hamiltonian $\bar{H}$ via convex duality, i.e. for $x \in \mathbb{R}^n$

$$\bar{L}(x,q) = \max_{p \in \mathbb{R}^n} \{p \cdot q - \bar{H}(x,p)\}.$$

Note that $\bar{L}(x,\cdot)$ is a convex nonnegative function such that $\bar{L}(x,0) = 0$ for all $x \in \mathbb{R}^n$, since $\bar{H}(x,\cdot)$ is convex nonnegative and $\bar{H}(x,0) = 0$ for all $x \in \mathbb{R}$.

# The rate function

Then the rate function is defined as follows

$$I(x;x_0,t) := \inf\left[\int_0^t \bar{L}(\xi(s),\dot{\xi}(s))\,ds \middle| \xi \in AC(0,t),\ \xi(0)=x_0, \xi(t)=x\right].$$

- I depends only on the volatility $\sigma$ and on the fast process $Y_t^\varepsilon$;
- $I$ does not depend on the drift $\phi$ of the log-price $X_t^\varepsilon$ and on the initial value $y_0$ of the process $Y_t$.
- $I$ satisfies the following growth condition for some $\nu, C > 0$ and all $x, x_0 \in \mathbb{R}^n$
$$\frac{1}{4C}\frac{|x-x_0|^2}{t} \le I(x;x_0,t) \le \frac{1}{4\nu}\frac{|x-x_0|^2}{t};$$
- if $\sigma$ does not depend on $x$, i.e. $\bar{H} = \bar{H}(p)$, the rate function is

$$I(x;x_0,t) = t\bar{L}\left(\frac{x-x_0}{t}\right).$$

# The rate function

- If $\alpha > 2$ and $n = 1$ and $\bar{H} = \bar{H}(p)$, then

$$I(x; x_0, t) = \frac{|x - x_0|^2}{4\bar{\sigma}^2 t} \qquad (3)$$

where

$$\bar{\sigma} = \sqrt{\int_{\mathbb{T}^m} \sigma(y)^2 d\mu(y)}$$

and $\mu$ is the invariant measure of the process $Y_t$ defined in the previous slides, i.e.

$$dY_t = b(Y_t)dt + \sqrt{2}\tau(Y_t)dW_t,$$

## Remark

*We observe that the rate function defined in (3) is the same as the rate function for the Black-Scholes model with constant volatility $\bar{\sigma}$. In other words, in the ultra fast regime, to the leading order, it is the same as averaging first and then taking the short maturity limit.*

Let $S_t^\varepsilon$ be the asset price, evolving according to the following stochastic differential system

$$\begin{cases} dS_t^\varepsilon = \varepsilon\xi(S_t^\varepsilon, Y_t^\varepsilon)S_t^\varepsilon dt + \sqrt{2\varepsilon}\zeta(S_t^\varepsilon, Y_t^\varepsilon)S_t^\varepsilon dW_t & S_0^\varepsilon = S_0 \in \mathbb{R}_+ \\ dY_t^\varepsilon = \varepsilon^{1-\alpha}b(Y_t^\varepsilon)dt + \sqrt{2\varepsilon^{1-\alpha}}\tau(Y_t^\varepsilon)dW_t & Y_0^\varepsilon = y_0 \in \mathbb{R}^m, \end{cases} \tag{4}$$

where $\alpha > 1$, $\tau, b$ are $\mathbb{Z}^m$-periodic in $y$ with $\tau$ non-degenerate and $\xi : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}$, $\zeta : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbf{M}^{1,r}$ are Lipschitz continuous bounded functions, periodic in $y$.
Observe that $S_t^\varepsilon > 0$ almost surely if $S_0 > 0$.
We consider out-of-the-money call option with strike price $K$ and short maturity time $T = \epsilon t$, by taking

$$S_0 < K \quad \text{or} \quad x_0 < \log K.$$

Similarly, by considering out-of-the-money put options, one can obtain the same for $S_0 > K$.

# Out-of-the-money option pricing

As an application of the Large Deviation Principle, we prove

## Corollary

*For fixed* $t > 0$

$$\lim_{\varepsilon \to 0^+} \varepsilon \log E\left[ \left( S_t^\varepsilon - K \right)^+ \right] = - \inf_{y > \log K} I\left( y; x_0, t \right).$$

When $\zeta(s, y) = \zeta(y)$, the option price estimate reads

$$\lim_{\varepsilon \to 0^+} \varepsilon \log E\left[ \left( S_t^\varepsilon - K \right)^+ \right] = -I\left( \log K; x_0, t \right)$$

# Implied volatility

We recall that given an observed European call option price for a contract with strike price $K$ and expiration date $T$, the implied volatility $\sigma$ is defined to be the value of the volatility parameter that must go into the Black-Scholes formula to match the observed price.

We consider out-of-the-money European call option, with strike price $K$, and we denote by $\sigma_\varepsilon(t, \log K, x_0)$ the implied volatility.

# Applications-An asymptotic formula for implied volatility

As a further application, we prove

## Corollary

$$\lim_{\varepsilon\to 0^+} \sigma_\varepsilon^2(t, \log K, x_0) = \frac{(\log K - x_0)^2}{2\inf_{y>\log K} I(y; x_0, t)t}.$$

Note that the infimum in the right-hand side, is always positive by the assumption on $S_0$ and by the growth of the rate function.

## Remark

*When $\alpha > 2$, the implied volatility is $\bar{\sigma}$ that is*

$$\bar{\sigma} = \sqrt{\int_{\mathbb{T}^m} \sigma^2(y) d\mu(y)}.$$

Thank you for the attention!

# The cell problem and the effective Hamiltonian, $\alpha = 2$

Pluggin in the equation the formal asymptotic expansion

$$v^\varepsilon(t,x,y) = v^0(t,x) + \varepsilon w(t,x,y).$$

we obtain

$$v_t^0 - |\sigma^T D_x v^0|^2 - 2(\tau\sigma^T D_x v^0)\cdot D_y w - b\cdot D_y w - |\tau^T D_y w|^2 - \mathrm{tr}(\tau\tau^T D^2_{yy} w) = O(\varepsilon).$$

## Proposition

*For any fixed $(\bar{x},\bar{p})$, there exists a unique $\bar{H}(\bar{x},\bar{p})$ for which the uniformly elliptic equation with quadratic nonlinearity in the gradient*

$$\bar{H}(\bar{x},\bar{p}) - |\sigma^T\bar{p}|^2 - (2\tau\sigma^T\bar{p} + b)\cdot D_y w(y) - |\tau^T D_y w(y)|^2 - tr(\tau\tau^T D^2_{yy} w(y)) = 0,$$

*has a periodic viscosity solution $w$.*

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# $\bar{H}$: First Representation formula

- $\bar{H}$ can be represented through stochastic control as

$$\bar{H}(\bar{x},\bar{p}) = \lim_{\delta\to 0}\sup_{\beta(\cdot)} \delta E\left[\int_0^{\infty}\left(|\sigma(\bar{x},Z_t)^T\bar{p}|^2 - |\beta(t)|^2\right)e^{-\delta t}dt \,|\, Z_0 = z\right]$$

and

$$\bar{H}(\bar{x},\bar{p}) = \lim_{t\to\infty}\sup_{\beta(\cdot)} \frac{1}{t}E\left[\int_0^{t}(|\sigma^T(\bar{x},Z_s)\bar{p}|^2 - |\beta(s)|^2)ds \,|\, Z_0 = z\right],$$

where $\beta(\cdot)$ is an admissible control process taking values in $\mathbb{R}^r$ for the stochastic control system

$$dZ_t = \left(b(Z_t) + 2\tau(Z_t)\sigma^T(\bar{x},Z_t)\bar{p} - 2\tau(Z_t)\beta(t)\right)dt + \sqrt{2}\tau(Z_t)dW_t; \quad (5)$$

# $\bar{H}$: Second representation formula

- Moreover

$$\bar{H} = \int_{\mathbb{T}^m} \left( |\sigma(\bar{x}, z)^T \bar{p}|^2 - |\tau(z)^T Dw(z)|^2 \right) d\mu(z),$$

where $w = w(\cdot; \bar{x}, \bar{p})$ is the smooth solution to

$$\bar{H}(\bar{x}, \bar{p}) - \mathrm{tr}(\tau\tau^T D^2_{yy} w) - |\tau^T D_y w|^2 + \\ - (2\tau\sigma^T \bar{p} + b) \cdot D_y w - |\sigma^T \bar{p}|^2 = 0 \quad \text{in } \mathbb{R}^m$$

and $\mu = \mu(\cdot; \bar{x}, \bar{p})$ invariant probability measure on the torus $\mathbb{T}^m$ of the process (5) with the feedback $\beta(z) = -\tau^T(z) Dw(z)$, i.e.

$$dZ_t = \big( b(Z_t) + 2\tau(Z_t)\sigma^T(\bar{x}, Z_t)\bar{p} + 2\tau(Z_t)\tau^T(Z_t) Dw(Z_t) \big) \, dt + \sqrt{2}\tau(Z_t) dW_t.$$

# $\bar{H}$: Third representation formula

- Moreover

$$\bar{H}(\bar{x},\bar{p}) = \lim_{t\to\infty} \frac{1}{t} \log E\left[e^{\int_0^t |\sigma^T(\bar{x},Y_s)\bar{p}|^2\,ds} \,|\, Y_0 = y\right],$$

where $Y_t$ is the stochastic process defined by

$$dY_t = \left(b(Y_t) + 2\tau(Y_t)\sigma^T(\bar{x},Y_t)\bar{p}\right)dt + \sqrt{2}\tau(Y_t)dW_t.$$

Sketch of the proof:

Take $v = v(t,x;\bar{x},\bar{p})$ a periodic solution of the $t$-cell problem and define the function $f(t,y) = e^{v(t,y)}$. Then $f$ solves the following equation

$$\begin{cases} \frac{\partial f}{\partial t} - f|\sigma^T\bar{p}|^2 - (2\tau\sigma^T\bar{p} + b)\cdot Df - \mathrm{tr}(\tau\tau^T D^2 f) = 0 & \text{in } (0,\infty)\times\mathbb{R}^m \\ f(0,z) = 1 & \text{in } \mathbb{R}^m. \end{cases}$$

and we conclude using the Feynam-Kac formula.

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# The cell problem and the effective Hamiltonian, $\alpha > 2$

Plugging the formal asymptotic expansion

$$v^\varepsilon(t,x,y) = v^0(t,x) + \varepsilon^{\alpha-1} w(t,x,y)$$

in the equation we get

$$v^0_t = |\sigma^T D_x v^0|^2 + b \cdot D_y w + \mathrm{tr}(\tau\tau^T D^2_{yy} w) + O(\varepsilon).$$

## Proposition

*For each $(\bar{x},\bar{p})$ fixed, there exists a unique constant $\bar{H}(\bar{x},\bar{p})$ such that the linear second order uniformly elliptic equation*

$$\bar{H}(\bar{x},\bar{p}) - tr(\tau\tau(y)^T D^2_{yy} w_\delta(y)) - b(y)\cdot D_y w_\delta(y) - |\sigma(\bar{x},y)^T \bar{p}|^2 = 0 \text{ in } \mathbb{R}^m,$$

*has a periodic smooth solution.*

# $\bar{H}$: Representation formula

$\bar{H}$ can be represented as

$$\bar{H} = \int_{\mathbb{T}^m} |\sigma(\bar{x}, y)^T \bar{p}|^2 \, d\mu(y),$$

where $\mu$ is the invariant probability measure on the torus $\mathbb{T}^m$ of the stochastic process

$$dY_t = b(Y_t)dt + \sqrt{2}\tau(Y_t)dW_t,$$

that is, the periodic solution of

$$-\sum_{i,j} \frac{\partial^2}{\partial y_i \partial y_j}((\tau\tau^T)_{ij}(y))\mu + \sum_i \frac{\partial}{\partial y_i}(b_i(y))\mu = 0 \qquad \text{in } \mathbb{R}^m,$$

with $\int_{\mathbb{T}^n} \mu(y)\, dy = 1.$

# $\bar{H}$: Representation formula

When $n = 1$,

$$\bar{H}(\bar{x}, \bar{p}) = (\bar{\sigma}\bar{p})^2$$

where

$$\bar{\sigma}(\bar{x}) = \sqrt{\int_{\mathbb{T}^m} \sigma^2(\bar{x}, y) d\mu(y)}$$

and $\mu$ is the invariant measure of the following process $Y_t$

$$dY_t = b(Y_t)dt + \sqrt{2}\tau(Y_t)dW_t,$$

# The cell problem and the effective Hamiltonian, $\alpha < 2$

We plug in the equation the formal asympthotic expansion

$$v^\varepsilon(t,x,y) = v^0(t,x) + \varepsilon^{\frac{\alpha}{2}} w(t,x,y).$$

and we obtain

$$v_t^0 = |\sigma^T D_x v^0|^2 + 2(\tau\sigma^T D_x v^0) \cdot D_y w + |\tau^T D_y w|^2 + O(\varepsilon).$$

## Proposition

*For any fixed $(\bar{x},\bar{p})$, there exists a unique constant $\bar{H}(\bar{x},\bar{p})$ such that the first order coercive equation*

$$\bar{H}(\bar{x},\bar{p}) - |\tau^T(y) D_y w(y) + \sigma^T(\bar{x},y)\bar{p}|^2 = 0 \quad \text{in } \mathbb{R}^m$$

*admits a (Lipschitz continuous) periodic viscosity solution $w$.*

# $\bar{H}$: Representation formulas

- $\bar{H}$ satisfies

$$\bar{H}(\bar{x},\bar{p}) = \lim_{\delta\to 0} \sup_{\beta(\cdot)} \delta \int_0^{+\infty} \left(|\sigma(\bar{x},y(t))^T \bar{p}|^2 - |\beta(t)|^2\right) e^{-\delta t}\, dt,$$

  where $\beta(\cdot)$ varies over measurable functions taking values in $\mathbb{R}^r$, $y(\cdot)$ is the trajectory of the control system

$$\begin{cases} \dot{y}(t) = 2\tau(y(t))\sigma^T(\bar{x},y(t))\bar{p} - 2\tau(y(t))\beta, & t > 0, \\ y(0) = y \end{cases}$$

  and the limit is uniform with respect to the initial position $y$ of the system;

- Moreover under the condition $\tau\sigma^T = 0$ of non-correlations among the components of the white noise acting on the slow and the fast variables in the system, we have

$$\bar{H}(x,p) = \max_{y\in\mathbb{R}^m} |\sigma^T(x,y)p|^2.$$

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# Remarks on the comparison principle for $\bar{H}$

- $\alpha = 2$: we relate the regularity in $x$ of $\bar{H}$ with that of the pseudo-coercive Hamiltonian $|\sigma^T(x,y)p|^2$, i.e. we prove for $0 < \mu < 1$

$$\mu \bar{H}(x, \frac{p}{\mu}) - \bar{H}(z,q) \geq \frac{1}{\mu - 1} \sup_{y \in \mathbb{R}^m} |\sigma^T(x,y)p - \sigma^T(z,y)q|^2.$$

  Then, we follow the same argument for pseudo-coercive Hamiltonian as in Barles-Perthame (1990) for the stationary case and M.Kobylanski for the evolutionary case (PhD thesis).
- $\alpha > 2$: as for the critical case (easier thanks to the explicit formula or $H$).
- $\alpha < 2$: let $H_0 = \sqrt{\bar{H}}$. Then

$$H_0(x, \lambda p) = |\lambda| H_0(x,p) \quad \forall \lambda \in \mathbb{R}$$

  and there exists $C > 0$ such that $|H_0(x,p)| \leq C|p|$ and

$$|H_0(x,p) - H_0(z,p)| \leq C(1 + |p|)|x - z| \quad \forall x, z \in \mathbb{R}^n, p \in \mathbb{R}^n.$$

  Then we can use the comparison result for evolutive non-coercive first-order HJ equations by Cutri, Da Lio (2007).

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# Without periodicity assumption (work in progress)

Main assumptions: ergodicity of the fast variables

- $\exists C > 0$ ad a compact set $K$ s.t.

$$b(y) \cdot y + 2\mathrm{tr}(\tau\tau^T(y)) < -C|y|^2 \quad \forall y \notin K.$$

  Main example: Ornstein-Uhlenbeck process

$$dY_t = (m - Y_t)dt + \tau dW_t.$$

- $\tau$ is non degenerate and $\sigma(x, \cdot)$ is bounded $\forall x$.

Recall that $v^\varepsilon$ solves

$$v_t^\varepsilon = |\sigma^T D_x v^\varepsilon|^2 + \varepsilon \left( \mathrm{tr}(\sigma\sigma^T D_{xx}^2 v^\varepsilon) + \phi \cdot D_x v^\varepsilon \right) + 2\mathrm{tr}(\sigma\tau^T D_{xy}^2 v^\varepsilon) + \\ + \frac{1}{\varepsilon} \left( (b + 2(\tau\sigma^T D_x v^\varepsilon)) \cdot D_y v^\varepsilon + \mathrm{tr}(\tau\tau^T D_{yy}^2 V^\varepsilon) \right) + \frac{1}{\varepsilon^2} |\tau^T D_y v^\varepsilon|^2.$$

and $v^\varepsilon$ is uniformly bounded in $\varepsilon$. We will use relaxed semilimits (Barles-Perthame procedure).
We want to prove that $v^\varepsilon$ converges locally uniformly to the unique solution to

$$v_t - \bar{H}(x, Dv) = 0.$$

# $\alpha = 2$, The ergodic problem

For every $x, p$ fixed, find the unique constant $\bar{H}(x,p)$ such that there is a solution $w$ to

$$\begin{cases} -(b + 2\tau\sigma^T p) \cdot Dw - \mathrm{tr}(\tau\tau^T D^2 w) + |\tau^T Dw|^2 + |\sigma^t p|^2 = -\bar{H} \\ \exists C > 0 \text{ s.t. } w(y) \leq C(1 + |y|^2) \end{cases} \tag{6}$$

- The main assumption on $b, \tau$ implies that $K(1 + |y|^2)$ is a superslolution to the previous problem outside a compact set.
- Without growth assumptions, no uniquenes of $\bar{H}$ (Ichiara 2011, Khaise-Sheu 2006).

We look at the ergodic approximation

$$\delta v_\delta - (b + 2\tau\sigma^t p) \cdot Dv_\delta - tr(\tau\tau^T D^2 v_\delta) - |\tau^T Dv_\delta|^2 - |\sigma^t p|^2 = 0,$$

and prove that $\delta v_\delta \to \bar{H}, v_\delta - v_\delta(0) \to w$.

# $\alpha = 2$, THE ERGODIC PROBLEM 2

We consider approximate ergodic problems in balls with singular boudary conditions:
for every $R >> 1$, for every $x, p$ fixed, find the unique constant $\bar{H}_R(x,p)$ such that there is a solution $w_R$ to

$$\begin{cases} -(b + 2\tau\sigma^T p) \cdot Dw_R - \mathrm{tr}(\tau\tau^T D^2 w_R) + |\tau^T Dw_R|^2 + |\sigma^t p|^2 = -\bar{H}_R & |y| < R \\ w_R \to -\infty & |y| \to R. \end{cases} \tag{7}$$

(Barles, Porretta, Tchamba, 2010).

Then $\bar{H}_R$ is increasing to $\bar{H}$ as $R \to +\infty$.

# $\alpha = 2$, Supersolution

Define

$$\underline{v}(t,x,y) = \inf(\liminf_{\varepsilon \to 0} v^{\varepsilon}(x_{\varepsilon}, y_{\varepsilon}, t_{\varepsilon}) \,|\, x_{\varepsilon} \to x, y_{\varepsilon} \to y, t_{\varepsilon} \to t).$$

- for all $t, x$ fixed, $\underline{v}(t,x,\cdot)$ is a (bdd) supersolution to

  $$-|\tau^T D_y \underline{v}|^2 \geq 0.$$

  So, it does not depend on $y$ (Liouville type theorem).
- If $\underline{v}(t,x) - f(t,x)$ has a strict minimum at a point $(\bar{t}, \bar{x})$, then $v^{\varepsilon}(t,x,y) - f(t,x) - \varepsilon w_R(y)$ has minima, for $\varepsilon$ small, $x_{\varepsilon} \to \bar{x}, t_{\varepsilon} \to \bar{t}, y_{\varepsilon} \in B(0,R)$.
- By the definition of viscosity solutions and the construction of $w_R$, we get that

  $$\underline{v}_t - \bar{H}_R(x, D\underline{v}) \geq 0,$$

  for every $R >> 1$.

# $\alpha = 2$, Subsolution

Define

$$\bar{v}(t,x) = \sup(\limsup_{\varepsilon\to 0} v^{\varepsilon}(x_{\varepsilon}, y_{\varepsilon}, t_{\varepsilon}) \,|\, x_{\varepsilon} \to x, t_{\varepsilon} \to t, (y_{\varepsilon})_{\varepsilon} \text{ is bounded}).$$

Note that

$$\bar{v}(t,x) \geq \sup(\limsup_{\varepsilon\to 0} v^{\varepsilon}(x_{\varepsilon}, y_{\varepsilon}, t_{\varepsilon}) \,|\, x_{\varepsilon} \to x, y_{\varepsilon} \to y, t_{\varepsilon} \to t).$$

- We construct a supersolution $\tilde{w}$ to the ergodic problem such that
  - $\tilde{w} \to +\infty$ as $|y| \to +\infty$
  - $-(b + 2\tau\sigma^T p) \cdot D\tilde{w} - \operatorname{tr}(\tau\tau^T D^2\tilde{w}) - |\tau^T D\tilde{w}|^2 \to +\infty$ as $|y| \to +\infty$.
- If $\bar{v}(t,x) - f(t,x)$ has a strict maximum at a point $(\bar{t}, \bar{x})$, then $v^{\varepsilon}(t,x,y) - f(t,x) - \varepsilon\tilde{w}(y)$ has maxima, for $\varepsilon$ small, $x_{\varepsilon} \to \bar{x}, t_{\varepsilon} \to \bar{t}$ and $y_{\varepsilon}$ that is for all $\varepsilon$ in a compact set independent of $\varepsilon$.
  Then, using the definition of viscosity solution,

$$\bar{v}_t - \bar{H}(x, D\bar{v}) \leq 0.$$

# Conclusion

By construction

$$\bar{v}(t,x) \geq \underline{v}(x,t).$$

We conclude using the comparison principle for the effective equation

$$v_t + \bar{H}(x, Dv) = 0$$

that

$$\begin{aligned} v(x,t) &= \sup(\limsup_{\varepsilon\to 0} v^{\varepsilon}(x_\varepsilon, y_\varepsilon, t_\varepsilon) \,|\, x_\varepsilon \to x, t_\varepsilon \to t, y_\varepsilon \text{ bounded}) \\ &= \inf(\liminf_{\varepsilon\to 0} v^{\varepsilon}(x_\varepsilon, y_\varepsilon, t_\varepsilon) \,|\, x_\varepsilon \to x, t_\varepsilon \to t, y_\varepsilon \text{ bounded}). \end{aligned} \tag{8}$$

# A related homogenization result

Let $\varepsilon > 0$ and consider $u^\varepsilon$ solution to

$$\begin{cases} u_t^\varepsilon - b\left(\frac{x}{\varepsilon}\right) Du^\varepsilon - \varepsilon \mathrm{tr}(\tau\tau^T\left(\frac{x}{\varepsilon}\right) - |\tau\left(\frac{x}{\varepsilon}\right) D^2 u^\varepsilon|^2 - l\left(x, \frac{x}{\varepsilon}\right) = 0 \\ u^\varepsilon(x,0) = h(x). \end{cases}$$

If the main assumptions hold for $b, \tau$ (the underlying process to the fast variables is ergodic), then $u^\varepsilon$ converges locally uniformly, as $\varepsilon \to 0$, to the solution $u$ of

$$u_t - \bar{H}(x, Du) = 0.$$