Optimal Control and MPC for the Fokker-Planck equation

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New Perspectives in Optimal Control and Games

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UNIVERSITÄT BAYREUTH

ITN SADCO
Initial Training Network
Sensitivity Analysis for Deterministic Controller Design

Imperial College London

Outline

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1. Motivation
2. Model Predictive Control
3. Existing Works
4. New Results
5. Outlook

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Aim/Application

Consider an Optimal Control Problem (OCP)

$$\min_u \tilde{J}(X, u)$$

constrained to a Itô Stochastic Differential Equation (SDE)

$$dX_t = b(X_t, t; u)dt + \sigma(X_t, t)dW_t, \qquad X(t = 0) = x_0$$

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X_t random $\Rightarrow$ deterministic objective results in a random variable, i.e. The cost functional $\tilde{J}(X, u)$ is a random variable!

Standard vs Alternative approaches

Remedy: consider the averaged objective

$$\min_u \mathbb{E}[\tilde{J}(X,u)] = \min_u \mathbb{E}\left[\int_0^{T_E} L\,(t, X_t, u(t))\ \mathrm{d}t + \psi(X_{T_E})\right] .$$

A–B (2013): *"This formulation is omnipresent in almost all stochastic optimal control problems considered in the scientific literature."*

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Related works: deterministic objectives defined by the Kullback-Leibler distance (G. Jumarie 1992, M. Kárný 1996) or the square distance (M.G. Forbes, M. Guay, J.F. Forbes 2004, Wang 1999) between the state PDF and a desired one. However, stochastic models needed to obtain the PDF by averaging or by an interpolation

The Fokker-Planck Equation /1

A new approach by Annunziato and Borzì (2010, 2013):
Reformulate the objective using the underlying PDF

$$y(x,t) := \int_\Omega \tilde{y}(x,t;z,0)\rho(z,0)\mathrm{d}z$$

$t > 0$, $\rho(z,0)$ given initial density probability,
$\tilde{y}$ transition density probability distribution function

$$\tilde{y}(x,t;z,s) := \mathbb{P}\{X(t) \in (x, x + \mathrm{d}x) : X(s) = z\}, \quad t > s$$

and control the PDF directly.

The next essential step:

the evolution of the PDF is governed by (cf Da Prato - Zabczyk) the Fokker-Planck Equation (deterministic parabolic PDE)

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The Fokker-Planck Equation /2

The evolution of the PDF is modeled by the

Fokker-Planck Equation

$$\begin{cases} \partial_t y(x,t) - \frac{1}{2}\partial_{xx}^2\Big(\sigma(x,t)^2 y(x,t)\Big) + \partial_x\Big(b(x,t;u))y(x,t)\Big) = 0 \\ y(\cdot,0) = y_0 \end{cases}$$

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where $y\colon \mathbb{R}\times[0,\infty[\to\mathbb{R}_{\geq 0}$ is the PDF $\left(\int_{\mathbb{R}} y(x,t)\mathrm{d}x = 1 \quad \forall t>0\right)$,

$y_0\colon \mathbb{R}\to\mathbb{R}_{\geq 0}$ is the initial PDF $\left(\int_{\mathbb{R}} y_0(x)\,\mathrm{d}x = 1\right)$, and

$\sigma\colon \mathbb{R}\times[0,\infty[\to\mathbb{R}$ and $b\colon \mathbb{R}\times[0,\infty[\times\mathbb{R}\to\mathbb{R}$
are given by the SDE

A MPC−Fokker-Planck control framework

Deterministic PDE-constrained Optimal Control Problem

$\min_u \mathbb{E}[\tilde{J}(X,u)]$		$\min_u J(y,u)$
s.t. Itô SDE	$\rightsquigarrow$	s.t. Fokker-Planck PDE

Rmk: the class of objectives described by $\min_u J(y,u)$ is larger then that expressed by $\min_u \mathbb{E}[\tilde{J}(X,u)]$, indeed

$$\mathbb{E}\left[\int_0^{T_E} L(t,X_t,u(t)) + \psi(X_{T_E})\right] = \int_{\mathbb{R}^d}\int_0^{T_E} L(t,x,u(t))y(t,x) + \int_{\mathbb{R}^d}\psi(x)y(T_E,x)$$

Control method: Model Predictive Control (MPC)

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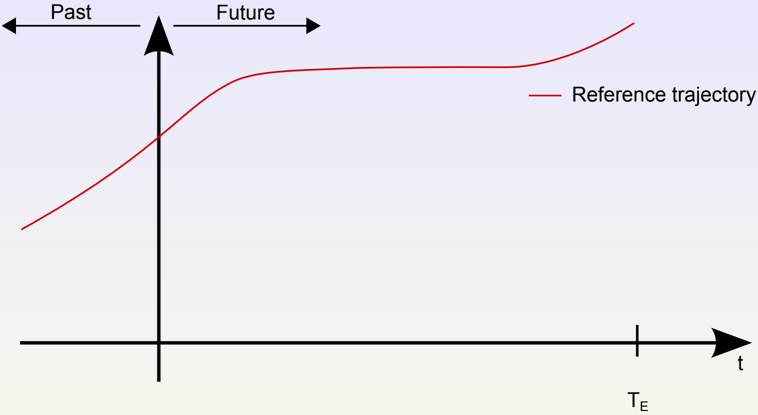
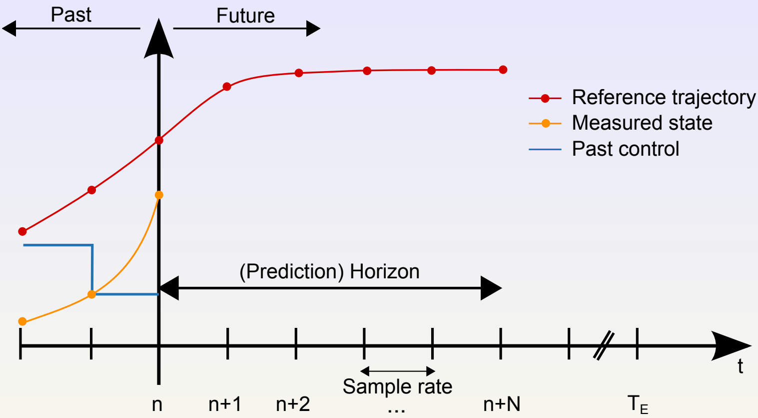
Outline

Basic idea of MPC

OCP on a long (possibly infinite) time horizon	$\rightsquigarrow$	Several iterative OCPs on (shorter) finite time horizons

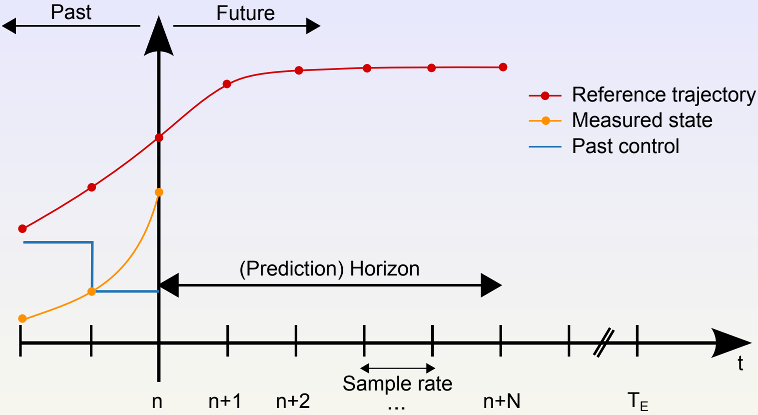


Consider an optimal control problem on $[0, T_E]$.

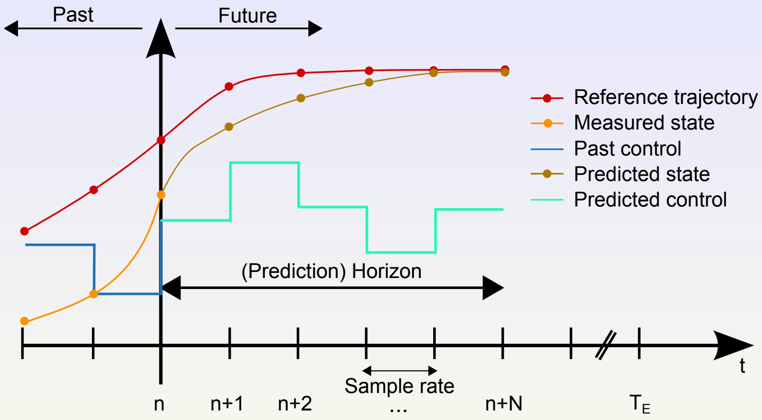


Choose a horizon $N \in \mathbb{N}$ and a sample rate $T > 0$.

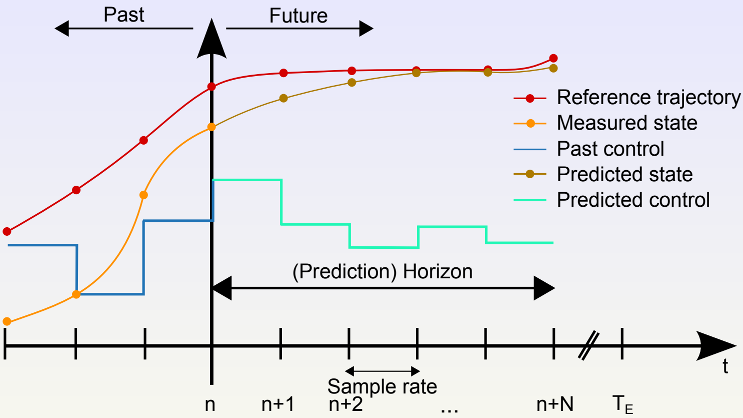
For each time $t_n := nT, n = 0, 1, 2, \ldots$:



1. Measure the current state $y(n)$.
2. Set $y_0 := y(n)$ and solve the optimal control problem on the current time horizon $[t_n, t_{n+N}]$.



3. Denote the calculated optimal control sequence by $u^*(\cdot)$ and apply its first value $u^*(0)$ on $[t_n, t_{n+1}]$.



4. If $t_{n+1} < T_E$, set $n := n + 1$ and go to 1. Otherwise end.

Adaption of http://en.wikipedia.org/wiki/File:MPC_scheme_basic.svg (CC BY-SA 3.0)

Outline

Existing Work

Problem [Annunziato and Borzì, 2010, 2013]

Track a desired PDF over a given time interval.

Optimal Control Problem

$\Omega \subset \mathbb{R}$ open interval, $u_a, u_b \in \mathbb{R}$ with $u_a < u_b$, $y_d \in L^2(\Omega)$ and $\lambda > 0$.
Consider the following OCP on $[t_n, t_{n+1}]$:

$$\min_u J(y, u) := \frac{1}{2} \|y(\cdot, t_{n+1}) - y_d(\cdot, t_{n+1})\|^2_{L^2(\Omega)} + \frac{\lambda}{2}|u|^2$$

s.t.

$$\begin{cases} \partial_t y - \frac{1}{2}\partial^2_{xx}\left(\sigma^2 y\right) + \partial_x\left(b(u)y\right) = 0 & \text{in } Q_n := \Omega \times (t_n, t_{n+1}) \\ y(\cdot, t_n) = y_n & \text{in } \Omega \\ y = 0 & \text{in } \Sigma_n := \partial\Omega \times (t_n, t_{n+1}) \end{cases} \tag{1}$$

$$u \in U_{ad} := \{u \in \mathbb{R} \mid u_a \le u \le u_b\}$$

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Theoretical results [Annunziato and Borzì, 2013]

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Assume $\sigma(x,t) \equiv \bar{\sigma} > 0$ and $b(x,t;u) := \gamma(x) + u(t)$ with $\gamma \in C^1(\Omega)$, sufficiently small $\bar{\gamma} := \max_{x\in\Omega}(|\gamma(x)|, |\gamma'(x)|)$, $y_n \in H_0^1(\Omega)$. Then:

- For every $u \in \mathbb{R}$ the initial boundary value problem (1) has a unique (weak) solution y.
- The OCP admits a (locally) optimal solution. Furthermore, it is unique if $\left\| y_n - y_{d,n} \right\|_{L^2(\Omega)}$ sufficiently small or λ sufficiently large.

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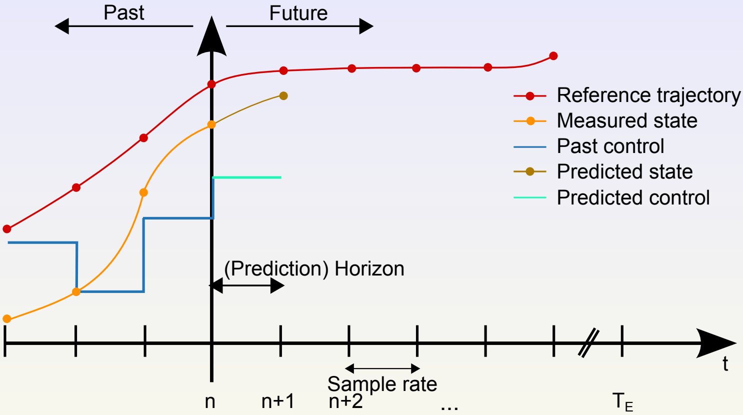

Adaption of http://en.wikipedia.org/wiki/File:MPC_scheme_basic.svg (CC BY-SA 3.0)

A Fokker-Planck optimality system

$N = 1$: the first order necessary optimality conditions solve the following optimality system

$$
\begin{aligned}
\partial_t y - \frac{1}{2}\partial_{xx}^2\left(\sigma^2 y\right) + \partial_x\left(b(u)y\right) &= 0 && \text{in } Q_n\\
y(\cdot, t_n) &= y_n && \text{in } \Omega\\
y &= 0 && \text{in } \Sigma_n\\
-\partial_t p - \frac{1}{2}\sigma^2\partial_{xx}^2 p - b(u)\partial_x p &= 0 && \text{in } Q_n\\
p(\cdot, t_{n+1}) &= y(\cdot, t_{n+1}) - y_d(\cdot, t_{n+1}) && \text{in } \Omega\\
p &= 0 && \text{in } \Sigma_n\\
\lambda u - \int_{Q_n} \partial_x\left(\frac{\partial b}{\partial u} y\right) p \,\mathrm{d}x\,\mathrm{d}t &= 0
\end{aligned}
$$

Discretization of the optimality system

- Since

$$\forall t \geq 0 : \int_\Omega y(x,t)\,\mathrm{d}x = 1, \quad \forall x,t : \; y(x,t) \geq 0$$

are required, a conservative space discretization scheme is needed.
  - ⇝ Preventive Lax-Friedrichs flux splitting (one-dimensional) [Annunziato and Borzì, 2010]
  - ⇝ Chang-Cooper scheme (multi-dimensional) [Annunziato and Borzì, 2013]

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First improvement

Joint work with Arthur Fleig and Lars Grüne
from University of Bayreuth

Implement an MPC scheme for a larger horizon $N > 1$
(small gain, yet effective) but

Note

Increasing N entails a different objective functional

$$J(y,u) := J_N(y,u) := \frac{1}{2}\sum_{n=0}^{N-1}\left(\|y - y_d\|^2_{L^2(Q_n)} + \lambda |u(t_n)|^2\right),$$

thus a different optimality system!

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Numerical Example

Consider the Ornstein-Uhlenbeck process with

$$\sigma(x,t) \equiv \bar{\sigma} = 0.8, \qquad b(x,t,u) := u - x$$

on $\Omega :=]-5, 5[$ with $u_a = -10$, $u_b = 10$, $\lambda = 0.1$, and $T_E = 5$.

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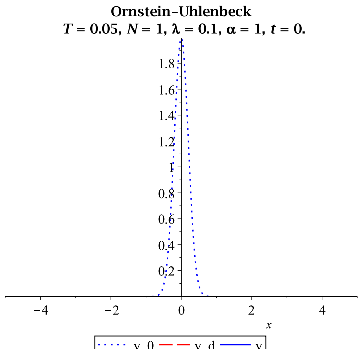
The target and initial PDF are given by

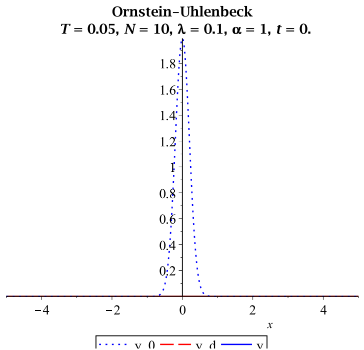
$$y_d(x,t) := \frac{\exp\left(-\frac{[x-2\sin(\pi t/5)]^2}{2\cdot 0.2^2}\right)}{\sqrt{2\pi \cdot 0.2^2}}$$

and

$$y_0(x) := y_d(x,0) = \frac{\exp\left(-\frac{x^2}{2\cdot 0.2^2}\right)}{\sqrt{2\pi \cdot 0.2^2}},$$

respectively.


Ornstein–Uhlenbeck
T = 0.05, N = 1, λ = 0.1, α = 1, t = 0.
x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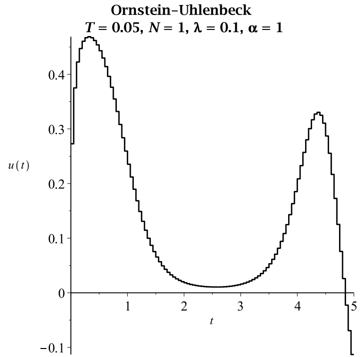
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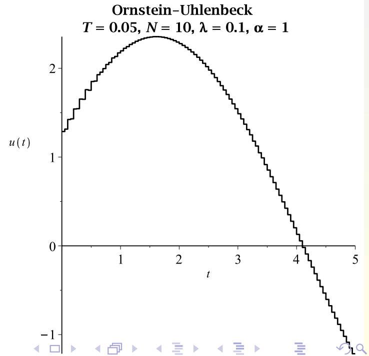
t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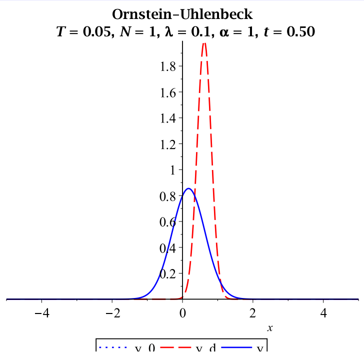
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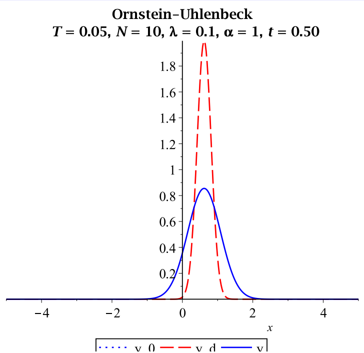
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
−4
−2
0
2
4
x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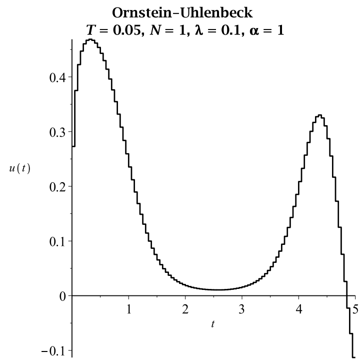
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T = 0.05, N = 1, λ = 0.1, α = 1
u(t)
0.4
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0
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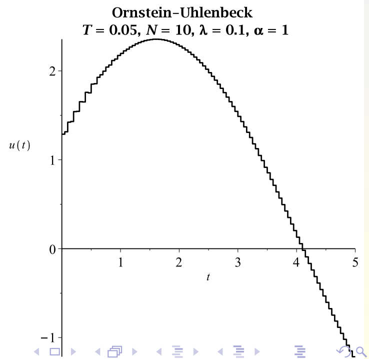
t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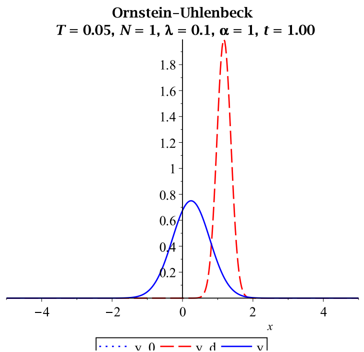
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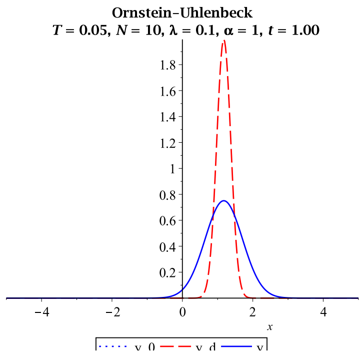
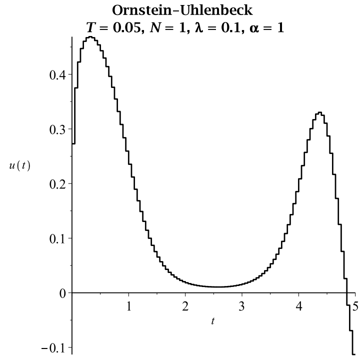
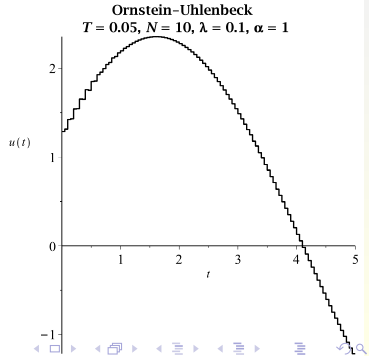
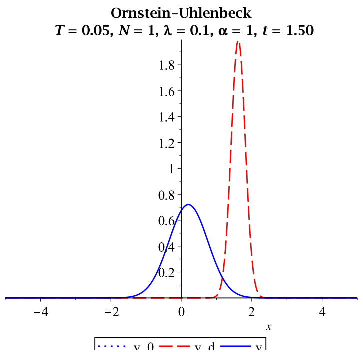
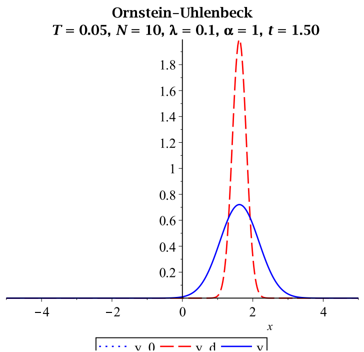
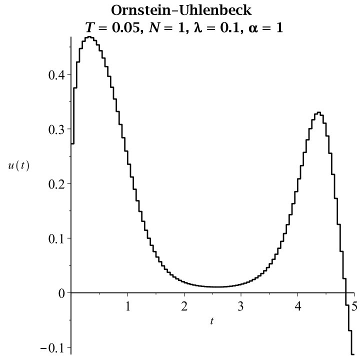

Ornstein–Uhlenbeck
T = 0.05, N = 10, λ = 0.1, α = 1, t = 1.00
x

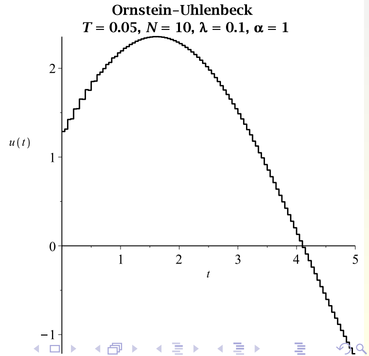

Ornstein–Uhlenbeck
T = 0.05, N = 1, λ = 0.1, α = 1
u(t)
t


Ornstein–Uhlenbeck
T = 0.05, N = 10, λ = 0.1, α = 1
u(t)
t


Ornstein–Uhlenbeck
T = 0.05, N = 1, λ = 0.1, α = 1, t = 1.50
x


Ornstein–Uhlenbeck
T = 0.05, N = 10, λ = 0.1, α = 1, t = 1.50
x


Ornstein–Uhlenbeck
T = 0.05, N = 1, λ = 0.1, α = 1
u(t)
t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T = 0.05, N = 10, λ = 0.1, α = 1
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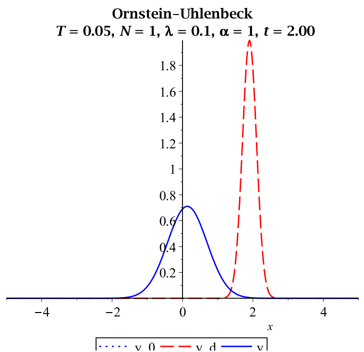
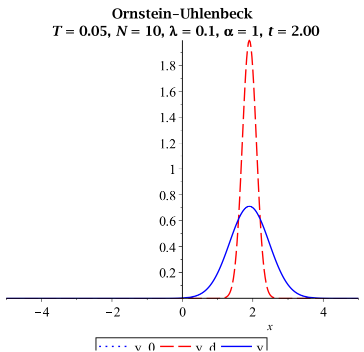
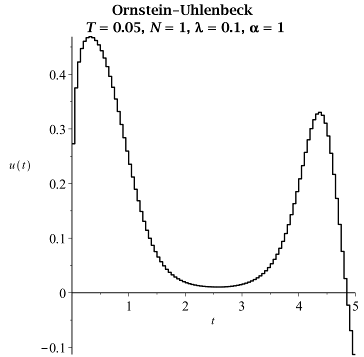

Ornstein–Uhlenbeck
T = 0.05, N = 1, λ = 0.1, α = 1, t = 2.00
x


Ornstein–Uhlenbeck
T = 0.05, N = 10, λ = 0.1, α = 1, t = 2.00
x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T = 0.05, N = 1, λ = 0.1, α = 1
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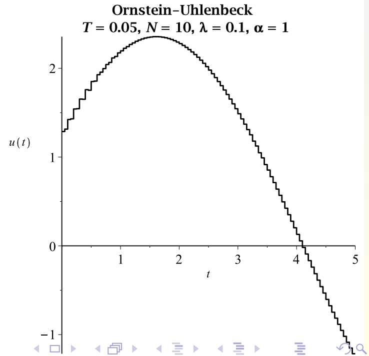
t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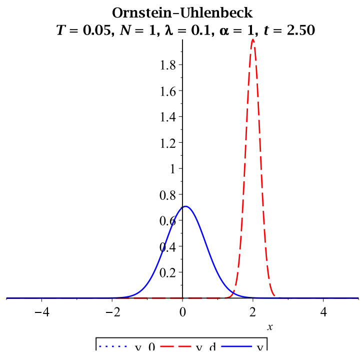
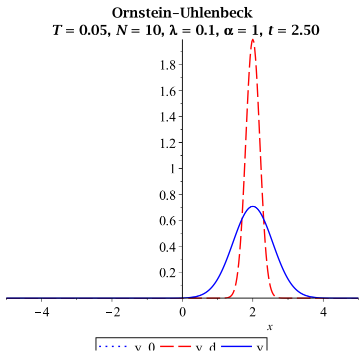
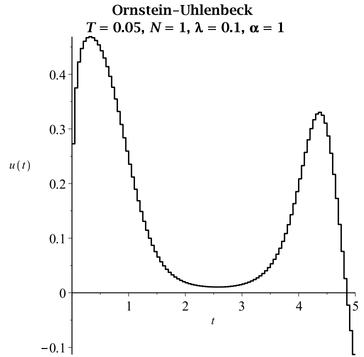
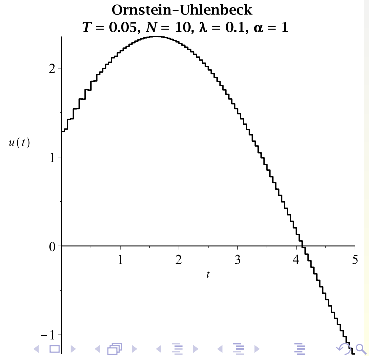
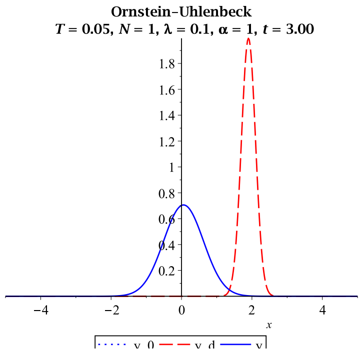
T = 0.05, N = 10, λ = 0.1, α = 1
u(t)
t

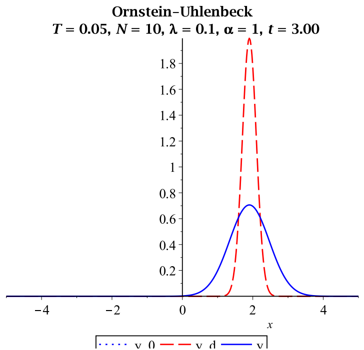

Ornstein–Uhlenbeck
T = 0.05, N = 1, λ = 0.1, α = 1, t = 2.50
x


Ornstein–Uhlenbeck
T = 0.05, N = 10, λ = 0.1, α = 1, t = 2.50
x


Ornstein–Uhlenbeck
T = 0.05, N = 1, λ = 0.1, α = 1
u(t)
t


Ornstein–Uhlenbeck
T = 0.05, N = 10, λ = 0.1, α = 1
u(t)
t


Ornstein–Uhlenbeck
$T = 0.05$, $N = 1$, $\lambda = 0.1$, $\alpha = 1$, $t = 3.00$
x
v 0
v d
v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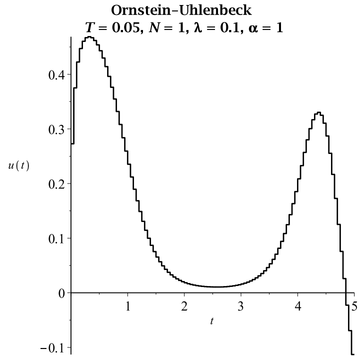
v 0
v d
v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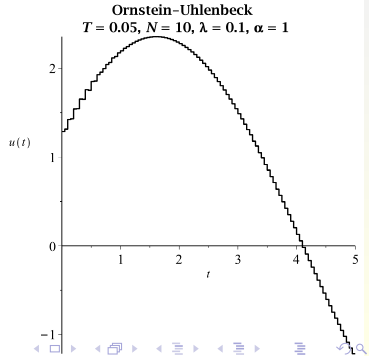
t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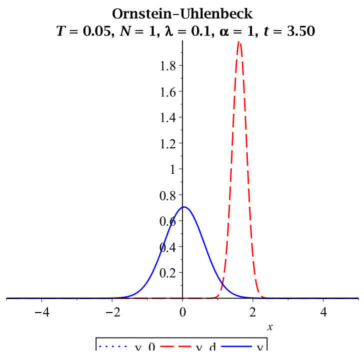
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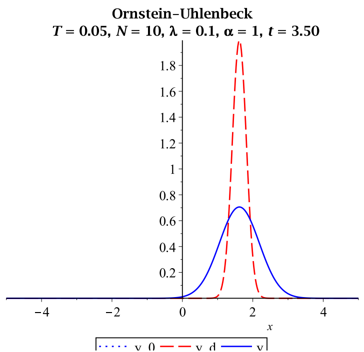
T = 0.05, N = 1, λ = 0.1, α = 1, t = 3.50
x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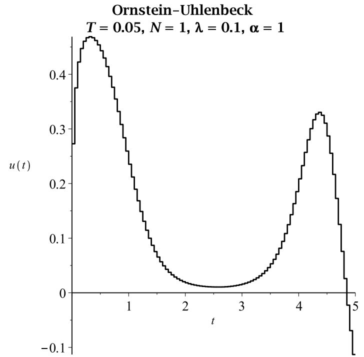
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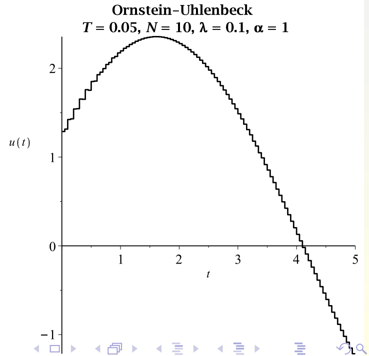
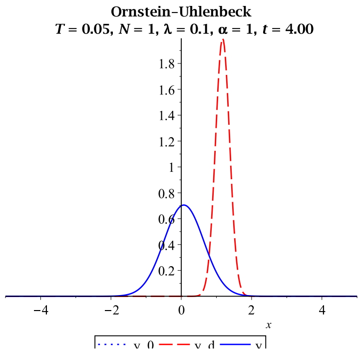
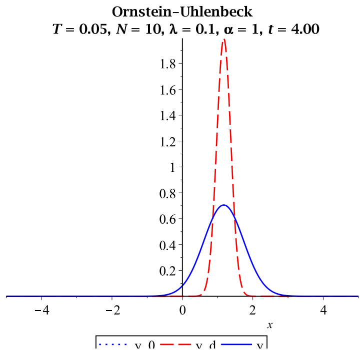
u(t)
t


Ornstein–Uhlenbeck
T = 0.05, N = 10, λ = 0.1, α = 1
u(t)
t


Ornstein–Uhlenbeck
T = 0.05, N = 1, λ = 0.1, α = 1, t = 4.00
x


Ornstein–Uhlenbeck
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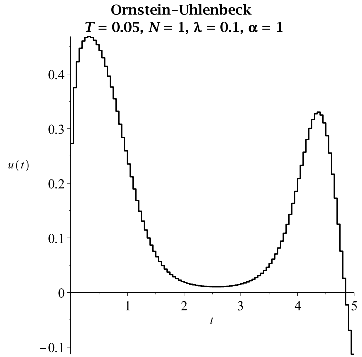
x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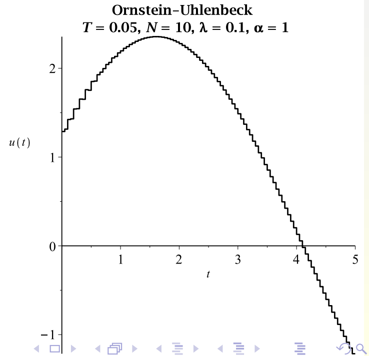
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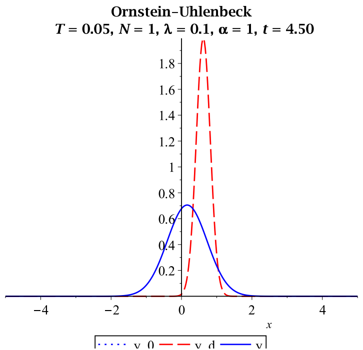
t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T = 0.05, N = 1, λ = 0.1, α = 1, t = 4.50
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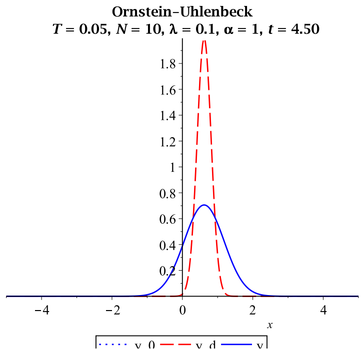
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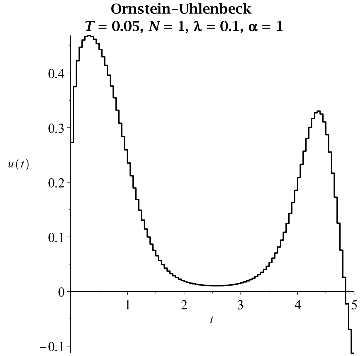
Ornstein–Uhlenbeck
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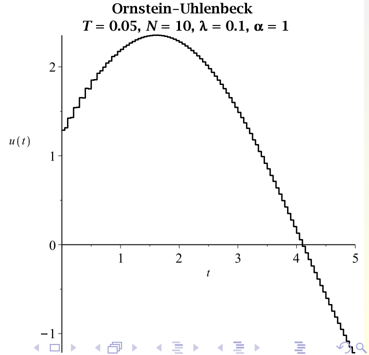
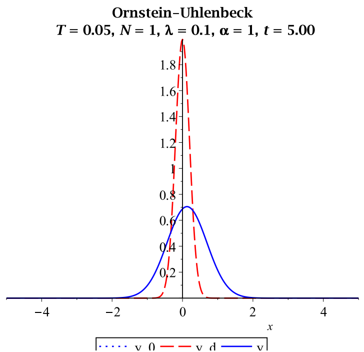
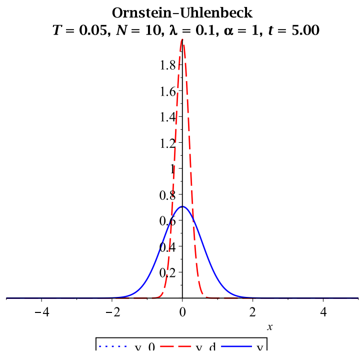
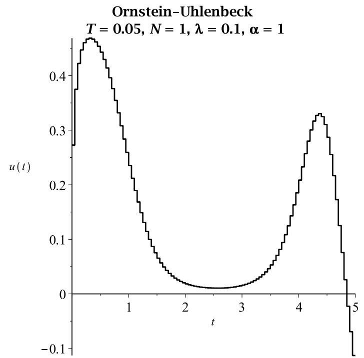
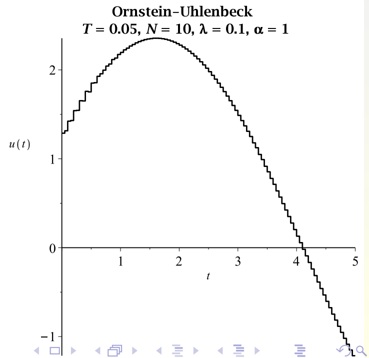
u(t)
t


Ornstein–Uhlenbeck
T = 0.05, N = 10, λ = 0.1, α = 1
u(t)
t


Ornstein–Uhlenbeck
T = 0.05, N = 1, λ = 0.1, α = 1, t = 5.00
x


Ornstein–Uhlenbeck
T = 0.05, N = 10, λ = 0.1, α = 1, t = 5.00
x


Ornstein–Uhlenbeck
T = 0.05, N = 1, λ = 0.1, α = 1
u(t)
t


Ornstein–Uhlenbeck
T = 0.05, N = 10, λ = 0.1, α = 1
u(t)
t

Main extension

Consider a control $u(x, t)$, depending also on the space variable x

Proposition

Let the following assumptions hold:

- $\sigma \in C^1(\Omega)$ such that

$$\sigma(x,t) \geq \theta \quad \forall (x,t) \in Q, \text{ for some constant } \theta > 0$$

- $b \in L^q(0, T_E; L^p(\Omega))$ with $2 < p, q \leq \infty$ and $\frac{1}{2p} + \frac{1}{q} < \frac{1}{2}$.
- $y_n \in L^2(\Omega)$ is nonnegative and bounded.

Then there exists a unique nonnegative weak solution y to the Fokker-Planck IBVP (1)

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Towards existence of Optimal Solutions

Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, $H := L^2(\Omega)$, $V := H_0^1(\Omega)$ and V' dual space of V.
The Fokker-Planck equation $\mathcal{E}(y_0, u, f)$ can be rewritten as

$$\begin{cases} \dot{y}(t) + Ay(t) + B(u(t), y(t)) = f(t) & \text{in } V',\ t \in (0, T) \\ y(0) = y_0 \,, \end{cases} \tag{2}$$

where $y_0 \in H$, $A : V \to V'$ linear and continuous operator,
$f \in L^2(0, T; V')$ and $B : \mathcal{U} \times L^\infty(0, T; H) \to L^2(0, T; V')$ is defined by

$$\langle B(u, y), \varphi \rangle_{V',V} = -\int_\Omega \sum_{i=1}^{d} b_i(u) y \, \partial_i \varphi \, \mathrm{d}x \qquad \forall u \in \mathcal{U}\,, y \in H\,, \varphi \in V\,,$$

with $\mathcal{U} := L^q(0, T; L^\infty(\Omega; \mathbb{R}^d))$, for some $q > 2$,

Rmk: optimization problem in a Banach space

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Rmk: optimization problem in a Banach space

A-priori estimates

Let $y_0 \in H$, $f \in L^2(0,T;V')$ and $u \in \mathcal{U}$. Then a solution y of the Fokker-Planck equation (2) satisfies the estimates

$$|y|^2_{L^\infty(0,T;H)} \leq Ce^{c|u|^2_{\mathcal{U}}}\left[|y(0)|^2_H + |f|^2_{L^2(0,T;V')}\right],$$

$$|y|^2_{L^2(0,T;V)} \leq C\max(1, |u|^2_{\mathcal{U}}e^{c|u|^2_{\mathcal{U}}})\left(|y(0)|^2_H + |f|^2_{L^2(0,T;V')}\right),$$

$$|\dot{y}|^2_{L^2(0,T;V')} \leq C(1 + |u|^2_{\mathcal{U}}e^{c|u|^2_{\mathcal{U}}})\left(|y(0)|^2_H + |f|^2_{L^2(0,T;V')}\right) + 2|f|^2_{L^2(0,T;V')},$$

for some positive constants c, C.

Existence and Uniqueness of Optimal Control

Let $y_0 \in V$, $y_d \in H$ and $\quad J(u) := |y - y_d|^2_{L^2(0,T;H)} + \lambda |u|^2_{\mathcal{U}}$, $\lambda > 0$

where y is the unique solution to

$$\begin{cases} \dot{y}(t) + Ay(t) + B(u(t), y(t)) = 0 & \text{in } V',\ t \in (0,T) \\ y(0) = y_0\,, \end{cases}$$

Then there exists a pair

$$(\bar{y}, \bar{u}) \in C([0,T], H) \times \mathcal{U}$$

such that $\bar{y}$ is a solution of $\mathcal{E}(y_0, \bar{u}, 0)$ and $\bar{u}$ minimizes J in $\mathcal{U}$.

Moreover, pair $(\bar{y}, \bar{u})$ unique for small $|y - y_d|_{L^2(0,T;H)}$ or for large λ

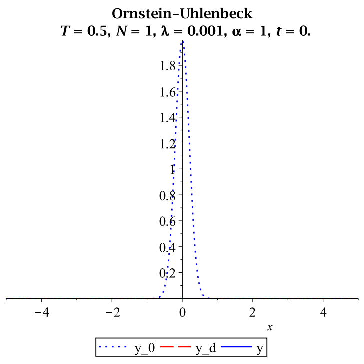
Necessary Optimality Conditions (for $u(x,t)$)

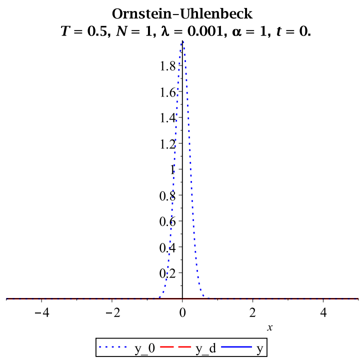
We derive the first-order necessary optimality system for $u(x,t)$

$$
\begin{aligned}
\partial_t y - \frac{1}{2}\partial^2_{xx}\left(\sigma^2 y\right) + \partial_x\left(by\right) &= 0 && \text{in } Q_n\\
y(\cdot, t_n) &= y_n && \text{in } \Omega\\
y &= 0 && \text{in } \Sigma_n\\
-\partial_t p - \frac{1}{2}\sigma^2\partial^2_{xx}p - b\partial_x p &= 0 && \text{in } Q_n\\
p(\cdot, t_{n+1}) &= y(\cdot, t_{n+1}) - y_d(\cdot, t_{n+1}) && \text{in } \Omega\\
p &= 0 && \text{in } \Sigma_n\\
\lambda u + \int_{t_n}^{t_{n+1}} D_3(b)\; y\; \partial_x p \,\mathrm{d}t &= 0
\end{aligned}
$$

Numerical Examples

Consider the Ornstein-Uhlenbeck process from before, but with space-dependent control $u(x, t)$ and $\lambda = 0.001$ instead of 0.1.


Ornstein-Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 0.
x
y_0
y_d
y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T = 0.5, N = 1, λ = 0.001, α = 1, t = 0.
x
y_0
y_d
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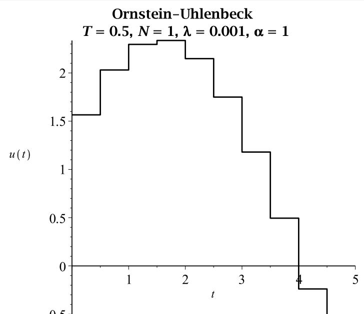
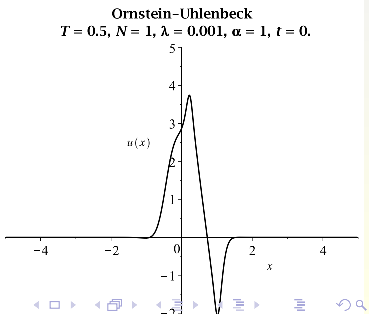
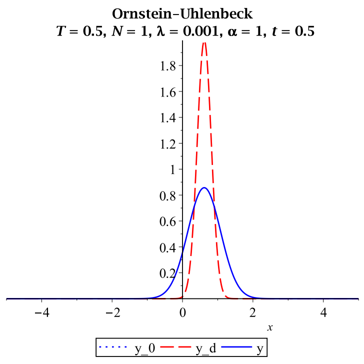
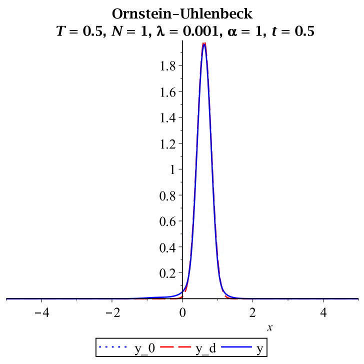
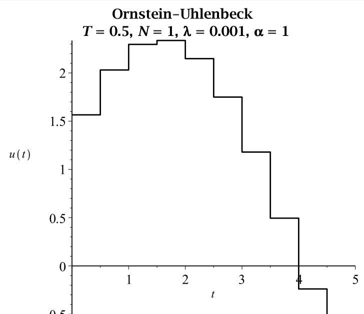

Ornstein-Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1
u(t)
t


Ornstein-Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 0.
u(x)
x


Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 0.5
x
y_0
y_d
y


Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 0.5
x
y_0
y_d
y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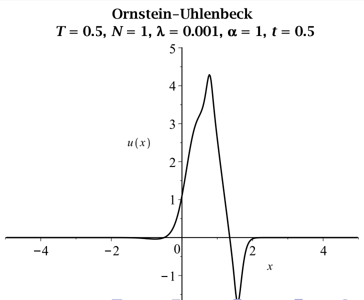
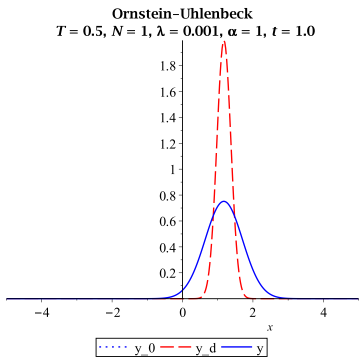
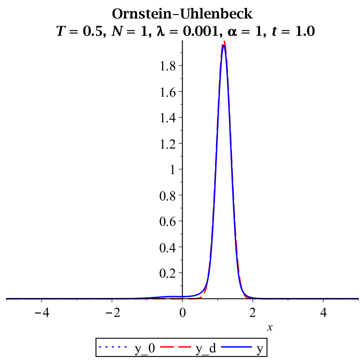
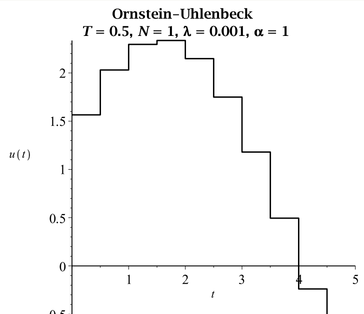
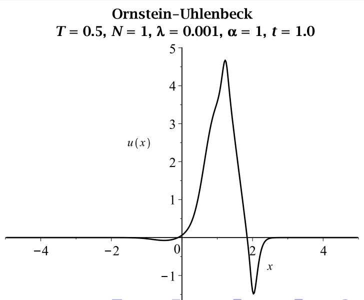
u(t)
t

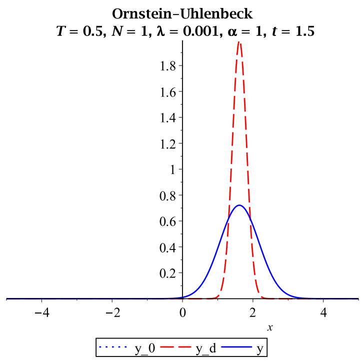

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 0.5
u(x)
x

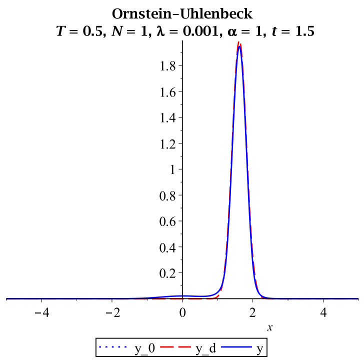

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 1.0
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
−4
−2
0
2
4
x
y_0
y_d
y

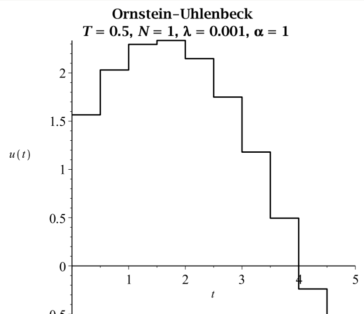

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 1.0
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
−4
−2
0
2
4
x
y_0
y_d
y

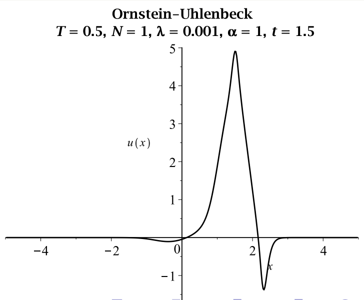

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1
2
1.5
u(t)
1
0.5
0
1
2
3
4
5
t

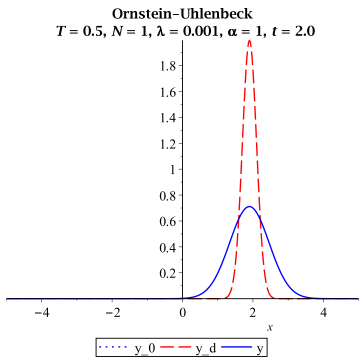

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 1.0
5
4
3
u(x)
2
1
−4
−2
0
2
4
x
−1

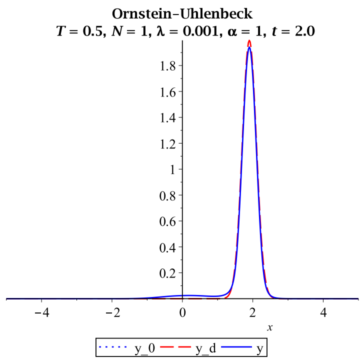

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 1.5
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
−4
−2
0
2
4
x
y_0
y_d
y

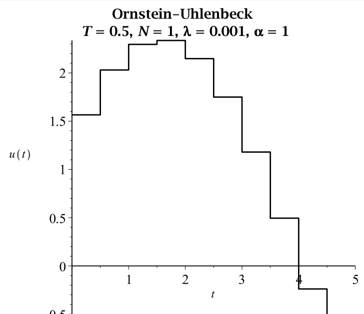

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 1.5
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
−4
−2
0
2
4
x
y_0
y_d
y

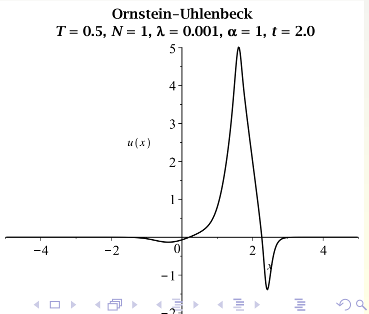

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1
2
1.5
u(t)
1
0.5
0
1
2
3
4
5
t

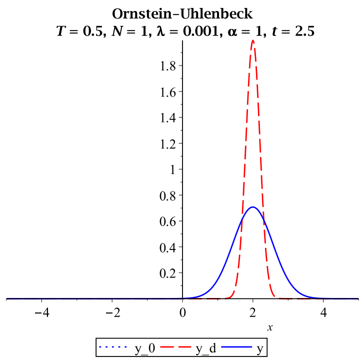

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 1.5
5
4
3
u(x)
2
1
−4
−2
0
2
4
x
−1

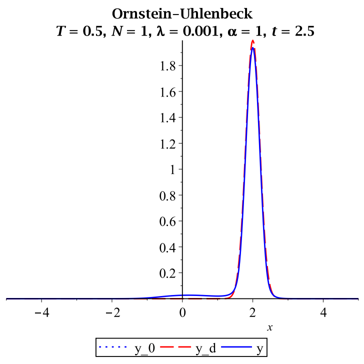

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 2.0
x
y_0 y_d y

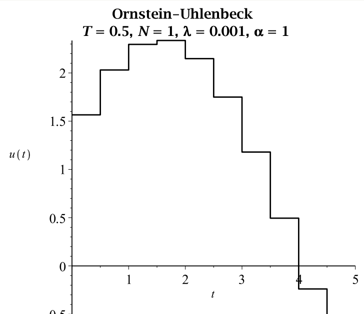

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 2.0
x
y_0 y_d y

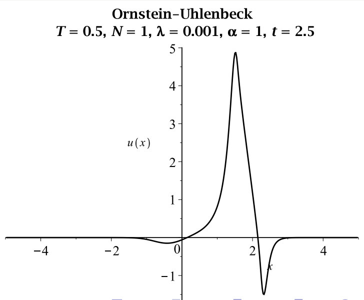

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1
u(t)
t

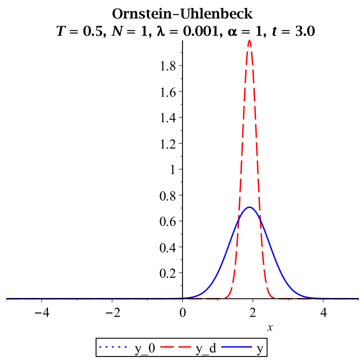

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 2.0
u(x)
x

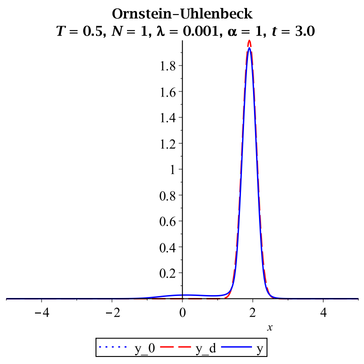

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 2.5
x
y_0 y_d y

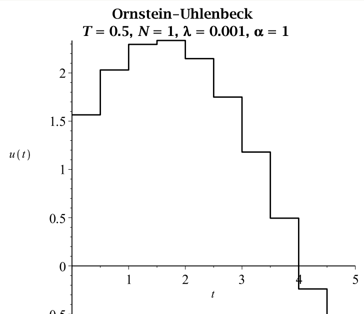

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 2.5
x
y_0 y_d y

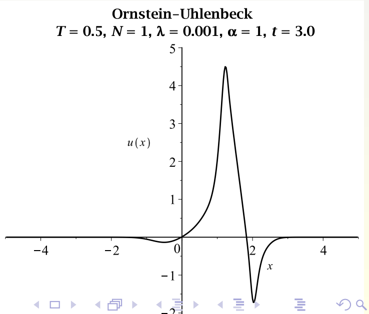

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1
u(t)
t

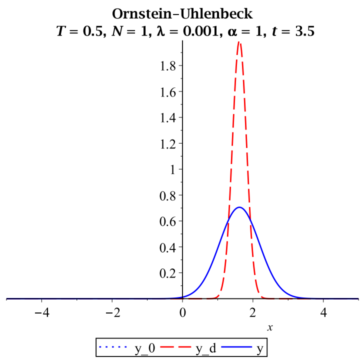

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 2.5
u(x)
x

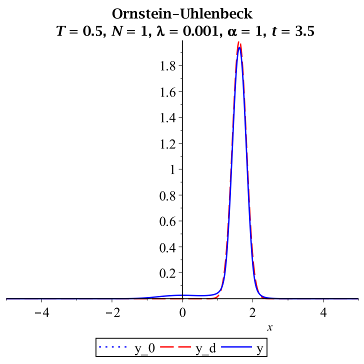

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 3.0
x
y_0
y_d
y

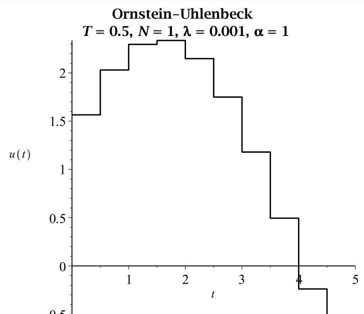

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 3.0
x
y_0
y_d
y

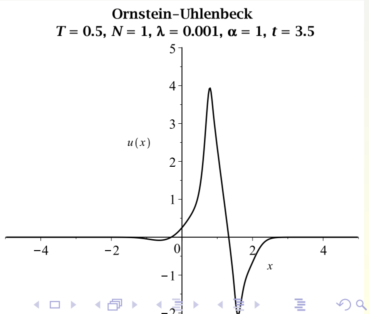

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1
u(t)
t

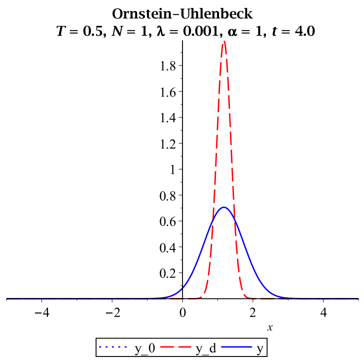

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 3.0
u(x)
x

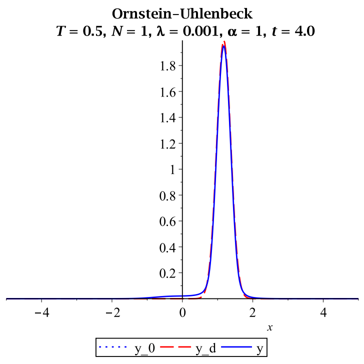

Ornstein-Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 3.5
x
y_0
y_d
y

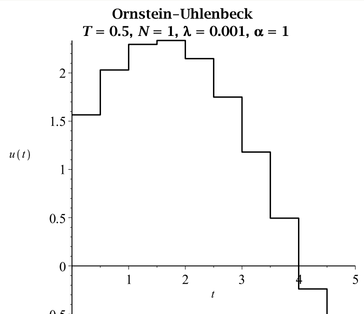

Ornstein-Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 3.5
x
y_0
y_d
y

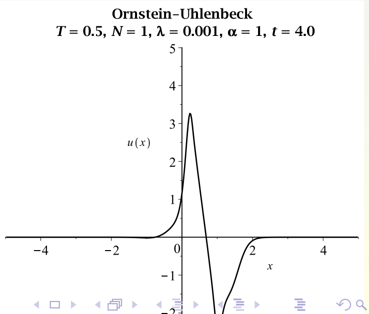

Ornstein-Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1
u(t)
t

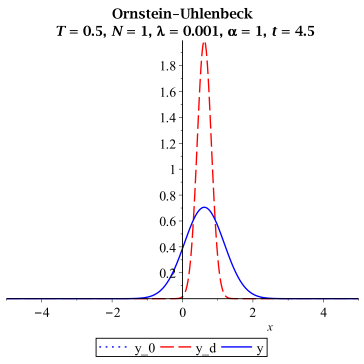

Ornstein-Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 3.5
u(x)
x

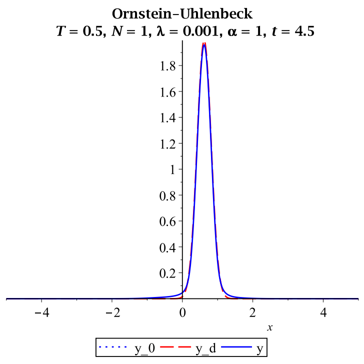

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 4.0
x
y_0 y_d y

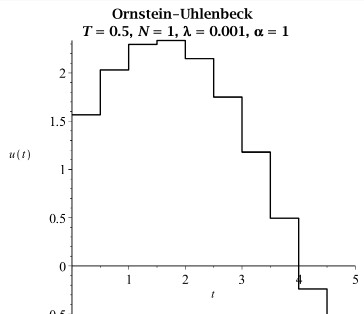

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 4.0
x
y_0 y_d y

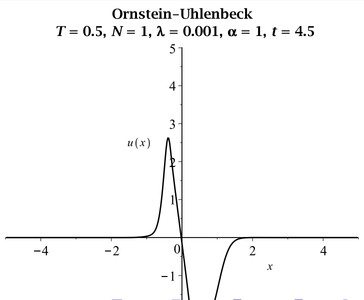

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1
u(t)
t

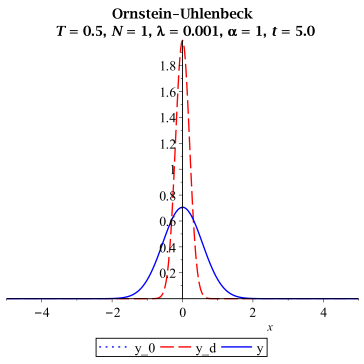

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 4.0
u(x)
x

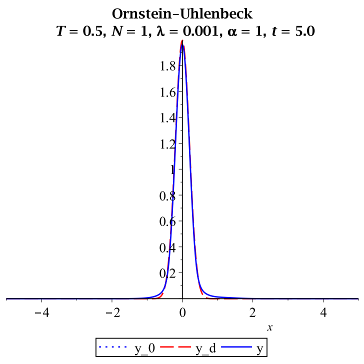

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 4.5
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
−4
−2
0
2
4
x
y_0
y_d
y

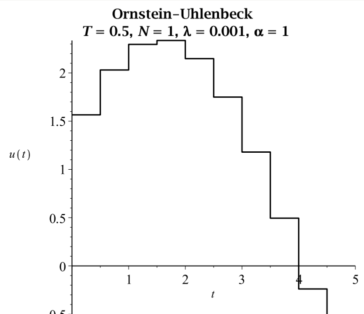

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 4.5
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
−4
−2
0
2
4
x
y_0
y_d
y

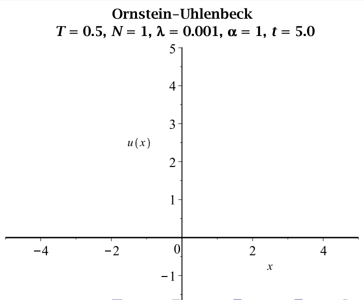

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1
2
1.5
u(t)
1
0.5
0
1
2
3
4
5
t


Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 4.5
5
4
3
u(x)
2
1
−4
−2
0
2
4
x
−1


Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 5.0
x
y_0
y_d
y


Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 5.0
x
y_0
y_d
y


Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1
u(t)
t

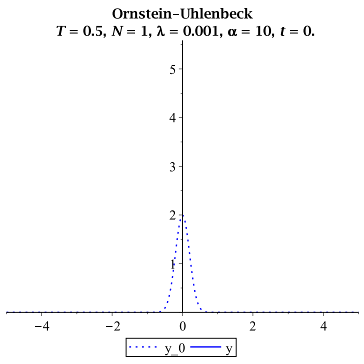

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 5.0
u(x)
x

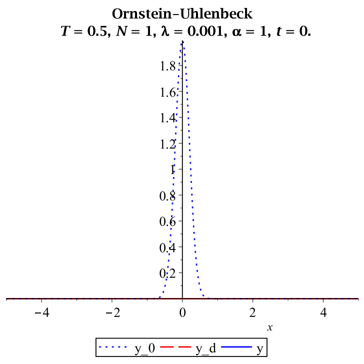
Numerical Examples (2)

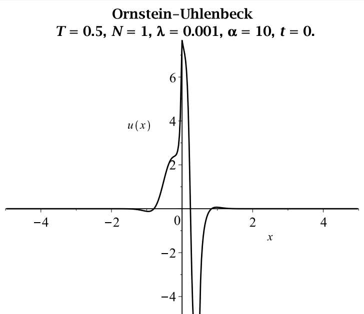
With space-dependent control, larger class of objectives possible:

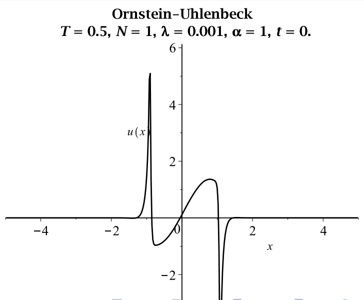
- region avoidance, without prescribing the shape of the PDF, e.g. try to force the state PDF into $[0, 0.5]$.
- Try to track non-smooth targets, e.g.

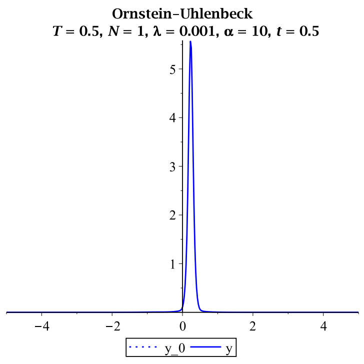
$$y_d(x,t) := \begin{cases} 0.5 & \text{if } x \in [-1 + 0.15t, 1 + 0.15t] \\ 0 & \text{otherwise.} \end{cases}$$

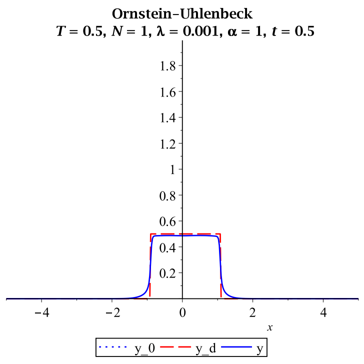

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 10, t = 0.
y_0
y

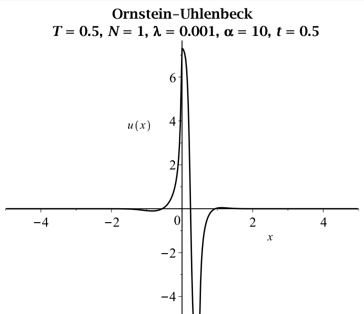

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 0.
x
y_0
y_d
y

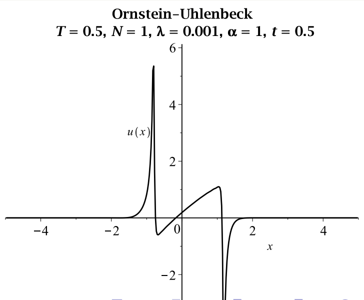

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 10, t = 0.
u(x)
x

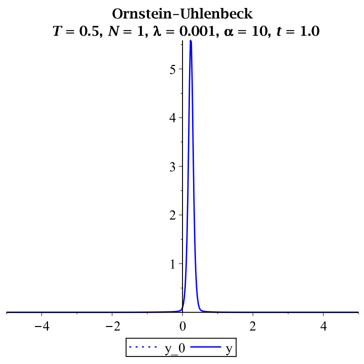

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 0.
u(x)
x

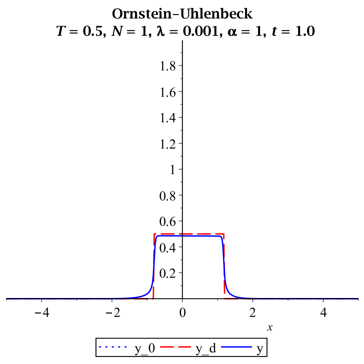

Ornstein-Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 10, t = 0.5
y_0
y

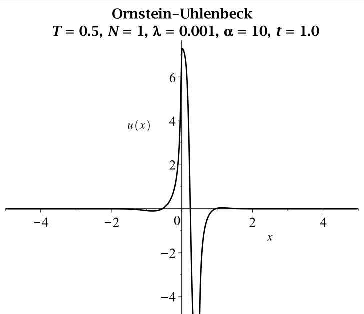

Ornstein-Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 0.5
x
y_0
y_d
y

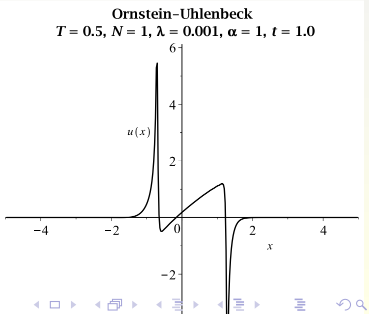

Ornstein-Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 10, t = 0.5
u(x)
x

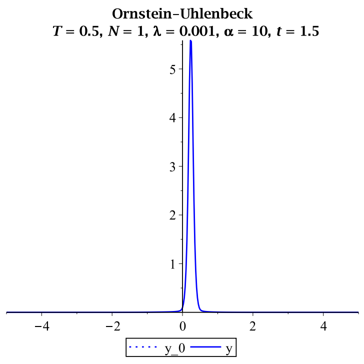

Ornstein-Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 0.5
u(x)
x

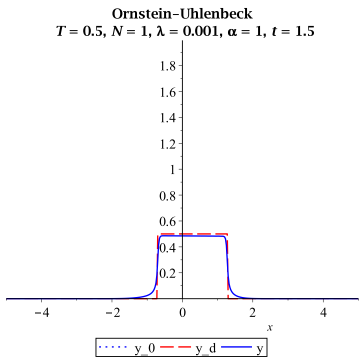

Ornstein-Uhlenbeck
$T = 0.5,\ N = 1,\ \lambda = 0.001,\ \alpha = 10,\ t = 1.0$
y_0
y

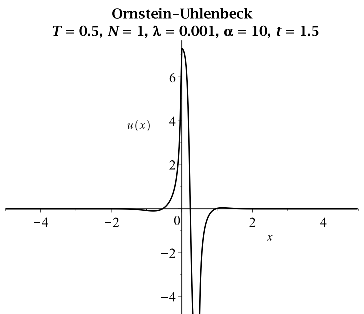

Ornstein-Uhlenbeck
$T = 0.5,\ N = 1,\ \lambda = 0.001,\ \alpha = 1,\ t = 1.0$
x
y_0
y_d
y

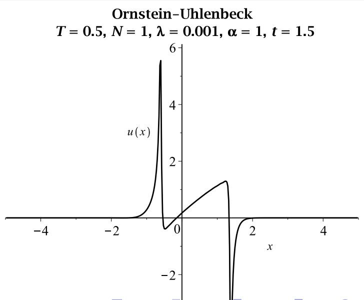

Ornstein-Uhlenbeck
$T = 0.5,\ N = 1,\ \lambda = 0.001,\ \alpha = 10,\ t = 1.0$
u(x)
x

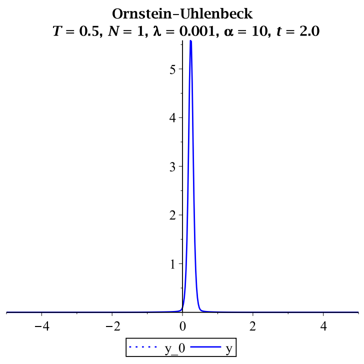

Ornstein-Uhlenbeck
$T = 0.5,\ N = 1,\ \lambda = 0.001,\ \alpha = 1,\ t = 1.0$
u(x)
x

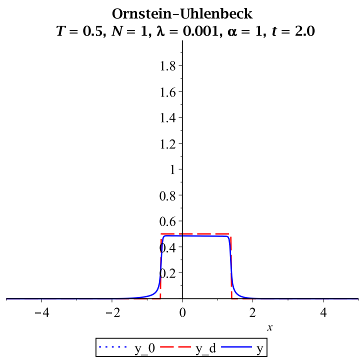

Ornstein-Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 10, t = 1.5
5
4
3
2
1
−4
−2
0
2
4
y_0
y

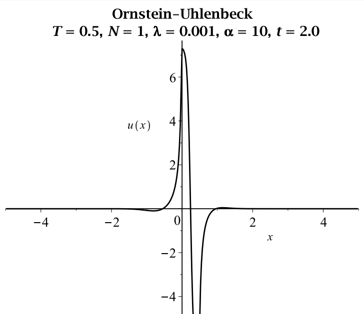

Ornstein-Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 1.5
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
−4
−2
0
2
4
x
y_0
y_d
y

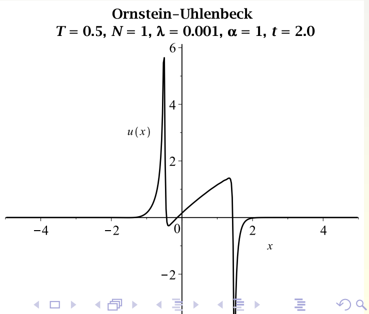

Ornstein-Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 10, t = 1.5
6
u(x)
4
2
−4
−2
0
2
4
x
−2
−4

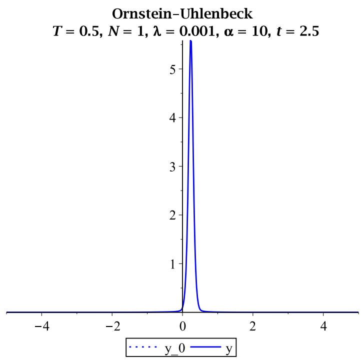

Ornstein-Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 1.5
6
4
u(x)
2
−4
−2
0
2
4
x
−2

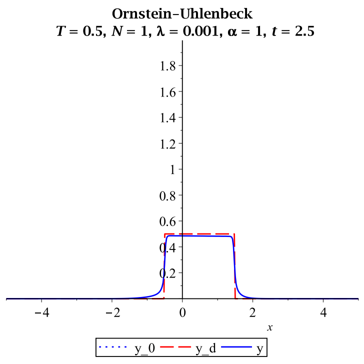

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 10, t = 2.0
5
4
3
2
1
−4
−2
0
2
4
y_0
y

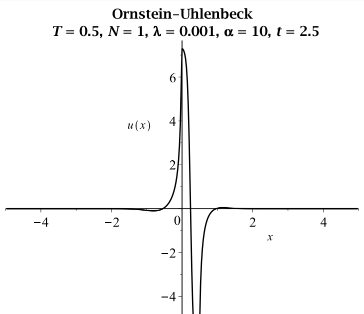

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 2.0
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
−4
−2
0
2
4
x
y_0
y_d
y

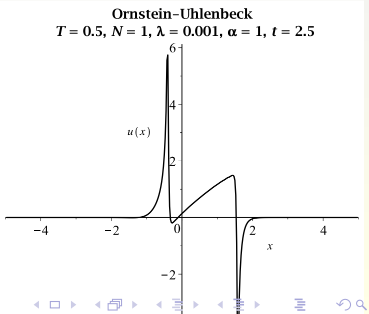

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 10, t = 2.0
6
u(x)
4
2
−4
−2
0
2
4
x
−2
−4

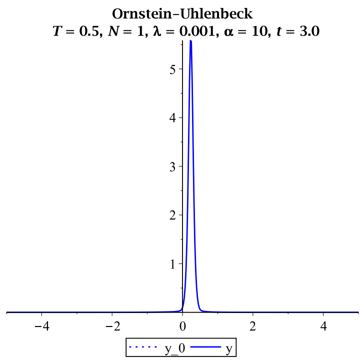

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 2.0
6
4
u(x)
2
−4
−2
0
2
4
x
−2

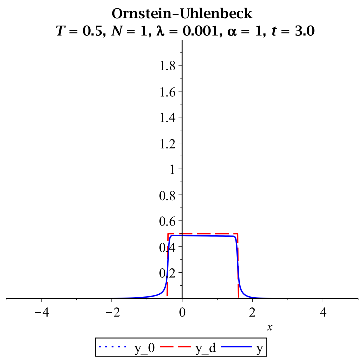

Ornstein–Uhlenbeck
$T = 0.5,\ N = 1,\ \lambda = 0.001,\ \alpha = 10,\ t = 2.5$
5
4
3
2
1
−4
−2
0
2
4
y_0
y

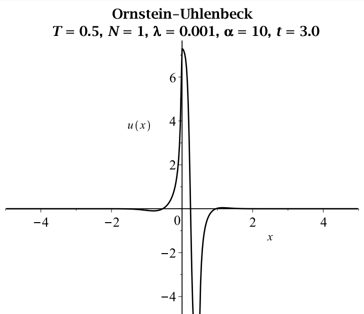

Ornstein–Uhlenbeck
$T = 0.5,\ N = 1,\ \lambda = 0.001,\ \alpha = 1,\ t = 2.5$
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
−4
−2
0
2
4
x
y_0
y_d
y

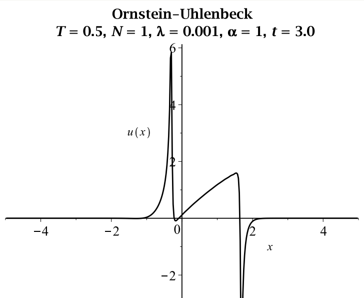

Ornstein–Uhlenbeck
$T = 0.5,\ N = 1,\ \lambda = 0.001,\ \alpha = 10,\ t = 2.5$
6
4
2
−2
−4
u(x)
−4
−2
0
2
4
x

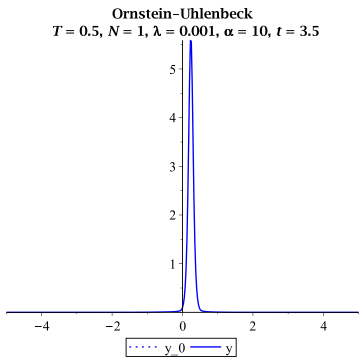

Ornstein–Uhlenbeck
$T = 0.5,\ N = 1,\ \lambda = 0.001,\ \alpha = 1,\ t = 2.5$
6
4
2
−2
u(x)
−4
−2
0
2
4
x

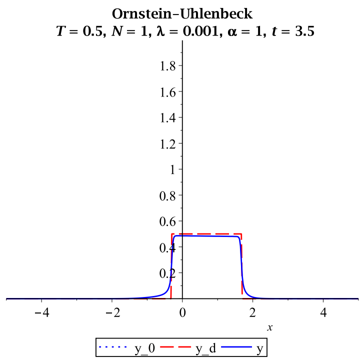

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 10, t = 3.0
5
4
3
2
1
−4
−2
0
2
4
y_0
y


Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 3.0
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
−4
−2
0
2
4
x
y_0
y_d
y


Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 10, t = 3.0
6
4
2
u(x)
−4
−2
0
2
4
x
−2
−4


Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 3.0
6
4
2
u(x)
−4
−2
0
2
4
x
−2


Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 10, t = 3.5
5
4
3
2
1
−4
−2
0
2
4
y_0
y


Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 3.5
1.8
1.6
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1.2
1
0.8
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−4
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2
4
x
y_0
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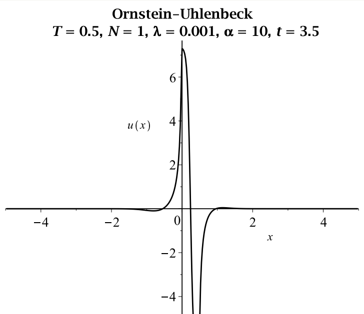
Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 10, t = 3.5
6
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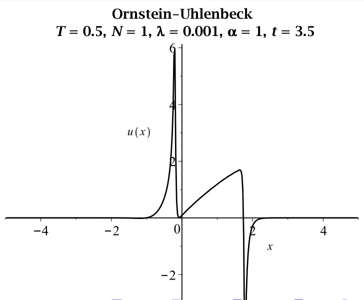
Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 3.5
6
4
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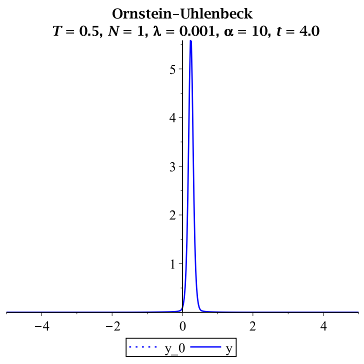
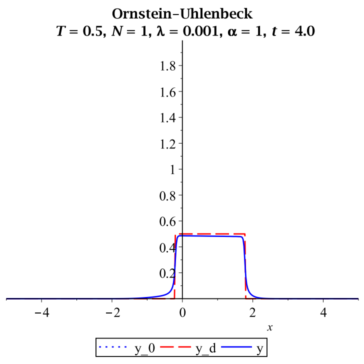
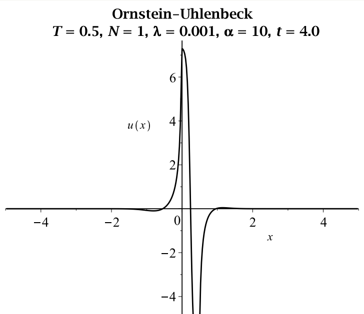
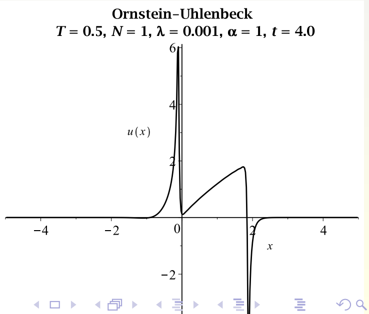
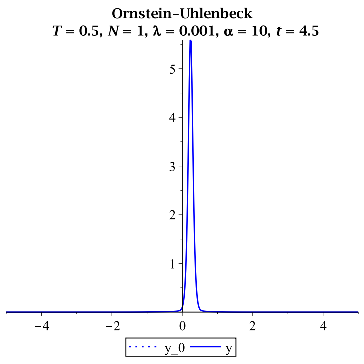
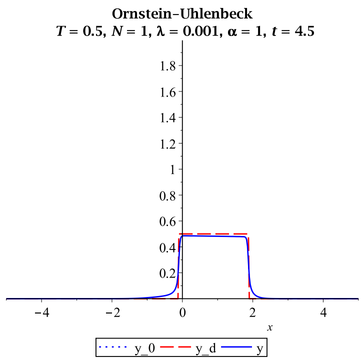

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 10, t = 4.0
y_0
y

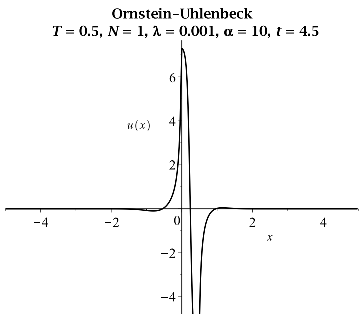

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 4.0
x
y_0
y_d
y

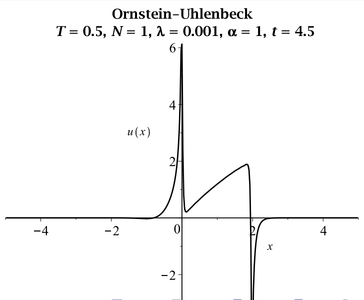

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 10, t = 4.0
u(x)
x

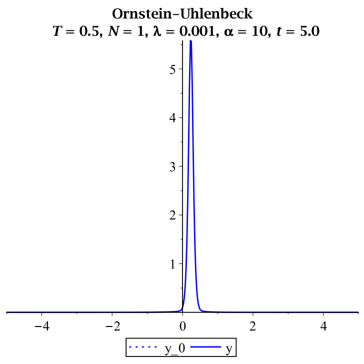

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 4.0
u(x)
x

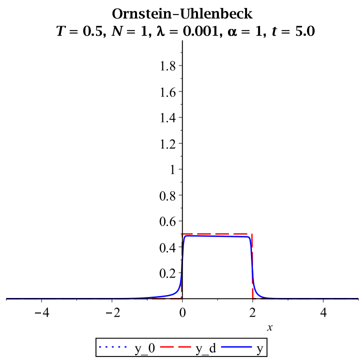

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 10, t = 4.5
y_0
y

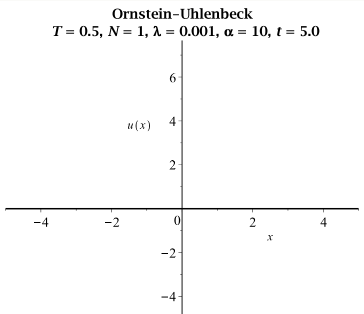

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 4.5
x
y_0
y_d
y

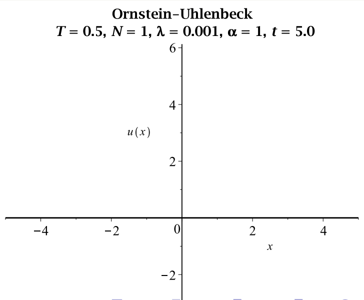

Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 10, t = 4.5
u(x)
x


Ornstein–Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 4.5
u(x)
x


Ornstein-Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 10, t = 5.0
y_0
y


Ornstein-Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 5.0
x
y_0
y_d
y


Ornstein-Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 10, t = 5.0
u(x)
x


Ornstein-Uhlenbeck
T = 0.5, N = 1, λ = 0.001, α = 1, t = 5.0
u(x)
x

Outline

Some remarks

– Numerical simulations also for the Geometric-Brownian process and the Shiryaev process

– The computed optimal control of the PDF is then applied to the stochastic process

– Right boundary conditions of Robin type

– The same Fokker-Planck Optimal Control framework applies to
  - the class of piecewise deterministic processes
  - optimal control of open quantum systems
  - subdiffusion processes

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Outlook

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  - Use known techniques from, e.g. [Altmüller and Grüne, 2012], in order to find estimates for horizons N that guarantee stability of the MPC closed-loop system.

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  - Controllability of the Fokker-Planck equation (2):
    Compare with previous work by
    Blaquière 1992
    ($\rightsquigarrow$ continuous initial datum and final target with compact support in 1D)
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Thank you for your attention!