A level-set crystalline mean curvature flow

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November 2014

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1. Introduction

1.1 Mean curvature flow

Let $\{\Gamma_t\}_{t\geq 0}$ be an evolving hypersurface in $\mathbf{R}^n$. The **mean curvature flow equation** is an equation for $\{\Gamma_t\}$ of the form

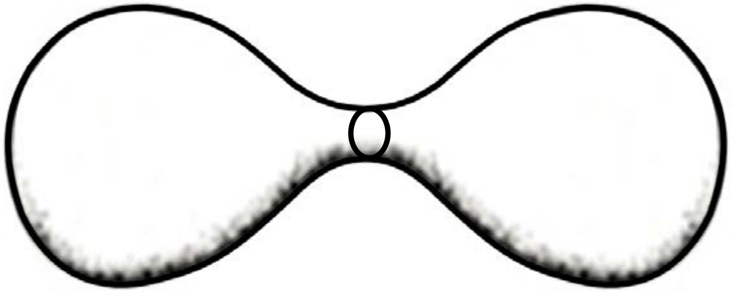
$$V = H \text{ on } \Gamma_t.$$

Here V is the normal velocity and H is ($n-1$ times) mean curvature.

Source: Materials Science [W. W. Mullins, 1957]
(motion of grain boundaries in annealing metals)

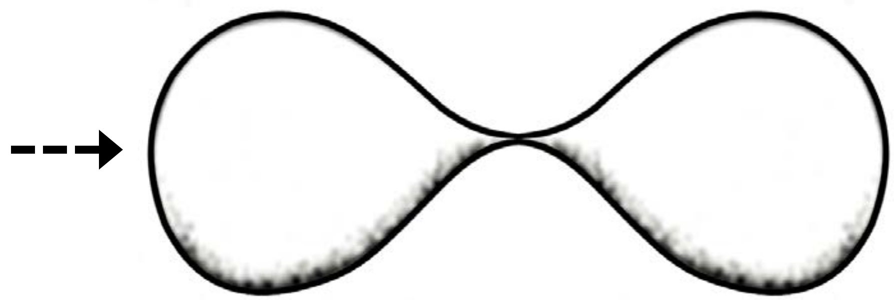
Formation of singularities

initial data | pinching



Γ_0

dumbbell with thin neck



Γ_t

[M. Grayson ’89]

Weak solutions including singularity

- Variational approach: K. Brakke ’72
  T. Ilmanen ’93, … Y. Tonegawa and et al ’14.
- A level set method: Y.-G. Chen – Y. G. – S. Goto ’91
  L. C. Evans – J. Spruck ’91

  [Book: Y. G., Surface evolution equations, Birkhäuser ’06]
  (encounter with the theory of viscosity solution, A deterministic game interpretation, R. V. Kohn – S. Serfaty ’06 …)

Extension to anisotropic flow

Is it possible to extend a level set approach to anisotropic curvature flow?

Yes, Y.-G. Chen – Y. G. – S. Goto '91 provided that the interfacial energy density is convex and **smooth**.

What happens the interface energy density is not C^1 (strong anisotropy)?

1.2 Crystalline mean curvature flow

A **crystalline mean curvature flow** is a typical example of **anisotropic** mean curvature flow, which for example describes motion of antiphase grain boundaries.

Anisotropic mean curvature is a change ratio of an interfacial energy with respect to variation of volume enclosed by a hypersurface Γ in $\mathbf{R}^n$.

Interfacial energy

Let γ_0 be a nonnegative continuous function defined on a unit sphere S^{n-1}, which is called an **interfacial energy density**. For a given hypersurface Γ we set

$$I(\Gamma) = \int_{\Gamma} \gamma_0(\vec{n})\, d\mathcal{H}^{n-1}$$

which is called an **interfacial energy**. Here $\vec{n}$ denotes the unit exterior normal of Γ and $d\mathcal{H}^{n-1}$ is the surface element.

Anisotropic mean curvature

The anisotropic mean curvature H_γ is defined by

$$H_\gamma = -\frac{\delta}{\delta\Gamma} I(\Gamma).$$

It is explicitly written as

$$H_\gamma = -\mathrm{div}_\Gamma\, \xi(\vec{n}) \quad \text{on} \quad \Gamma,$$

where $\xi(p) = \nabla_p \gamma(p)$ and γ is the homogenization of γ_0 i.e., $\gamma(p) = \gamma_0(p/|p|)|p|$. Here div_Γ denotes the surface divergence. The vector field $\xi(\vec{n})$ is often called the Cahn–Hoffman vector field.

Wulff shape
— a substitute for the sphere

$$W_\gamma = \bigcap_{|\vec{m}|=1} \{\, x \in \mathbf{R}^n |\, x \cdot \vec{m} \le \gamma(\vec{m})\}$$

$$\Rightarrow\ H_\gamma = -(n-1) \quad \text{on} \quad \Gamma = \partial W_\gamma$$

for smooth γ. [The converse is true provided that Γ is compact and embedded and that γ_0 is smooth and "strictly convex". Anisotropic version of Alexandrov's theorem. (Y. He – H. Li – H. Ma – J. Ge '09)]

Crystalline mean curvature

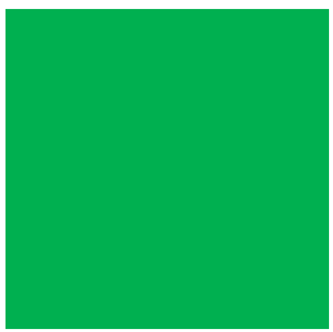
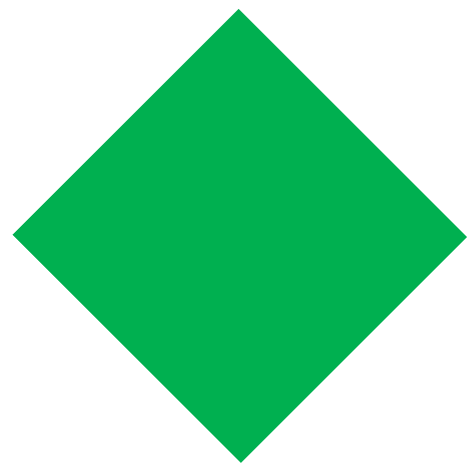
If $\gamma_0 \equiv 1$, then $I(\Gamma)$ is nothing but the area of Γ and H_γ is $(n-1\,\mathrm{times})$ the mean curvature. In this case $\gamma(p)=|p|$. In general, γ may not be convex nor smooth.

We say that γ_0 (or γ) is **crystalline** if γ is convex and piecewise linear. An anisotropic mean curvature H_γ is a **crystalline mean curvature** if γ is crystalline.

Wulff shape for a crystalline energy

$$F_\gamma = \{\, p \in \mathbf{R}^n \,|\, \gamma(p) \leq 1 \,\}$$ Frank diagram

($W_\gamma =$ polar of F_γ)

F_γ	W_γ
	
regular polyhedron	its dual

Anisotropic mean curvature flow

Let V denote the normal velocity of an evolving (hyper)surface Γ_t. A general form of anisotropic mean curvature flow equation is

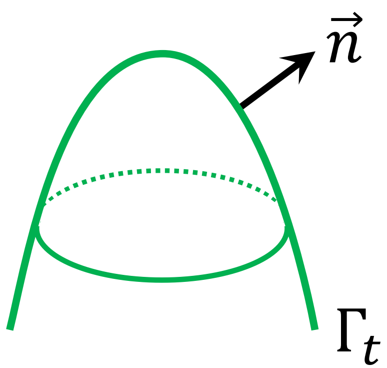
$$V = f\left(\vec{n}, H_\gamma\right) \quad \text{on} \quad \Gamma_t\,, \qquad \text{(ACF)}$$

where f is a given function. Example includes

(i) $V = H_\gamma$

(ii) $V = M(\vec{n})\left(H_\gamma + C\right),\ C \in \mathbf{R}\,,$

where $M(\vec{n}) > 0$ is a given function.



Crystalline mean curvature flow

We say that (ACF) is a **crystalline mean curvature flow equation** if f is continuous and non-decreasing in H_γ and H_γ is a crystalline mean curvature. It is formally a degenerate parabolic equation of the second order. However, as we see later it is a very singular equation.

1.3 Problems

Problem 1 (Existence and Uniqueness). Consider the crystalline mean curvature flow equation, for example,

(i) $$V = H_\gamma \quad \text{on} \quad \Gamma_t.$$

Does the initial value problem admit a unique solution in $\mathbf{R}^3$?

[For a given closed surface $\Gamma_0 \subset \mathbf{R}^3$ are there a unique family $\{\Gamma_t\}_{t \geq 0}$ solving (i)?]

Problem 2 (Stability). Is this solution (if exists) approximable by smoothed anisotropic mean curvature flow?

[If interfacial energy γ is approximable by γ_ε, does the corresponding solution $\{\Gamma_t^\varepsilon\}_{t\geq 0}$ approximate Γ_t?]

2. Unique existence of a level-set flow and its stability

2.1 Known results

- Well-studied for planar motion
  S. B. Angenent – M. Gurtin ’89, J. Taylor ’91
  Level set method: M.-H. Giga – Y. G. ’01 ARMA
- Higher dimension
  Even local existence was not known unless initial data is convex.
  A unique existence of a global flow provided that initial data is **convex**.
  G. Bellettini – V. Caselles – A. Chambolle – M. Novaga ’06 ARMA

2.2 Level set flow

This is a **generalized solution** of curvature flow equations which allows topological change of the flow Γ_t. We consider a level-set equation of (ACF). A level-set of the solution is regarded as a level-set flow.

Example

For the mean curvature flow equation, its level-set equation is of the form

$$v_t - |\nabla v| \operatorname{div}\left(\frac{\nabla v}{|\nabla v|}\right) = 0.$$

We consider this equation in $\mathbf{R}^n \times (0, \infty)$ not only on Γ_t.

- Unique solvability for the initial value problem in the sense of viscosity solutions with uniformly continuous initial data.
- $\Gamma_t = \{x | v(x,t) = 0\}$ is uniquely determined by Γ_0.
- $D_t = \{x | v(x,t) > 0\}$ is uniquely determined by D_0.

cf. S. Osher – J. Sethian ’89 (numerics), Y. G. Chen – Y. G. – S. Goto ’91, L. C. Evans – J. Spruck ’91, … Y. G. Surface Evolution Equations ’06

2.3 Main Results

Theorem 2.1 (Existence and Uniqueness). For a given initial data (a bounded open set) $D_0 \subset \mathbf{R}^3$ with its boundary Γ_0 there exists a global level-set flow D for **crystalline** mean curvature flow equation (ACF) (provided that f is at most linear growth in H_γ).

Level set equation for $V = f\left(\vec{n}, H_\gamma\right)$ on Γ_t:

$$v_t - |\nabla v| \ f\left(-\frac{\nabla v}{|\nabla v|}, -\mathrm{div}\big(\nabla\gamma(-\nabla v)\big)\right) = 0.$$

Theorem 2.2 (Stability). Assume the same hypotheses of Theorem 2.1. Let γ_ε be a smooth convex interfacial energy density approximate γ uniformly. Then the level-set flow D_ε converges to D in the Hausdorff distance sense in space-time provided that no fattening occurs.

(Underlying structure: comparison principle)

(work in progress, M.-H. Giga – Y. G. – N. Pozar)

3. Nonlocal mean curvature like quantity

3.1 Evolution of curves

If the space dimension equals two so that the evolving hypersurface Γ_t is a curve, then the corresponding results are known [M.-H. Giga – Y. G. '01]. The problem is nonlocal. Even curve evolution is nontrivial.

Consider $V = H_\gamma$ on $\Gamma_t = \{(x, y) | \ y = u(x, t)\}$ and $\gamma(p) = |p_1| + |p_2|$. Then one gets a total variation flow

$$u_t = (\mathrm{sgn}\, u_x)_x \ .$$

Speed is nonlocal even for curve evolution

Consider simplest eq

$$u_t = (\operatorname{sgn} u_x)_x.$$

What is the speed of the facet (flat part)?

Assume "facet stays as facet"

$$\int_{a-\delta}^{b+\delta} u_t = \operatorname{sgn}\left(-\bar{\delta}\right) - \operatorname{sgn}\left(+\tilde{\delta}\right)$$

$$\Leftrightarrow u_t = \frac{-2}{b-a} \, (\delta \downarrow 0)$$

$$\tilde{\delta} = u_x(a-\delta), -\bar{\delta} = u_x(b+\delta)$$

⋯⋯▶ Consistent with subdifferential formulation

T. Fukui – Y. G. '96

Crystalline flow for an admissible polygon

In the curve evolution the assumption “facet stays as facet” is compatible with comparison principle. (This does not apply for higher dimension.)

If one restricts the evolution on “admissible” polygons, then the evolution (ACF) is reduced to a system of ODEs.

S. B. Angenent – M. Gurtin ’89, J. Taylor ’91.

Crystalline algorithm (curve case)

A. Anisotropic curvature flow is approximated by crystalline algorithm.

P. M. Girão – R. V. Kohn ’94 (heat equation) (convergence rate),
P. M. Girão ’95 (curve shortening equation) (convergence rate),
T. Fukui – Y. G. ’96 (graph, divergence type) (no rate)

B. Crystalline algorithm is approximated by a smoother problem.

T. Fukui – Y. G. ’96 / M.-H. Giga – Y. G. ’00

Strategy and difficulty

- We need establish a notion of viscosity solutions since the equation is of non-divergence type, degenerate and very singular. The theory of maximal monotone operators does not apply directly.
- Nevertheless, the theory of maximal monotone operators is useful to identify nonlocal curvature like quantity .
- We use this quantity to define nonlocal mean curvature for test functions.

3.2 Theory of maximal monotone operators

This approach applies to a gradient flow $u_t \in -\partial E(u)$.

Example 1. Total variation flow

$H := L^2(\mathbf{T}^n)$ and $E(u)$:= total variation energy

Example 2. Its fourth order version

$H := \dot{H}^{-1}(\mathbf{T}^n)$ and $E(u)$:= total variation energy

(Y. G. – R. V. Kohn '12, Finite time extinction)

(M.-H. Giga – Y. G. '10, Instant loss of continuity)

Speed of Evolution

Theorem 3.1 (Y. Kōmura '67, H. Brezis – A. Pazy '70).

H: Hilbert space, E: convex, lower semicontinuous, $u_0 \in \overline{D(E)}$

⇒ There exists a **unique** solution $u \in C([0,\infty),H) \cap AC([\delta,T],H)$ solving

$$\frac{du}{dt} \in -\partial E(u) \quad \text{a.e.} \quad t > 0, \;\; u(0) = u_0.$$

Moreover, u is right differentiable for all $t > 0$ and

$$\frac{d^+u}{dt} = -\partial^0 E(u).$$

canonical restriction / minimal section:

$$\partial^0 E(u) = \arg\min\{\|f\|_H \,;\, f \in \partial E(u)\}$$

Solution knows how to evolve!

Subdifferentials

$$\partial E(f) = \{\, w \in H \,|\, E(f+h) - E(f) \geq \langle h, w \rangle \ \text{ for all } \ h \in H \,\}$$

This is not a singleton in general. It is a closed convex set in H. It can be empty.

Canonical restriction (minimal section)

$$\partial^0 E(f) = \arg\min \{ \|w\|_H \,|\, w \in \partial E(f) \}$$

Uniquely determined.

3.3 Characterization of subdifferential

A rigorous interpretation of gradient flow

In the case of total variation flow

$$H = L^2(\mathbf{T}^n), \qquad \langle f, g \rangle = \int_{\mathbf{T}^n} fg dx,$$

$$E(f) = \begin{cases} \int_{\mathbf{T}^n} |\nabla f|, & f \in BV \cap H, \\ \infty, & \text{otherwise} \end{cases}$$

Equation: $u_t \in -\partial_{L^2} E(u)$ (formally, $u_t = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$)

The total variation is defined as

$$\int_{\mathbf{T}^n} |\nabla f| := \sup\left\{ \int_{\mathbf{T}^n} f \operatorname{div} \varphi \,\middle|\, \|\varphi\|_\infty \leq 1, \varphi \in C_0^1(\mathbf{T}^n) \right\}.$$

$$BV(\mathbf{T}^n) := \left\{ f \in L^1(\mathbf{T}^n) \,\middle|\, \int_{\mathbf{T}^n} |\nabla f| < \infty \right\}.$$

Characterization of subdifferential in L^2

$w \in \partial E(f)$

$\Leftrightarrow {}^{\exists} z \in L^\infty(\mathbf{T}^n), z(x) \in \partial M(\nabla f)$ a.e. in $\mathbf{T}^n$,

$w = -\mathrm{div}\, z$ in $\mathbf{T}^n$, where $M(p) = |p|$.

[F. Andreu-Vaillo – V. Caselles – J. M. Mazon '04]

In the place $\nabla f \neq 0$, w formally equals

$$w = -\,\mathrm{div} \frac{\nabla f}{|\nabla f|}$$

since $\partial M(p) = \left\{ \frac{p}{|p|} \right\}$ for $p \neq 0$.

Obstacle problem and nonlocal curvature

We are interested in calculating the minimal section of subdifferentials for the second order problem. We consider periodic setting.

We consider a pair (A_-, A_+) of **disjoint open** sets in $\mathbf{T}^n$. We say that $\psi \in \mathrm{Lip}(\mathbf{T}^n)$ is a **support function** of a pair if

$$x \in A_\pm \Leftrightarrow \psi(x) > 0 \ (< 0).$$

Definition 3.2. We say that a pair (A_-, A_+) is **smooth** if

(i) $\operatorname{dist}(A_-, A_+) > 0$,

(ii) $\partial A_-, \partial A_+$ is smooth.

[A_-, A_+ can be empty]

[$\operatorname{dist}(x, y) := \inf\{|\bar{x} - \bar{y}| \,|\, \bar{x} \in x + \mathbf{Z}^n, \bar{y} \in y + \mathbf{Z}^n\}$]

Definition 3.3. A pair is **admissible** if there exists its support function ψ in $D(\partial E)$.

Obstacle problem

Lemma 3.4. A smooth pair is admissible.

Lemma 3.5. Let $\psi \in D(\partial E)$ be a support function of a smooth pair (A_-, A_+). Then $-\partial^0 E(\psi)$ on a facet

$$D = A_+^c \cap A_-^c$$

is independent of the choice of ψ.

We denote $-\partial^0 E(\psi)$ on D by $\Lambda[A_-, A_+]$ and call Λ a **nonlocal mean curvature**.

Nonlocal mean curvature

Lemma 3.6 (Obstacle problem).

$$\Lambda[A_-, A_+] = \arg\min \left\{ \int_D |w|^2 \, dx \right|$$

$$w = -\operatorname{div} z, \quad \|z\|_\infty \leq 1 \text{ (constraint)},$$

$$z \cdot \nu_{\partial A_\pm} = \mp 1 \text{ on } \partial A_\pm \text{ (B.C.)} \left. \right\}$$

if (A_-, A_+) is a smooth pair.

M.-H. Giga – Y. G. – N. Pozar ’12, ’13

Whether or not $w = \text{const}$?

Proposition 3.7. If the minimizer w_0 is constant in D, then

$$w_0 = \{\mathcal{H}^{n-1}(\partial A_+) - \mathcal{H}^{n-1}(\partial A_-)\}/\mathcal{L}^n(D).$$

This is trivial since

$$\int_D w_0 = -\int_D \operatorname{div} z = -\int_{\partial A_\pm} z \cdot \nu_{\partial A_\pm}$$

$$= \mathcal{H}^{n-1}(\partial A_+) - \mathcal{H}^{n-1}(\partial A_-).$$

Proposition 3.8. The minimizer w_0 is constant if and only if there is a vector field z $(\operatorname{div} z \in L^2(D))$ such that

$$-\operatorname{div} z = \lambda\, (= \text{const}) \quad \text{in} \quad D$$

satisfying the constraint and B.C.

If such z exists, D (or (A_-, A_+)) is called **calibrable** (or a **Cheeger** set).

Proof. Note that

$$\mathcal{L}^n(D)\lambda = \int_D w \leq \|w\| \left(\mathcal{L}^n(D)\right)^{1/2} \text{ (Schwarz).}$$

If w_0 is constant, the min value of $\|w\|$ is attained.

Examples $(\mathbf{T}^n = (\mathbf{R}/\mathbf{Z})^n)$

(1) $(A_-, A_+) = \left((B_R(0))^c, \emptyset\right) \quad 0 < R < 1/2$ so that facet is a ball

$z = -x/R$ fulfills BC and the constraint.

Thus $w_0 = -n/R$.

(2) (annulus) $(A_-, A_+) = \left(B_{R_-}(0)^c, \operatorname{int} B_{R_+}(0)\right)$

$0 < R_+ < R_- < 1/2.$

Take z of the form $z = \nabla\varphi$ and solve

$$\begin{cases} -\Delta\varphi = \lambda \\ \dfrac{\partial\varphi}{\partial\nu} = \pm 1 \quad \text{on} \quad |x| = R_\pm . \end{cases}$$

- In general, a facet may not be calibrable. What is known in general is that $\Lambda \in L^\infty \cap BV$.

  [G. Bellettini – M. Novaga – M. Paolini, ARMA01]

  It can be discontinuous.

- There are several criteria so that the set is calibrable.

  [G. Bellettini et al, IFB, ’01]

  [B. Kawohl – T. Lachand-Robert ’06]

Monotonicity of Λ

Although it is difficult to calculate Λ in general, we have monotonicity with respect to a domain.

Theorem 3.9. Let $\left(A_-^i, A_+^i\right)$ be a smooth pair with $i = 1, 2$. Assume that $\overline{A_-^2} \subset A_-^1$, $\overline{A_+^1} \subset A_+^2$. Then

$$\Lambda[A_-^1, A_+^1] \leq \Lambda[A_-^2, A_+^2] \quad \text{a.e.}$$

on $D^1 \cap D^2$ where $D^i = \left(A_-^i\right)^c \cap \left(A_+^i\right)^c$.

Idea of proof

We use the resolvent approximation

$$f_a^i = (I + a\partial E)^{-1} f^i$$

$$\frac{f_a^i - f^i}{a} \to -\partial^0 E\left(f^i\right) \quad \text{in} \quad L^2$$

- We construct a support function having an order $f^1 \le f^2$ with $f^i \in D(\partial E)$ for $i = 1, 2$.
- Since the problem is 2nd order we have order preserving property $f_a^1 \le f_a^2$. $\square$

(cf. Y. G. – M. Gurtin – J. Matthias ’98 using semigroup. It also constructs a crystalline algorithm, which may not satisfy comparison principle)

Order preserving property for resolvent problem

Lemma 3.10. If $f^1 \leq f^2$ a.e. and $f^i \in L^2(\mathbf{T}^n)$

$$f_a^1 \leq f_a^2 \text{ a.e.}$$

V. Caselles – A. Chambolle '06

A simple idea is that we approximate E by smooth energy so that the resolvent problem is a uniformly elliptic problem which preserves ordering.

Corollary 3.11. If f is Lipschitz, then f_a is Lipschitz. Moreover,

$$\|\nabla f_a\|_\infty \leq \|\nabla f\|_\infty.$$

This is a simple application of Lemma 3.10 by taking

$$f^1 = f, f^2(x) = f(x+h) + Lh.$$

4. Viscosity approach in the case of total variation flow

We should discuss for total variation flow for non-divergence type. Typically

$$u_t = \sqrt{1+|\nabla u|^2}\left(\mathrm{div}\,\frac{\nabla u}{|\nabla u|}\right) \qquad (*)$$

which is the graph representation of

$$V = H_\gamma \ \text{ on } \ \Gamma_t \subset \mathbf{R}^{n+1}$$

with $\gamma(p, p_{n+1}) = |p| + |p_{n+1}|$.

Main unique existence result
(M.-H. Giga, Y. Giga, N. Pozar '13)

We impose a periodic boundary condition:

$\mathbf{T}^n = \prod_{i=1}^{n}(\mathbf{R} / \omega_i \mathbf{Z}), \omega_i > 0$ periodic cell.

Theorem 4.1 [GGPo13, JMPA]. For $u_0 \in C(\mathbf{T}^n)$ there is a unique viscosity solution $u \in C(\mathbf{T}^n \times [0, \infty))$ (defined later) for (*) with initial data u_0.

If $u_0 \in \mathrm{Lip}(\mathbf{T}^n)$, then

$$\|\nabla u\|_\infty(t) \le \|\nabla u_0\|_\infty.$$

Remark. (i) More general equation

$$u_t + F\left(\nabla u, \operatorname{div}\frac{\nabla u}{|\nabla u|}\right) = 0 \qquad \text{(E)}$$

can be handled provided that it is degenerate parabolic.

(ii) $\operatorname{div}\frac{\nabla u}{|\nabla u|} = -\frac{\delta E}{\delta u}$ can be generalized by $E(u) = \int e(\nabla u)$ with $\partial e(0) = \{|p| \leq 1\}$.

e.g. $e(p) = |p| + |p|^q, q > 1$.

Assumptions on F

(F1) $F: \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ is continuous

(F2) $X \longmapsto F(p, X)$ is nonincreasing

(⇒ the equation is at least degenerate parabolic.)

Example

$$u_t - \sqrt{1 + |\nabla u|^2} \left\{ \operatorname{div} \frac{\nabla u}{|\nabla u|} + \sigma \right\} = 0$$

σ: a given constant

Unique Solvability

Theorem 4.2 [GGPo13, JMPA]. Assume (F1), (F2). Then their exists a unique global-in-time continuous **viscosity** solutions of (E) for any initial data $u_0 \in C(\mathbf{T}^n)$. If $u_0 \in \mathrm{Lip}(\mathbf{T}^n)$ then

$$\|\nabla u\|_\infty(t) \leq \|\nabla u_0\|_\infty$$

Remark. $\mathrm{div}\,\dfrac{\nabla u}{|\nabla u|}$ can be generalized to $\mathrm{div}\nabla_p W(\nabla u)$ where $W \in C^2(\mathbf{R}^n \setminus \{0\})$ is convex and 1-homogeneous [GGPo13, AMSA].

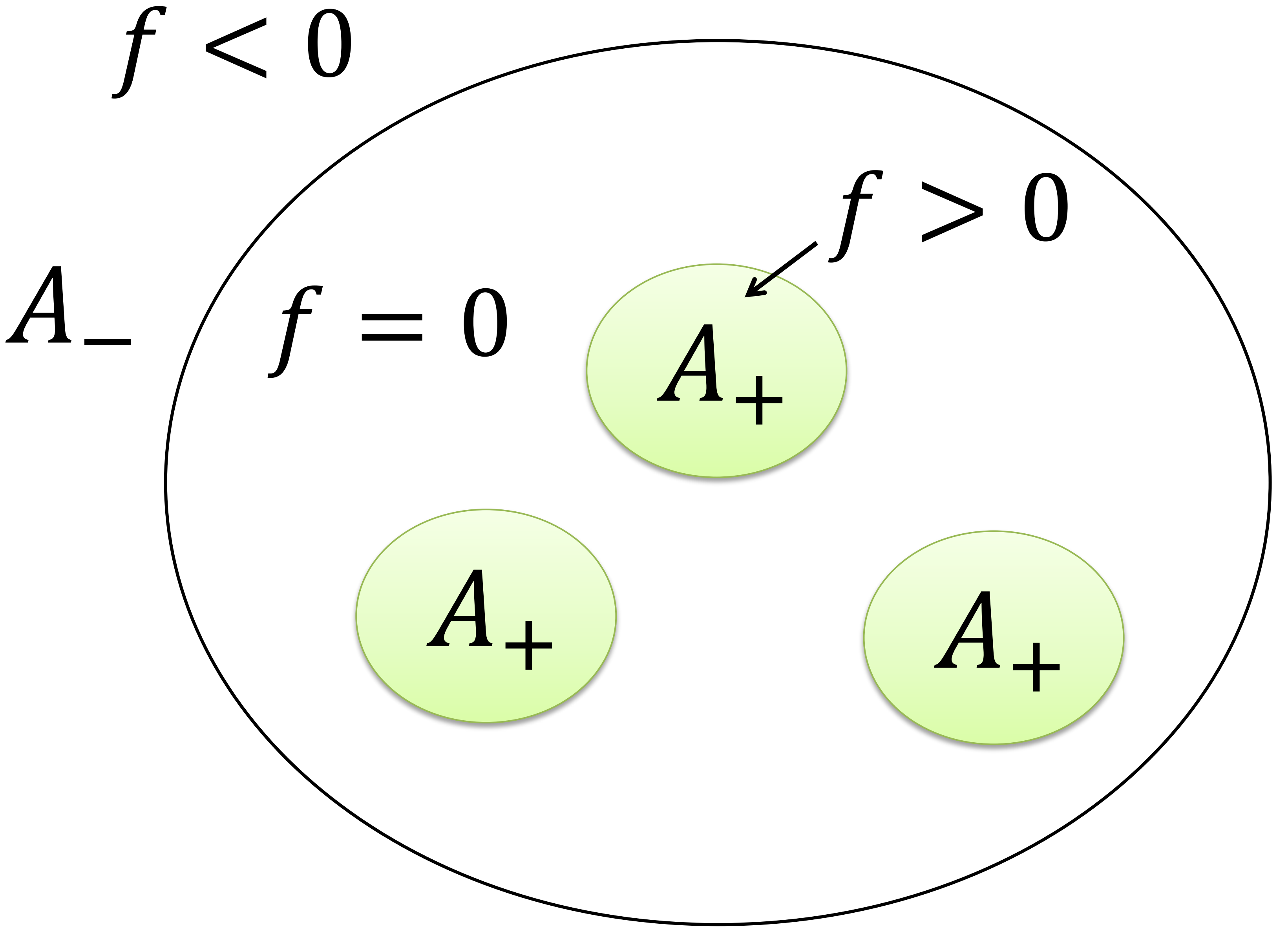
4.1 Definition of viscosity solutions

Support function

Let (A_-, A_+) be an (open) pair in $\mathbf{T}^n$. A Lipschitz function f is said to be a **support function** of (A_-, A_+) if

$$A_- = \{x | f(x) < 0\}, A_+ = \{x | f(x) > 0\}.$$

We also write $\mathrm{Pair}\, f = (A_-, A_+)$.


$f < 0$
A_-
$f = 0$
$f > 0$
A_+
A_+
A_+

Admissible and smooth pair

A pair is said to be **admissible** if there is a support function f belonging to the domain of ∂E, i.e. $f \in D(\partial E)$ where E is the total variation energy. A pair (A_-, A_+) is said to be **smooth** if $\mathrm{dist}(A_-, A_+) > 0$ and $\partial A_-, \partial A_+$ are smooth.

Lemma 4.3. A smooth pair is admissible.

Idea of proof. Take f as a signed distance function of "facet" $A_-^c \cap A_+^c$ near the boundary of the facet.

Admissible faceted test function

For a given (admissible) pair (A_-, A_+) a function $\varphi(x,t) = f(x) + g(t)$, $f \in \mathrm{Lip}(\mathbf{T}^n)$, $g \in C^1(\mathbf{R})$ is called an **admissible faceted test function at** $x_0 \in \mathrm{int}\,(A_-^c \cap A_+^c)$ **with a pair** (A_-, A_+) if $f \in D(\partial E)$ is a support function of (A_-, A_+).

Definition of a subsolution

Definition 4.4. We say that $u \in USC(\mathbf{T}^n \times (0,T))$ is a subsolution of

$$\text{(E)}\ u_t + F\big(\nabla u, \operatorname{div}(\nabla u \,/\, |\nabla u|)\big) = 0$$

$$\text{in} \quad \mathbf{T}^n \times (0,T)$$

($F\colon \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ is assumed to be continuous and nonincreasing in the second variable.)

if $\max\limits_{\mathbf{T}^n \times (0,T)} (u-\varphi) = (u-\varphi)(x_0, t_0)$ with an admissible faceted test function $\varphi = f + g$ at (x_0, t_0) with a smooth pair (A_-, A_+) always implies

(a) $g'(t_0) + F\left(0, \textbf{ess.inf}_{B_{\eta/2}(x_0)} \boldsymbol{\Lambda}[A_-, A_+]\right) \le 0$ when (u, φ) is in **general position** at (x_0, t_0) with $\eta > 0$ or

(b) $g'(t_0) + F\big(\nabla f(x_0), \operatorname{div}(\nabla f(x_0) \,/\, |\nabla f(x_0)|)\big) \le 0$ when $f \in C^2$ near x and $\nabla f(x_0) \ne 0$.

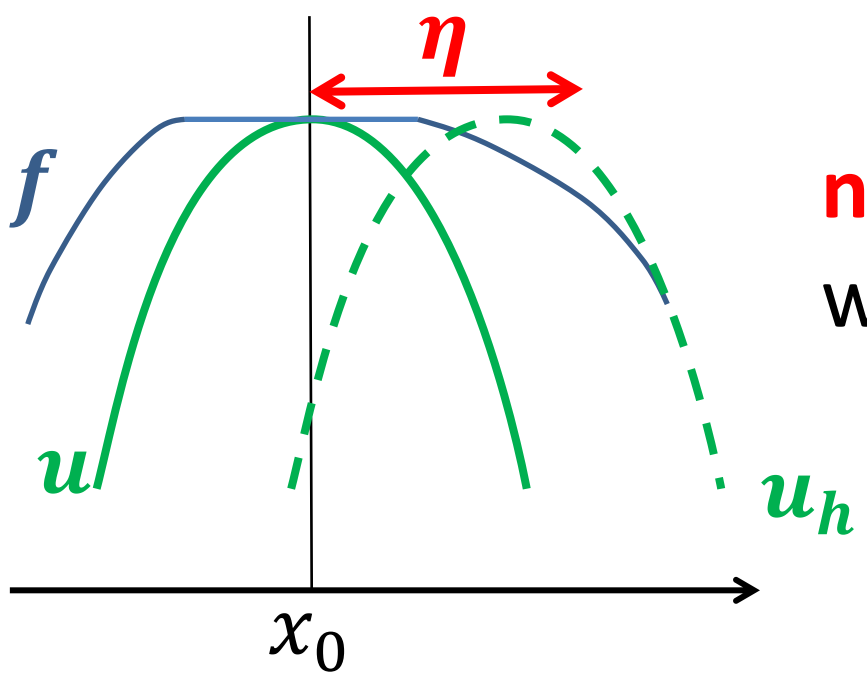
[Notion of general position is convenient to prove "stability" and no severe restriction for "comparison".]

General position

We say that (u, φ) is in **general position** at (x_0, t_0) with $\eta > 0$ if

$$\max(u_h - \varphi) = (u - \varphi)(x_0, t_0)$$

for all $|h| \leq \eta$, where $u_h(x, t) = u(x - h, t)$ and max is taken over $\mathbf{T}^n \times [t_0 - \eta, t_0 + \eta]$.

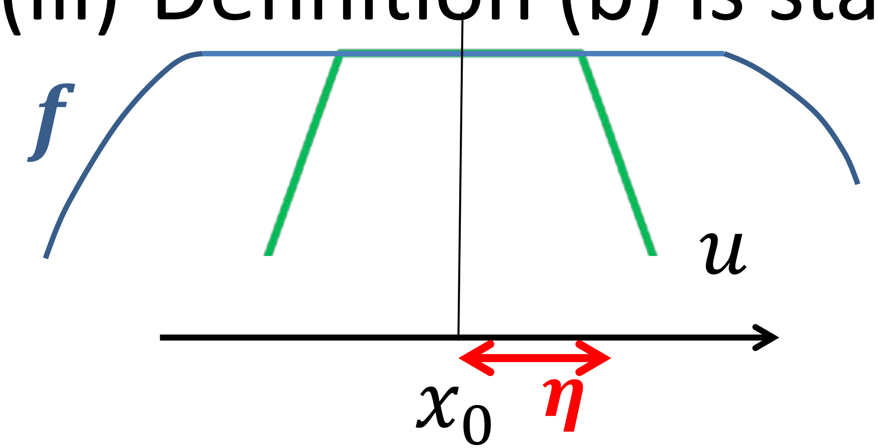


not in general position with respect to η

Remark. (i) Definition for a supersolution is symmetric. Replace max by min and $\leq$ by ≥ 0. If u is both sub- and supersolution, we say that u is a **viscosity** solution.

(ii) We say that $u \in C(\mathbf{T}^n \times (0,T))$ is a solution of (E) if u is both a sub- and supersolution.

(iii) Definition (b) is standard.



at $t = t_0$
(general position with **small** η)

4.2 Comparison principle

Only periodic case is proved.

Theorem 4.5. Let $F \in C(\mathbf{R}^n \times \mathbf{R})$ be non increasing in the second variable (so that (E) is degenerate parabolic). Let $u \in USC\,(\bar{Q})$, $Q = \mathbf{T}^n \times (0, T)$ and $v \in LSC(\bar{Q})$ be a sub- and super-solution of (E). If $u \leq v$ at $t = 0$, then $u \leq v$ in Q.

The proof is far from trivial.

A flavor of the proof

The proof is very involved.

Argument by contradiction

1. Doubling variable procedure (standard)
2. Flattening procedure

   Exclude the situation which easily yields a contradiction by a classical method to prove the standard comparison principle for a second order problem.

   Y. G. Chen – Y. G. – S. Goto ’91, S. Goto ’94,
   M.-H. Giga – Y. G. ’98.
3. Construction of facets in between

   Flattening procedure leaves a space to construct a facet between those of u and v. Monotonicity of nonlocal curvature yields a contradiction.

A few words for the existence

1. Approximation by a smoother problem
2. Stability

"Let u_m be a subsolution of the approximate problem (associate with energy E_m). Then the limit as $m \to \infty$ is subsolution of the original problem."

We approximate a test function $\varphi(x,t) = f(x) + g(t)$ by $\varphi_{a,m} = f_{a,m} + g$, where

$$f_a = (I + a\,\partial E)^{-1} f,$$

$$f_{a,m} = (I + a\,\partial E_m)^{-1} f.$$

We first send $m \to \infty$ and then send $a \to 0$. (We need to replace f by εf to make the slope small as we like near a facet.)

4.3 Further difficulty for crystalline flow

$$E(f) := \begin{cases} \int_\Omega W(\nabla f) & f \in BV \cap L^2(\Omega) \\ \infty & \text{otherwise} \end{cases}$$

We have to consider anisotropic total variation. Here W is convex and 1-homogeneous, piecewise linear.

Instead of considering smooth pair, we have to consider admissible pair.

Definition 3.12. Let Ω be a bounded domain in $\mathbf{R}^n$. We say that (A_-, A_+) is **admissible** if

(i) $\operatorname{dist}(A_-, A_+) > 0$

(ii) There is $\psi \in D(\partial E)$ which is Lipschitz and a support function of (A_-, A_+).

(iii) Moreover, $\partial A_-, \partial A_+$ in $\mathbf{R}^n$does not touch $\partial\Omega$.

Further difficulty for crystalline flow

- One has to approximate a general pair by admissible pair. This is so far only successful in the case when the pair is in $\mathbf{R}^2$.
- We have to stratify the definition.
- Construction is by approximation by smoothing W. Perron's method does not work.

5. Other approaches

5.1 Other notions of solutions

So far we have explained two notions of solutions

- viscosity solution
- gradient flow based by the theory of monotone operators

There are a few other notions of solutions where comparison principle has been proved

- crystalline flow for curves
- a solution based on distance functions (explained below)

ϕ-regular flow
(G. Bellettini – M. Novaga ’00, MMMA)

Equation $V = \gamma H_\gamma,\ H_\gamma = -\mathrm{div}\nabla_p \gamma(\vec{n})$
(anisotropic curvature flow in $\mathbf{R}^n$)

$\phi = \gamma^0$: support function of $\{p | \gamma(p) \leq 1\}$
(Frank diagram)

$$\gamma^0(x) = \sup\{\langle x, p\rangle | \gamma(p) \leq 1\}$$

Here γ: convex, 1-homogeneous function
(given interfacial energy density)

ϕ-regular flow (continued)

Definition 5.1. Let $T > 0$. We say the mapping $t(\in [0,T]) \to E(t) \subset \mathbf{R}^n$ is ϕ-regular flow if (i), (ii) hold.

(i) There exists an open set $A \subset \mathbf{R}^n \times [0,\infty)$ such that $\bigcup_{t\in[0,T]}(\partial E(t) \times \{t\}) \subset A$ and $d_\phi(z,t) = \mathrm{dist}_\phi\big(z, E(t)\big) - \mathrm{dist}_\phi(z, E(t)^c)$ is Lipschitz in A (space-time).

ϕ-regular flow (continued)

(ii) There exists a bounded vector field $m: A \to \mathbf{R}^n$ such that $m \in \partial\gamma(\nabla d_\phi)$ a.e. in A and there exists $\lambda > 0$ such that

$$\left|\frac{\partial d_\phi}{\partial t}(z,t) - \operatorname{div} m(z,t)\right| \leq \lambda \left|d_\phi(z,t)\right|$$

for a.e. $(z,t) \in A$.

Remark. Comparison principle has been established by a reaction diffusion approximation. However, there is no general existence result for $n \geq 3$ except convex initial data.

5.2 Related topics

5.2.1 Graph case

Spatially inhomogeneous problem

Even one dimensional problem is quite involved. Consider

$$u_t + F(u_x, (W'(u_x))_x + \sigma) = 0 \text{ in } \mathbf{T} \times (0, \infty)$$

where σ is Lipschitz in x and uniformly in time. Here W is convex and W' has a discrete set P of jumps ($\sup_{K \setminus P} W''$ is bounded for every compact set K).

M.-H. Giga – Y. G. – P. Rybka '11 comparison and example

M.-H. Giga – Y. G. – A. Nakayasu '13

existence by Perron – Ishii's method

Boundary value problem

This is so far not well studied because of difficulty. Comparison principle for a domain to total variation flow of non-divergence type is not known.

Approximation

Deterministic game approximation for crystalline curve evolution.

(R. V. Kohn – S. Serfaty ’06)

(M.-H. Giga – Y. G. work in progress)

5.2.2 Approximation of solution

(1) Allen – Cahn type approximation

$$u_t - \Delta u + \frac{\varphi'(u)}{\varepsilon^2} = 0$$

φ: double well potential

$$\varphi(u) = \frac{1}{4}(1 - u^2)^2$$

"Anisotropic version converges to anisotropic men curvature flow", The convergence is uniform with respect to γ provided $\lambda|p| \leq \gamma(p) \leq \Lambda|p|$

Y. G. – T. Ohtsuka – R. Schätzle '06.

(2) Almgren – Taylor – Wang approach and Chambolle scheme

- F. Almgren – J. Taylor – L. Wang '93

$K \subset \mathbf{R}^2$ bounded measurable set

$$F_h(K) := \arg\min \left\{ \mathrm{Per}(\partial L) + \frac{1}{h} \int_{L \Delta K} \mathrm{dist}(x, \partial K) dx \right\}$$

L is of finite perimeter, bounded and measurable

"$K^t = F_h^{[t/h]}(K_0)$ approximates the solution of the mean curvature flow equation starting from K_0" as $h \to 0$.

$F_h(K)$ may not be unique!

- Chambolle's scheme '04

$$E_0 \subset \mathbf{R}^n \text{ compact. } E_0 \subset \Omega$$

$$J_h(v) = \int_\Omega |\nabla v| + \frac{1}{2h} \left\| v - d_{E_0} \right\|_{L^2}^2$$

$$d_{E_0}(x) = \operatorname{dist}(x, E_0) - \operatorname{dist}(x, \mathbf{R}^n \setminus E_0).$$

$$T_h(E_0) := \left\{ w_{E_0}^h \leq 0 \right\}$$

when $w_{E_0}^h = \arg\min J_h$ (This minimizer is unique).

This is one selection of $F_h(E_0)$; moreover, $T_h(E_0)$ is **uniquely** determined. (Convex minimization)

"$E^t = T_h^{[t/h]}(E_0)$ approximate the solution"

Extension to unbounded set

Consider the resolvent equation

$$w - h\,\mathrm{div}\left(\frac{\nabla v}{|\nabla v|}\right) \ni d_E \quad \text{in} \quad \mathbf{R}^n.$$

Let $u^h(x,t) = \sup\left\{ \mu \in \mathbf{R} \,\middle|\, x \in T_h^{[t/h]}\left(\{u_0 \geq \mu\}\right)\right\}$ for a given uniformly continuous u_0.

Theorem 5.2 (T. Eto – Y. G. – K. Ishii). $u^h \to u$ locally uniformly in $\mathbf{R}^n \times [0,\infty)$ as $h \to 0$ and u solves

$$u_t - |\nabla u| = \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right)|\nabla u|$$

(Anisotropic extension is possible but singular case is not well understand except curve case (K. Ishii '13, NoDEA) or convex case (V. Caselles – A. Chambolle '06).)

5.3 Summary

- We show the well-posedness of the initial value problem for a level-set crystalline flow equation in $\mathbf{R}^3$ by extending the theory of viscosity solution.
- The difficulty stems from the fact that the speed is nonlocal and not constant on a “facet”.
- This difficulty has already arisen for total variation flow of divergence type. To handle crystalline flow a further difficulty for approximation arises.
- The existence is also nontrivial because the Perron method does not work.

Main Results (work in progress, GGPo)

Theorem 2.1 (Existence and Uniqueness). For a given initial data (a bounded open set) $D_0 \subset \mathbf{R}^3$ with its boundary Γ_0 there exists a global level-set flow D for **crystalline** mean curvature flow equation (ACF) (provided that f is at most linear growth in H_γ).

Theorem 2.2 (Stability). Assume the same hypotheses of Theorem 2.1. Let γ_ε be a smooth convex interfacial energy density approximate γ uniformly. Then the level-set flow D_ε converges to D in the Hausdorff distance sense in space-time provided that no fattening occurs.

(Underlying structure: comparison principle)

(work in progress, M.-H. Giga – Y. G. – N. Pozar)