Exact Penalization Applied To First Order State Constrained Problems

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New Perspectives in Optimal Control and Games
November 10-13, 2014, Rome

ITN SADCO
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Motivation

- Measures appear as multipliers in Necessary Conditions for State Constrained Problems
- We would like to avoid it (such multipliers are hard to treat)
- Is there a class of problems where such measures are absolutely continuous w.r.t. Lebesgue measure?
- Can we use Exact Penalization to
  - a) Identify an appropriate class of problems?
  - b) Obtain the NC for it?

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State Constrained Control Problem

$$
(P) \quad \begin{cases} \text{Minimize } l(x(a), x(b)) \\ \text{subject to} \\ \quad \dot{x}(t) = f(t, x(t), u(t)) & \text{a.e. } t \in [a, b] \\ \quad h(x(t)) \leq 0 & \text{for all } \; t \in [a, b] \\ \quad u(t) \in U & \text{a.e. } t \in [a, b] \\ \quad (x(a), x(b)) \in E, \end{cases}
$$

where

$l : \; \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$,
$f : \; [\mathbf{a}, \mathbf{b}] \times \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}^n$,
$h : \; \mathbf{R}^n \to \mathbf{R}$, (for simplicity, independent of t)
$U \subset \mathbf{R}^k$,
$E \subset \mathbf{R}^n \times \mathbf{R}^n$.

subject to a standard set of assumptions

Necessary Conditions of Optimality

Let (x^*, u^*) be an optimal solution. Then

- $(p, \mu, \lambda_0) \neq (0, 0, 0)$

- $-\dot{p}(t) = f_x^T(t, x^*(t), u^*(t))q(t)$ a.e.,

  BV function $q(t) = \begin{cases} p(t) + \int_{[a,t)} \nabla h(s, x^*(s))\, \mu(ds), & t \in [a, b) \\ p(t) + \int_{[a,b]} \nabla h(s, x^*(s))\, \mu(ds), & t = b \end{cases}$

- $\forall \quad u \in U,$

  $\langle q(t), f(t, x^*(t), u)\rangle \leq \langle q(t), f(t, x^*(t), u^*(t))\rangle$ a.e.

- $(p(a), -q(b)) = \lambda_0 \nabla l(x^*(a), x^*(b)) + N_E^L(x^*(a), x^*(b)),$

- $\text{supp}\{\mu\} \subset \{t : \; h(x^*(t)) = 0\}$

Well-Behaved Measures

If there exists an integrable function ξ such that

$$q(t) = p(t) + \int_{[a,t)} \nabla h(x^*(s))\, \mu(ds) = p(t) + \int_{[a,t)} \nabla h(x^*(s)) \xi(s)\, ds$$

Then measure μ is AC w.r.t. Lebesgue measure, and

$$\dot{q}(t) = \dot{p}(t) + \xi(t) \nabla h(x^*(t)).$$

The adjoint equation

$$-\dot{p}(t) = f_x^T(t, x^*(t), u^*(t)) q(t)$$

becomes

$$-\dot{q}(t) = f_x^T(t, x^*(t), u^*(t)) q(t) - \xi(t) \nabla h(x^*(t))$$

Well-Behaved Measures (2)

Question: Identify a class of problems with measures

$$\int_{[a,t)} \nabla h(x^*(s))\, \mu(ds) = \int_{[a,t)} \nabla h(x^*(s)) \xi(s)\, ds$$

A first guess: If $h(x^*(t)) < 0$ for all $t \in [a, b]$, then

$$\mu \equiv 0, \qquad q = p$$

Is there a larger class of problems?

One idea is to identify such class by **Exact Penalization**

Distance Function

Define

$$\Phi := \{y \in \mathbf{R} \; : \; y \leq 0\}, \qquad S := \{x \in \mathbf{R}^n \; : \; h(x) \in \Phi\}.$$

Observe

$$h(x) \leq 0 \quad \Longleftrightarrow \quad x \in S$$

Definition of the distance function

$$d_S(x) := \inf\left\{\left|x - x'\right| \; : \; x' \in S\right\}.$$

Then

$$\boxed{d_S(x^*(t)) = 0 \quad \Longleftrightarrow \quad h(x^*(t)) \leq 0.}$$

Question

When is (x^*, u^*) a strong minimum of (P), also a strong minimum of an exact penalization problem (Q)?

Exact Penalization Problem

$$(Q) \quad \begin{cases} \text{Minimize } l(x(a), x(b)) + K \int_a^b d_S(x(t))\, dt \\ \text{subject to} \\ \dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. } t \in [a, b], \\ u(t) \in U \quad \text{a.e. } t \in [a, b], \\ (x(a), x(b)) \in E. \end{cases}$$

State constraint $h(x(t)) \leq 0$ in (P) is substituted by

From a minimizer of (P) to a minimizer of (Q)

- We know (x^*, u^*) is admissible for (Q).
- Suppose (x^*, u^*) is not a solution to (Q).
- Let (x', u') be an admissible process for (Q) such that

$$l(x'(a), x'(b)) + K \int_a^b d_S(x'(t))dt < l(x^*(a), x^*(b)).$$

- Set $\rho = l(x^*(a), x^*(b)) - l(x'(a), x'(b)) - K \int_a^b d_S(x'(t))dt$.
- Choose $\delta \in (0, \frac{\rho}{2K})$.
- We obtain

$$l(x'(a), x'(b)) + K \int_a^b d_S(x'(t))dt < l(x^*(a), x^*(b)) - K\delta.$$

From a minimizer of (P) to a minimizer of (Q) (2)

Suppose that there exists an admissible process (z, v) for (P) s.t.

$$(\mathrm{H}*) \qquad \max_{t\in[a,b]} \{|z(t) - x'(t)|\} \leq \frac{K}{2}\int_a^b d_S(x'(t))\,dt$$

- Supposing that l is Lipschitz:

$$l(z(a), z(b)) - l(x'(a), x'(b)) \leq K_l \Big|(z(a), z(b)) - (x'(a), x'(b))\Big|.$$

- Assuming $K > K_l$:

$$l(z(a), z(b)) - l(x'(a), x'(b)) \leq K \Big|(z(a), z(b)) - (x'(a), x'(b))\Big|$$

$$\leq K\int_a^b d_S(x'(t))dt < K\int_a^b d_S(x'(t))dt + K\delta < K\int_a^b d_S(x'(t))dt + \rho$$

$$= K\int_a^b d_S(x'(t))dt + l(x^*(a), x^*(b)) - l(x'(a), x'(b)) - K\int_a^b d_S(x'(t))dt$$

$$= l(x^*(a), x^*(b)) - l(x'(a), x'(b))$$

From a minimizer of (P) to a minimizer of (Q) (3)

This gives us:

$$l(z(a), z(b)) \leq l(x^*(a), x^*(b))$$

a contradiction to (x^*, u^*) being the optimal solution to (P)!

From a minimizer of (P) to a minimizer of (Q) (4)

Conclusion

If

- (H*) holds, i.e. $\max_{t\in[a,b]}\{|z(t)-x'(t)|\} \leq \frac{K}{2}\int_a^b d_S(x'(t))\,dt$
- Cost l is Lipschitz continuous with K_l,
- $K > K_l$,

Then:

A strong loc. minimum of (P) is also a strong loc. minimum of (Q).

NC for Penalized Problem

Exact Penalization Problem in Mayer Form:

$$
(\mathrm{Q_M}) \quad \begin{cases} \text{Minimize } l(x(a), x(b)) + y(b) \\ \text{subject to} \\ \quad \dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. } t \in [a, b], \\ \quad \dot{y}(t) = K \displaystyle\int_a^t d_S(x(s))\, ds \quad \text{a.e. } t \in [a, b], \\ \quad u(t) \in U \quad \text{a.e. } t \in [a, b], \\ \quad (x(a), x(b), y(a), y(b)) \in E \times \{0\} \times \mathbb{R}. \end{cases}
$$

Idea:

- Obtain Necessary Conditions for (Q) via ($\mathrm{Q_M}$).
- Apply them obtain a set of candidates for (P).

NC for Penalized Problem (2)

A useful result:

Suppose that $h \in C^1$ and $\nabla h(x) \neq 0$ for $x \in \mathbf{R}^n$ such that $h(x) = 0$. As defined earlier, $\Phi = \{y \in \mathbf{R} \ : \ y \leq 0\}$ and $S = \{x \in \mathbf{R}^n \ : \ h(x) \in \Phi\}$. Then,

$$\forall\, \zeta \in \partial^C d_S(x) \quad \exists\, \alpha \in N_\Phi^C(h(x)) : \quad \zeta = \alpha \nabla h(x^*(t)).$$

Idea of a proof:

- $N_\Phi^L(y) = N_\Phi^C(y)$ since Φ is convex.
- If $h(x) = 0$, then $\alpha \in N_\Phi^C(h(x)) \implies \alpha \geq 0$
- If $\zeta \in \partial^C d_S(x) \implies \zeta \in N_S^C(x)$.

NC for Penalized Problem (3)

Assumptions (reminder):

- l and $(x,u) \to f(t,x,u)$ are K_l-, K_f-Lipschitz continuous
- $h \in C^1$ and $\nabla h(x) \neq 0 \;\; \forall x \in \mathbf{R}^n$ with $h(x) = 0$;
- U compact, E closed;
- (H*) holds

Apply MP to $(\mathrm{Q_M})$ to obtain NC for (P):

$\exists\, p \in W^{1,1}$, a meas. function ξ and a scalar $\lambda \geq 0$:

(i) $$||p||_\infty + \lambda > 0,$$

(ii) $$-\dot{p}(t) \in \partial_x^C \langle p(t), f(t,x^*(t),u^*(t))\rangle - \lambda\xi(t)\nabla h(x^*(t)) \;\text{ a.e.},$$

(iii) $$u \in U \implies \langle p(t), f(t,x^*(t),u)\rangle \leq \langle p(t), f(t,x^*(t),u^*(t))\rangle \;\text{ a.e.},$$

(iv) $$(p(a), -p(b)) \in N_E^L(x^*(a),x^*(b)) + \lambda\partial^L l(x^*(a),x^*(b)),$$

(v) $$\xi(t) \geq 0 \text{ and } \xi(t)h(x^*(t)) = 0 \;\text{ a.e.}$$

"Measure-free" Necessary Conditions for (P)!

The Remaining Question

- When does (H*) hold? – No answer (Hypothesis is hard to verify)
- Take a simple state constrained problem, test if the NC hold!

A simple problem with a first-order state constraint

$$(FO) \quad \begin{cases} \text{Minimize } \displaystyle\int_a^b \langle c, x(t)\rangle + u^2(t)\, dt \\ \text{subject to} \\ \quad \dot{x}(t) = f(x(t)) + g(x(t))u(t) \quad \text{a.e. } t \in [a,b], \\ \quad h(x(t)) \le 0 \quad \text{for all } t \in [a,b], \\ \quad u(t) \in U(t) \quad \text{a.e. } t \in [a,b], \\ \quad (x(a), x(b)) \in E. \end{cases}$$

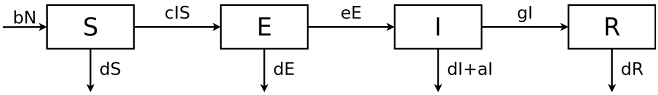
SEIR Compartmental Model

SEIR Model

The total population N is divided into four compartments:

- S susceptible,
- E exposed (not yet infectious),
- I infectious,
- R recovered;

follows a system of ODEs.



SEIR Control Problem

Minimize $\int_0^T \left(AI(t) + u^2(t)\right)\, dt$

subject to

$$\left.\begin{aligned}
&\dot{S} = bN(t) - dS(t) - cS(t)I(t) - u(t)S(t),\\
&\dot{E}(t) = cS(t)I(t) - (e+d)E(t),\\
&\dot{I}(t) = eE(t) - (g+a+d)I(t),\\
&\dot{N}(t) = (b-d)N(t) - aI(t),
\end{aligned}\right\}$$

$S(t) - 1100 \le 0$ for all $t \in [0, T]$,

$u(t) \in [0, 1]$ a.e. $t \in [0, T]$,

$(x(a), x(b)) \in E$.

SEIR Control Problem

Minimize $\displaystyle\int_0^T \left(AI(t) + u^2(t)\right)\, dt$

subject to

$$\left.\begin{array}{l} \dot{S} = bN(t) - dS(t) - cS(t)I(t) - u(t)S(t), \\ \dot{E}(t) = cS(t)I(t) - (e+d)E(t), \\ \dot{I}(t) = eE(t) - (g+a+d)I(t), \\ \dot{N}(t) = (b-d)N(t) - aI(t), \end{array}\right\}$$

$\dot{x} = f(x) + g(x)u$

where

$g(x) = (S, 0, 0, 0)^T$

$S(t) - 1100 \leq 0$ for all $t \in [0, T]$,

$u(t) \in [0, 1]$ a.e. $t \in [0, T]$,

$(x(a), x(b)) \in E$.

SEIR Control Problem

Minimize $\int_0^T \left(AI(t) + u^2(t)\right)\, dt$

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\dot{I}(t) &= eE(t) - (g+a+d)I(t),\\
\dot{N}(t) &= (b-d)N(t) - aI(t),
\end{aligned}\right\}
$$

$\dot{x} = f(x) + g(x)u$

where

$g(x) = (S, 0, 0, 0)^T$

$S(t) - 1100 \le 0$ for all $t \in [0, T]$,

$u(t) \in [0,1]$ a.e. $t \in [0, T]$,

$(x(a), x(b)) \in E$.

$h(x(t)) = \langle (1,0,0,0), (S, E, I, N) \rangle$

$$\frac{dh}{dt} = \nabla h(x)\dot{x} = \dot{S} = bN - dS - cSI - uS$$

The state constraint is of order 1!

Necessary Conditions SEIR

Inward Pointing Condition [Rampazzo & Vinter '99] verified $\Longrightarrow$ $\lambda = 1$ (**Normality**)

(i) $(p, \lambda, \mu) \neq (0, 0, 0)$;

(ii) $-\dot{p}(t) = f_x^T(x^*(t))q(t) + u^*(t)g_x^T(x^*(t))q(t) - \lambda c$;

(iii) $\forall\, u \in U$,
$\langle g(x^*(t))u^*(t), q(t)\rangle - \lambda(u^*)^2(t) \geq \langle g(x^*(t))u, q(t)\rangle - \lambda u^2$;

(iv) $-q(b) = 0$;

(v) $supp\{\mu\} \subset \{t :\ h(x^*(t)) = 0\}$.

where

$$q(t) = p(t) + \int_{[a,t)} \nabla h(x^*(s))\, \mu(ds),$$

$$q(b) = p(b) + \int_{[a,b]} \nabla h(x^*(s))\, \mu(ds).$$

Necessary Conditions SEIR (2)

(i) $(p, \lambda, \mu) \neq (0, 0, 0)$;

(ii) $-\dot{p}(t) = f_x^T(x^*(t))q(t) + u^*(t)g_x^T(x^*(t))q(t) - \lambda c$;

(iii) $\forall\, u \in U$,

$$\langle g(x^*(t))u^*(t), q(t)\rangle - \lambda(u^*)^2(t) \geq \langle g(x^*(t))u, q(t)\rangle - \lambda u^2,$$

(iv) $-q(b) = 0$;

(v) $supp\{\mu\} \subset \{t :\ h(x^*(t)) = 0\}$.

where

$$q(t) = p(t) + \int_{[a,t)} \nabla h(x^*(s))\, \mu(ds),$$

$$q(b) = p(b) + \int_{[a,b]} \nabla h(x^*(s))\, \mu(ds).$$

Closed form of the optimal control

$u^*(t) = \max\left\{0, \min\left\{1, -\frac{q_s(t)S^*(t)}{2}\right\}\right\}$.

State constraint characterization w.r.t x^*

- A *boundary interval* if $\exists [t_0^b, t_1^b]: \quad h(t, x^*(t)) = 0 \quad \forall t \in [t_0^b, t_1^b]$.

Boundary interval in the SEIR case

We show:

$$S^*(t) = S_{max}$$
$$\implies \quad \dot{S}^*(t) = bN^*(t) - dS^*(t) - cS^*(t)I^*(t) - u^*(t)S^*(t) = 0$$
$$\implies \quad \boxed{u^*(t) = b\frac{N^*(t)}{S^*(t)} - d - cI^*(t).}$$

- u^* is an AC function on all (t_0^b, t_1^b) (when $t_b^1 < T$)
- u^* is continuous in t_0^b and t_1^b.
- Conclusion: Measure μ has no atoms on interior intervals including the contact points.

Earlier results: [Shvartsman & Vinter '06], [Frankowska '06]

We already know that the measure

$$\nu(t) = \int_{[0,t)} \mu(ds)$$

is absolutely continuous for all $t \in [0, t] \subset [0, T]$. Thus the multiplier

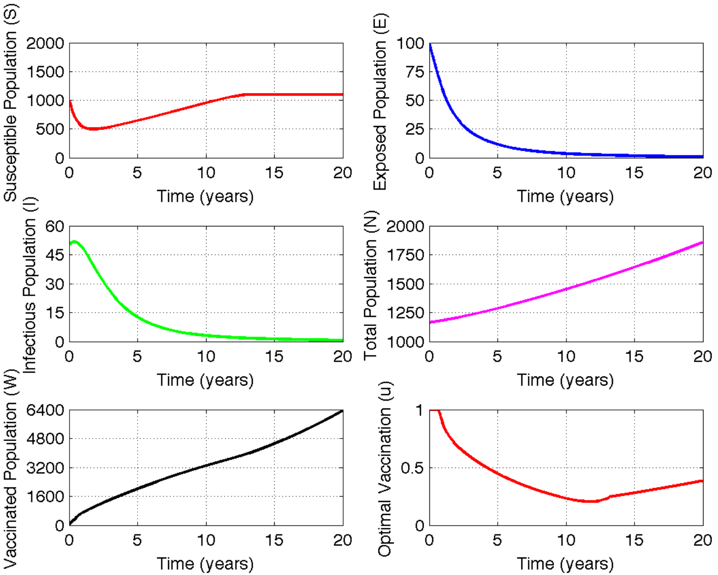
$$q_S(t) = p_S(t) + \nu(t)$$

is absolutely continuous.

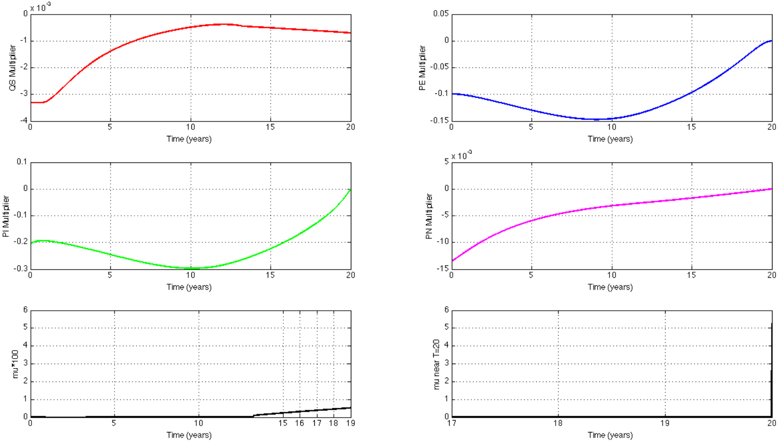
But the measure may have an atom in the end point T

This is exactly the case with SEIR problem.

Numerical Solution

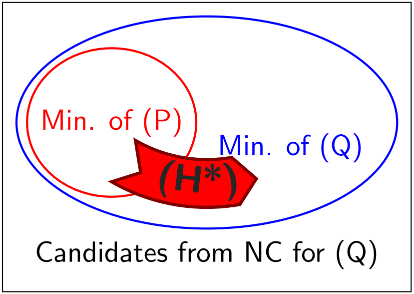


Numerical Multipliers



NOTE:
The multiplier p_S is not 0 at the end;
Measure μ has an atom at $T = 20$.

Conclusion



- NC for state constrained problems are "easy to handle" if the measure is AC w.r.t Lebesgue measure.
- Idea of an Exact Penalization to ensure such NC.

- Introduced a hypothesis (H*) to guarantee Exact Penalization of (P) via (Q).
- SEIR problem as a counterexample that Exact Penalization does not work (if it did, the measure would be AC in the entire interval $[0, T]$)

Thank You
For Your Attention

His name is SADKO (too)