Speeding up Model Predictive Control via Al'brekht's Method and its Extensions

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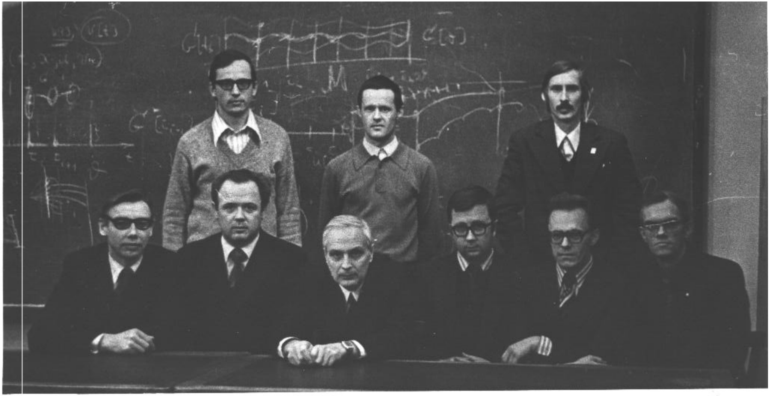
Альбрехт Э.Г. Об оптимальной стабилизации нелинейных систем. ПММ, т. 25, вып. 5, 1961, с. 836 - 844.

E. G. Al'brekht and N. N. Krasovski



Optimal Stabilization

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He was also influenced by the optimization techniques of L. S. Pontryagin and R. E. Bellman.

Stabilization Problem

Given a nonlinear system

$$\dot{x} = f(x, u)$$

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WLOG $x^e = 0, \ u^e = 0$ **.**

Infinite Horizon Optimal Control Problem

$$\min_{u(0:\infty)} \int_0^\infty l(x,u)\ dt$$

$$\dot{x} = f(x,u), \qquad x(0) = x^0$$

$$x \in I\!R^{n\times 1}, \qquad u \in I\!R^{m\times 1}$$

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Optimal Cost $\pi(x)$ and Optimal Feedback $u = \kappa(x)$

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Hamiltonian, a function of x, u and a new variable $p \in I\!R^{1\times n}$

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Hamilton Jacobi Bellman Equations

$$0 \;=\; \mathcal{H}\left(\frac{\partial \pi}{\partial x}(x), x, \kappa(x)\right)$$
$$\kappa(x) \;=\; \text{argmin}_u \mathcal{H}\left(\frac{\partial \pi}{\partial x}(x), x, u\right)$$

Lyapunov Argument

If the optimal cost $\pi(x)$ and optimal feedback $\kappa(x)$ can be found then a basin of attraction can be verified by a Lyapunov argument.

$$\frac{d}{dt}\pi(x(t)) = \frac{\partial \pi}{\partial x}(x(t))f(x(t),\kappa(x(t))) = -l(x(t),\kappa(x(t))) < 0$$

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Practical optimal control problems usually have state dimension larger than 2 or 3. For example, the attitude control problem for a spacecraft has state dimension $n = 6$ and control dimension at least $m = 3$. The position and attitude control problem for an airplane has state dimension $n = 12$ and control dimension at least $m = 4$.

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Consider trying to apply a grid based method. For the solution to be reasonably accurate we would need a substantial number of grid points in each coordinate direction, e.g., 10^2. Then the total number of nodes is 10^{12} for attitude control and 10^{24} for position and attitude control. If we can process 100 nodes a second that works out to about 300 years for attitude control and $3 \cdot 10^{14}$ years for position and attitude control.

Exception: Linear Quadratic Regulator

If the dynamics is linear and the Lagrangian quadratic

$$f(x,u) = Fx + Gu, \qquad l(x,u) = \frac{1}{2}\left(x'Qx + u'Ru\right)$$

then the optimal cost is quadratic and optimal feedback is linear

$$\pi(x) = \frac{1}{2}x'Px, \qquad \kappa(x) = Kx$$

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Theorem: If $Q \geq 0$, $R > 0$, (F,G) stabilizable and $(Q^{1/2}, F)$ detectable then there exist a unique nonnegative definite solution P to the Riccati equation and the feedback $u = Kx$ is asymptotically stabilizing, i.e., all the poles of $F + GK$ are in the open left half plane.

Al'brekht's Method

Al'brecht developed the power series method for solving the HJB equations for smooth systems that have Taylor series expansions.

$$
\begin{aligned}
f(x,u) &= Fx + Gu + f^{[2]}(x,u) + f^{[3]}(x,u) + \ldots \\
l(x,u) &= \frac{1}{2}\left(x'QX + u'Ru\right) + l^{[3]}(x,u) + l^{[4]}(x,u) + \ldots
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He assumed that the optimal cost and optimal feedback had similar expansions

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He plugged these expansions into HJB. At the lowest degrees he got the familiar LQR equations

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\begin{aligned}
0 &= F'P + PF + Q - PGR^{-1}G'P \\
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Al'brekht's Method

Next the unknown degree three terms $\pi^{[3]}(x)$ of the cost and the unknown degree two terms $\kappa^{[2]}(x)$ of the feedback satisfy

$$0 = \frac{\partial \pi^{[3]}}{\partial x}(x)(F + GK)x + x'Pf^{[2]}(x, Kx) + l^{[3]}(x, Kx)$$

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Notice the linear triangular structure. Under the standard LQR assumptions the first linear equation is always solvable for $\pi^{[3]}(x)$ because the eigenvalues of the map

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Al'brekht's method works in reasonable dimensions. For example, the HJB equations can be solved to degree 4 in $\pi(x)$ and degree 3 in $\kappa(x)$ for systems with state dimension $n = 25$ and control dimension $m = 8$ on this four year old laptop.

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Al'brekht's method is fast. This laptop took 0.082334 seconds to solve the HJB equations for the satellite attitude problem, $(n = 6,\ m = 3)$, to degree 4 in $\pi(x)$ and degree 3 in $\kappa(x)$.

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- **Going to higher degree approximations requires more memory. There are $n + d - 1$ choose d monomials of degree d in n variables, approximately $n^d/d!$.**

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- **Al'brekht's Method can be used to speed up Model Predictive Control (MPC)!**

Discrete Time Infinite Horizon Optimal Control

Minimize

$$\sum_{t=0}^{\infty} l(x(t), u(t))$$

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subject to

$$\begin{aligned} x^+ &= f(x, u) \\ x(0) &= x^0 \end{aligned}$$

where $x^+(t) = x(t+1)$.

Dynamic Programming Equations

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Bellman's Dynamic Programming Equations

$$\begin{aligned} \pi(x) &= \pi(f(x,\kappa(x)) + l(x,\kappa(x)) \\ \kappa(x) &= \mathrm{argmin}_u \{\pi(f(x,u)) + l(x,u)\} \end{aligned}$$

Al'brekht's Method in Discrete Time

Expand everything in power series and collect terms of lowest degree. This yields the familiar discrete time LQR equations,

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\begin{aligned}
P &= F'PF + Q - F'PG\left(R + G'PG\right)^{-1} G'PF \\
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At the next degrees we get

$$
\begin{aligned}
0 &= \pi^{[3]}((F+GK)x) - \pi^{[3]}(x) \\
&\quad + l^{[3]}(x,Kx) + (f^{[2]}(x,Kx))'P(F+GK)x
\end{aligned}
$$

$$
\begin{aligned}
0 &= \frac{\partial \pi^{[3]}}{\partial x}((F+GK)x)G + \frac{\partial l^{[3]}}{\partial u}(x,Kx) \\
&\quad + (f^{[2]}(x,Kx)'PG + ((F+GK)x)'P\frac{\partial f^{[2]}}{\partial u}(x,Kx) \\
&\quad + (\kappa^{[2]}(x))'(R+G'PG)
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Again these equations are linear and triangular as only the unknown $\pi^{[3]}$ appears in the first one.

This linear equation is always solvable because the eigenvalues of the map

$$\pi^{[3]}(x) \mapsto \pi^{[3]}(x) - \pi^{[3]}((F + GK)x)$$

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Under the standard discrete LQR assumptions all the eigenvalues of $F + GK$ are strictly inside the unit disk so their triple products are also strictly inside the unit disk.

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Then the second linear equation is always solvable for the other unknown $\kappa^{[2]}$ because R is assumed to be invertible.

The higher degree terms are found in a similar fashion.

Al'brekht with Equality Constraints

We now show how Al'brekht's method can be extended to handle an equality constraint of the form

$$\mathbf{0} = \boldsymbol{\alpha}(x, u)$$

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Because the constraint must hold at $x = 0,\ u = 0$

$$a(x,u) = Ax + Bu + a^{[2]}(x,u) + a^{[3]}(x,u) + \dots$$

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$$a(x, u) = Ax + Bu + a^{[2]}(x, u) + a^{[3]}(x, u) + \dots$$

We assume that $\frac{\partial \alpha}{\partial u}(x, u)$ is of full row rank. If this does not hold then at least one of the constraints can be expressed in terms of x alone. Then we just reduce the state dimension.

Al'brekht with Equality Constraints

We attach this constraint to the second HJB equation with a state dependent Lagrange multiplier $\lambda(x) \in I\!R^{k\times 1}$

$$\kappa(x) = \operatorname{argmin}_{u,\lambda} \left\{ \frac{\partial \pi}{\partial x}(x)(f(x) + g(x)u) + l(x,u) + \lambda'(x)a(x,u) \right\}$$

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Because this is strictly convex in u and linear in λ it reduces to

$$\begin{aligned} 0 &= \frac{\partial \pi}{\partial x}(x)g(x) + \frac{\partial l}{\partial u}(x,\kappa(x)) + \lambda'(x)\frac{\partial a}{\partial u}(x,\kappa(x)) \\ 0 &= a(x,\kappa(x)) \end{aligned}$$

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$$\begin{aligned} 0 &= \frac{\partial \pi}{\partial x}(x)g(x)+\frac{\partial l}{\partial u}(x,\kappa(x))+\lambda'(x)\frac{\partial a}{\partial u}(x,\kappa(x)) \\ 0 &= a(x,\kappa(x)) \end{aligned}$$

Assume

$$\lambda(x) = Lx+\lambda^{[2]}(x)+\lambda^{[3]}(x)+\dots$$

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Assume

$$\lambda(x) = Lx+\lambda^{[2]}(x)+\lambda^{[3]}(x)+\dots$$

Plug this and the other expansions into the HJB equations and collect terms of lowest degree.

Al'brekht with Equality Constraints

This leads to an unusual Riccati equation for P and L

$$0 = PF + F'P + Q - (PG + S)R^{-1}(G'P + S') + L'BR^{-1}B'L$$

The optimal feedback linear gain is

$$K = -R^{-1}\left(G'P + S' + B'L\right)$$

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Since B has full row rank we can reorder the controls so that the last k columns of B form an invertible $k \times k$ matrix B^2. Partition accordingly

$$\begin{aligned} u &= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ B &= \begin{bmatrix} B^1 & B^2 \end{bmatrix} \\ G &= \begin{bmatrix} G^1 & G^2 \end{bmatrix} \\ S &= \begin{bmatrix} S^1 & S^2 \end{bmatrix} \\ R &= \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \end{aligned}$$

Al'brekht with Equality Constraints

The linear part of the constraint forces the partial feedback

$$u_2 \;=\; -(B^2)^{-1}\left(Ax + B^1 u_1\right)$$

This leads to an unconstrained LQR problem in the free control u_1 which is of dimension $m-k$. We solve this problem for P, K_1 and compute L from the linear part of the second HJB equation.

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The quadratic part of the cost and the linear part of the feedback are

$$\begin{aligned}
\pi^{[2]}(x) &= \frac{1}{2}x'Px \\
\kappa^{[1]}(x) &= \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} x = \begin{bmatrix} K_1 \\ -(B^2)^{-1}\left(A + B^1 K_1\right)x \end{bmatrix}
\end{aligned}$$

Al'brekht with Equality Constraints

The cubic part of the first HJB equation is

$$
\begin{aligned}
0 \;=\;& \frac{\partial \pi^{[3]}}{\partial x}(x)(F+GK)x + x'P\left(f^{[2]}(x,Kx) + G\kappa^{[2]}(x)\right) \\
& + l^{[3]}(x,Kx) + x'K'R\,\kappa^{[2]}(x)
\end{aligned}
$$

with the unknowns in red.

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This reduces to an equation for $\pi^{[3]}(x)$ alone,

$$\begin{aligned} 0 &= \frac{\partial \pi^{[3]}}{\partial x}(x)(F+GK)x + x'Pf^{[2]}(x,Kx) \\ &\quad + l^{[3]}(x,Kx) + x'La^{[2]}(x,Kx) \end{aligned}$$

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The cubic part of the first HJB equation is

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If $F+GK$ is Hurwitz then this equation is uniquely solvable for $\pi^{[3]}(x)$.

Al'brekht with Equality Constraints

The quadratic part of the second HJB equation is

$$\begin{aligned} 0 &= \frac{\partial \pi^{[3]}}{\partial x}(x)G + x'P\frac{\partial f^{[2]}}{\partial u}(x,Kx) + \frac{\partial l^{[3]}}{\partial u}(x,Kx) \\ &\quad + x'L'\frac{\partial a^{[2]}}{\partial u}(x,Kx) + (\kappa^{[2]}(x))'R + (\lambda^{[2]}(x))'B \end{aligned}$$

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The quadratic part of the constraint is

$$0 = B\kappa^{[2]}(x) + a^{[2]}(x,Kx)$$

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$$\begin{bmatrix} R & B' \\ B & 0 \end{bmatrix} \begin{bmatrix} \kappa^{[2]}(x) \\ \lambda^{[2]}(x) \end{bmatrix} = \text{Known Terms}$$

Because B is assumed to be of full row rank this linear equation is uniquely solvable.

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The higher degree terms are found in a similar way.

Al'brekht with Inequality Constraints

Suppose we have the constraint

$$\mathbf{0} \quad \geq \quad \boldsymbol{\beta}(\boldsymbol{x}, \boldsymbol{u})$$

which we assume is not active at the origin $\boldsymbol{\beta}(\mathbf{0}, \mathbf{0}) < \mathbf{0}$.

Al'brekht with Inequality Constraints

Suppose we have the constraint

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Frequently such constraints can be handle by adding penalty terms to the Lagrangian $l(x, u)$.

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Frequently such constraints can be handle by adding penalty terms to the Lagrangian $l(x,u)$.

Here are two simple examples.

Al'brekht with an Inequality State Constraint

Unstable linear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

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Lagrangian

$$l(x,u) = \frac{1}{2}\left(x_1^2 + x_2^2 + u^2\right)$$

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Lagrangian

$$l(x, u) = \frac{1}{2} \left(x_1^2 + x_2^2 + u^2 \right)$$

State Constraint

$$x_1 \leq 0.5$$

Al'brekht with an Inequality State Constraint

Unstable linear system

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Lagrangian

$$l(x,u) = \frac{1}{2}\left(x_1^2 + x_2^2 + u^2\right)$$

State Constraint

$$x_1 \leq 0.5$$

Initial Condition

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.7 \end{bmatrix}$$

Al'brekht with a State Inequality Constraint

Linear Feedback

$$\begin{aligned} l(x,u) &= \frac{1}{2}\left(x_1^2 + x_2^2 + u^2\right) \\ u &= -2.4142x_1 - 2.4142x_2 \end{aligned}$$

Al'brekht with a State Inequality Constraint

Linear Feedback

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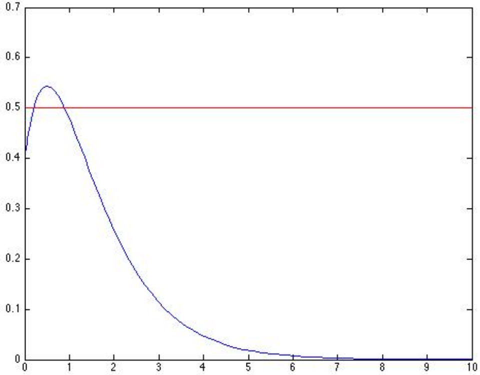
l(x,u) &= \frac{1}{2}\left(x_1^2 + x_2^2 + u^2\right) \\
u &= -2.4142x_1 - 2.4142x_2
\end{aligned}
$$



Al'brekht with a State Inequality Constraint

Quintic Feedback

$$
\begin{aligned}
l(x,u) &= \frac{1}{2}\left(x_1^2 + x_2^2 + u^2\right) + 32x_1^5 + 64x_1^6 \\
u &= -2.41x_1 - 2.41x_2 \\
&\quad -22.62x_1^4 - 28.49x_1^3x_2 - 15.36x_1^2x_2^2 - 4.00x_1x_2^3 - 0.41x_2^4 \\
&\quad -45.25x_1^5 - 67.01x_1^4x_2 - 45.60x_1^3x_2^2 - 16.93x_1^2x_2^3 - 3.34x_1x_2^4 - 0.27x_2^5
\end{aligned}
$$

Al'brekht with a State Inequality Constraint

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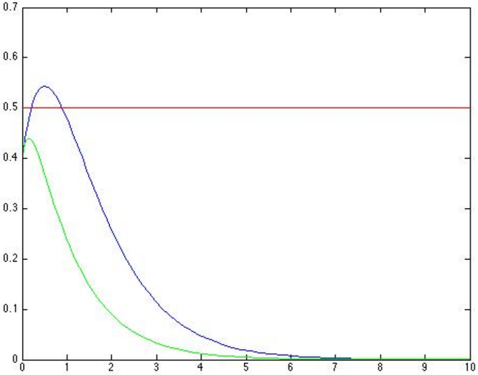
$$



Al'brekht with a Control Inequality Constraint

$$
\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\
l(x,u) &= \frac{1}{2}\left(x_1^2 + x_2^2 + u^2\right)
\end{aligned}
$$

Al'brekht with a Control Inequality Constraint

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$$l(x, u) = \frac{1}{2}\left(x_1^2 + x_2^2 + u^2\right)$$

Control Constraint

$$|u| \leq 1$$

Al'brekht with a Control Inequality Constraint

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$$l(x, u) = \frac{1}{2}\left(x_1^2 + x_2^2 + u^2\right)$$

Control Constraint

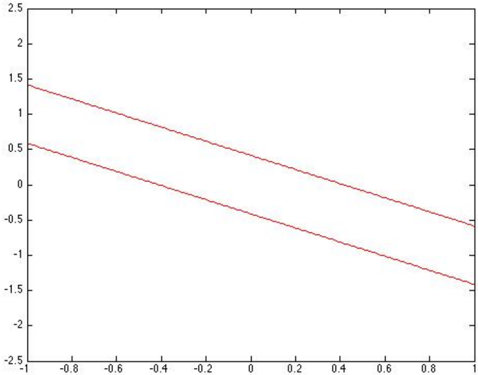
$$|u| \leq 1$$

Linear Feedback

$$u = -2.4142(x_1 + x_2)$$

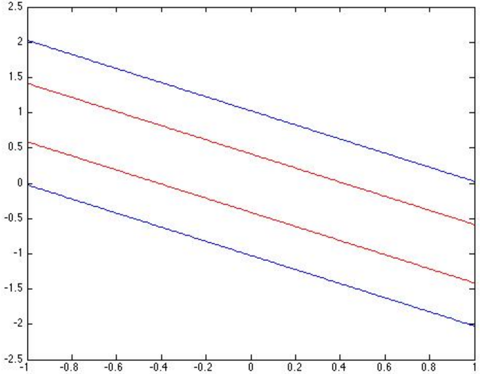
Feasible Region of Linear Feedback

$$u = -2.4142(x_1 + x_2)$$



Feasible Region of Cubic Feedback

$$
\begin{aligned}
l(x,u) &= \frac{1}{2}\left(x_1^2 + x_2^2 + u^2\right) + \frac{1}{10}u^4 \\
u &= -2.4142(x_1 + x_2) - 3.2263(x_1 + x_2)^3
\end{aligned}
$$



Finite Horizon Optimal Control Problem

Minimize

$$\sum_{t=t_0}^{t_f-1} l(t, x(t), u(t)) + \pi_f(x(t_f))$$

subject to

$$\begin{aligned} x^+ &= f(t, x, u) \\ x(t_0) &= x^0 \end{aligned}$$

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subject to

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The terminal cost is $\pi_f(x(t_f))$.

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subject to

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The terminal cost is $\pi_f(x(t_f))$.

The optimal cost is $\pi(t_0, x^0)$.

The optimal feedback is $u(t_0) = \kappa(t_0, x^0)$.

Finite Horizon Optimal Control Problem

The Dynamic Promming Equations for this problem are

$$\begin{aligned}
\pi(t,x) &= \pi(t+1, f(t,x,\kappa(t,x))) + l(t,x,\kappa(t,x)) \\
\kappa(t,x) &= \operatorname{argmin}_u \{\pi(t+1, f(t,x,u)) + l(t,x,u)\} \\
\pi(t_f,x) &= \pi_f(x)
\end{aligned}$$

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Assuming the Hamiltonian is strictly convex in u then $\kappa(t,x)$ is the solution of

$$0 = \frac{\partial \pi}{\partial x}(t+1, f(t,x,\kappa(t,x)))\frac{\partial f}{\partial u}(t,x,\kappa(t,x)) + \frac{\partial l}{\partial u}(t,x,\kappa(t,x))$$

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Assuming the Hamiltonian is strictly convex in u then $\kappa(t,x)$ is the solution of

$$
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$$

Henceforth we shall assume that f is linear in u and l is quadratic in u which ensures strict convexity,

$$
\begin{aligned}
f(t,x,u) &= f(t,x) + g(t,x)u \\
l(t,x,u) &= \frac{1}{2}\left(x'Q(t,x)x + 2x'S(t,x)u + u'R(t,x)u\right)
\end{aligned}
$$

where $R(t,x) > 0$, $\begin{bmatrix} Q(t,x) & S(t,x) \\ S'(t,x) & R(t,x) \end{bmatrix} \geq 0.$

Finite Horizon Optimal Control Problem

If this can be solved for $\kappa(t, x)$ then the result is plugged into the first DPE. It becomes a difference equation for $\pi(t, x)$ that is solved backward in time from the final condition $\pi(t_f, x) = \pi_f(x)$.

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Of course this is easier said than done!

Al'brekht's Method around an Optimal Trajectory

Let $x^*(t),\ u^*(t)$ be an optimal trajectory and define variational coordinates

$$\begin{aligned} z &= x - x^*(t) \\ v &= u - u^*(t) \end{aligned}$$

Al'brekht's Method around an Optimal Trajectory

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$$\begin{aligned} z &= x - x^*(t) \\ v &= u - u^*(t) \end{aligned}$$

In these coordinates the variational dynamics and the variational Lagrangian are given by

$$\begin{aligned} \tilde{f}(t,z,v) &= f(t,x^*(t)+z,u^*(t)+v) - f(t,x^*(t),u^*(t)) \\ \tilde{l}(t,z,v) &= l(t,x^*(t)+z,u^*(t)+v) - l(t,x^*(t),u^*(t)) \end{aligned}$$

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\tilde{l}(t, z, v) &= l(t, x^*(t) + z, u^*(t) + v) - l(t, x^*(t), u^*(t))
\end{aligned}
$$

The optimal variational cost and optimal variational feedback are given by

$$
\begin{aligned}
\tilde{\pi}(t_0, z^0) &= \pi(t, x^*(t) + z) - \pi(t, x^*(t)) \\
v(t_0) &= \tilde{\kappa}(t_0, z^0) = \kappa(t_0, x^*(t_0) + z^0) - u^*(t)
\end{aligned}
$$

Al'brekht's Method around an Optimal Trajectory

Following Al'brekht we expand everthing in power series in $z,\ v$

$$
\begin{aligned}
\tilde{f}(t,z) &= \tilde{F}(t)z + \tilde{f}^{[2]}(t,z) + \dots \\
\tilde{g}(t,z)v &= \tilde{G}(t)v + \tilde{g}^{[1]}(t,z)v + \dots \\
\tilde{l}(t,z,v) &= \tilde{\lambda}(t)z + \tilde{\mu}(t)v + \frac{1}{2}\left(z'\tilde{Q}(t)z + 2z'\tilde{S}(t)v + v'\tilde{R}(t)v\right) \\
&\quad + \tilde{l}^{[3]}(t,z,v) + \dots \\
\tilde{\pi}_f(z) &= \tilde{\rho}_f z + \frac{1}{2}z'\tilde{P}_f z + \tilde{\pi}_f^{[3]}(z) + \dots \\
\tilde{\pi}(t,z) &= \tilde{\rho}(t)z + \frac{1}{2}z'\tilde{P}(t)z + \tilde{\pi}^{[3]}(t,z) + \dots \\
\tilde{\kappa}(t,z) &= \tilde{K}(t)z + \tilde{\kappa}^{[2]}(t,z) + \dots
\end{aligned}
$$

Al'brekht's Method around an Optimal Trajectory

Following Al'brekht we expand everthing in power series in $z, \ v$

$$
\begin{aligned}
\tilde{f}(t,z) &= \tilde{F}(t)z + \tilde{f}^{[2]}(t,z) + \dots \\
\tilde{g}(t,z)v &= \tilde{G}(t)v + \tilde{g}^{[1]}(t,z)v + \dots \\
\tilde{l}(t,z,v) &= \tilde{\lambda}(t)z + \tilde{\mu}(t)v + \frac{1}{2}\left(z'\tilde{Q}(t)z + 2z'\tilde{S}(t)v + v'\tilde{R}(t)v\right) \\
&\quad + \tilde{l}^{[3]}(t,z,v) + \dots \\
\tilde{\pi}_f(z) &= \tilde{\rho}_f z + \frac{1}{2}z'\tilde{P}_f z + \tilde{\pi}_f^{[3]}(z) + \dots \\
\tilde{\pi}(t,z) &= \tilde{\rho}(t)z + \frac{1}{2}z'\tilde{P}(t)z + \tilde{\pi}^{[3]}(t,z) + \dots \\
\tilde{\kappa}(t,z) &= \tilde{K}(t)z + \tilde{\kappa}^{[2]}(t,z) + \dots
\end{aligned}
$$

What is different is the presence of linear terms in $\tilde{l}(t,z,v), \ \tilde{\pi}_f(z), \ \tilde{\pi}(t,z)$.

Al'brekht's Method around an Optimal Trajectory

Variational Dynamic Programming Equations (VDPE)

Al'brekht's Method around an Optimal Trajectory

Variational Dynamic Programming Equations (VDPE)

VDPE 1

$$\tilde{\pi}(t,z) \;=\; \tilde{\pi}(t+1,\tilde{f}(t,z,\tilde{\kappa}(t,z))) + \tilde{l}(t,z,\tilde{\kappa}(t,z))$$

VDPE 2

$$0 = \frac{\partial \tilde{\pi}}{\partial x}(t+1,f(t,x,\tilde{\kappa}(t,x)))\frac{\partial \tilde{f}}{\partial v}(t,x,\tilde{\kappa}(t,x)) + \frac{\partial \tilde{l}}{\partial v}(t,x,\tilde{\kappa}(t,x))$$

VDPE 3

$$\tilde{\pi}(t_f,x) \;=\; \tilde{\pi}_f(x)$$

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VDPE 3

$$\tilde{\pi}(t_f,x) \;=\; \tilde{\pi}_f(x)$$

We plug the above expansions into these equations and start collecting terms of lowest degree.

Al'brekht's Method around an Optimal Trajectory

VDPE 1, Degree 0

$$\mathbf{0} = \mathbf{0}$$

VDPE 2, Degree 0

$$\mathbf{0} = \tilde{\mu}(t) + \tilde{\rho}(t+1)G(t)$$

VDPE 1, Degree 1

$$\tilde{\rho}(t) = \tilde{\lambda}(t) + \tilde{\rho}(t+1)\tilde{F}(t)$$

VDPE 2, Degree 1

$$\mathbf{0} = z'\tilde{S}(t) + z'\tilde{P}(t+1)\tilde{G}(t) + \tilde{\rho}(t+1)\tilde{g}^{[1]}(t,z) + z'\tilde{K}'(t)\tilde{R}(t)$$

Al'brekht's Method around an Optimal Trajectory

VDPE 1, Degree 2

$$\begin{aligned} z'\tilde{P}(t)z &= \tilde{\mu}(t)\tilde{\kappa}^{[2]}(t,z) + (\tilde{\kappa}^{[2]}(t,z))'\tilde{\mu}'(t) \\ &\quad + z'\tilde{Q}(t)z + z'\tilde{S}(t)\tilde{K}(t)z + z'\tilde{K}'(t)\tilde{S}'(t)z \\ &\quad + z'\tilde{K}'(t)\tilde{R}'(t)\tilde{K}(t)z \\ &\quad + z'(\tilde{F}(t) + \tilde{G}(t)\tilde{K}(t))'\tilde{P}(t+1)(\tilde{F}(t) + \tilde{G}(t)\tilde{K}(t))z \end{aligned}$$

VDPE 2, Degree 2

$$\begin{aligned} 0 &= \frac{\partial \tilde{l}^{[3]}}{\partial v}(t,z,\tilde{K}(t,z)) \\ &\quad + \tilde{\rho}(t+1))\tilde{g}^{[2]}(t+1,(\tilde{F}(t) + \tilde{G}(t)\tilde{K}(t))z) \\ &\quad + z'(\tilde{F}(t) + \tilde{G}(t)\tilde{K}(t))'\tilde{P}(t+1)\tilde{g}^{[1]}(t+1,(\tilde{F}(t) \\ &\quad + \tilde{G}(t)\tilde{K}(t))z) + \tilde{G}(t)\tilde{K}(t))z) + \frac{\partial \tilde{\pi}^{[3]}}{\partial z}(t+1,(\tilde{F}(t) + \tilde{G}(t)\tilde{K} \\ &\quad + (\tilde{\kappa}^{[2]}(t,z)))'\tilde{R}^{(}t) + \frac{\partial \tilde{\pi}^{[3]}}{\partial z}(t+1,(\tilde{F}(t) + \tilde{G}(t)\tilde{K}(t))z)\tilde{G} \\ &\quad + (\tilde{\kappa}^{[2]}(t,z)))'\tilde{R}(t) \end{aligned}$$

Al'brekht's Method around an Optimal Trajectory

There are four equations in the four unknowns in $\tilde{\rho}(t), \tilde{P}(t), \tilde{K}(t), \tilde{\kappa}^{[2]}(t,z)$ at time t .

$$
\begin{aligned}
\tilde{\rho}(t) &= \tilde{\lambda}(t) + \tilde{\rho}(t+1)\tilde{F}(t) \\
0 &= z'\tilde{S}(t) + z'\tilde{P}(t+1)\tilde{G}(t) + \tilde{\rho}(t+1)\tilde{g}^{[1]}(t,z) \\
&\quad +\tilde{K}'(t)\tilde{R}(t) \\
z'\tilde{P}(t)z &= \tilde{\mu}(t)\tilde{\kappa}^{[2]}(t,z) + (\tilde{\kappa}^{[2]}(t,z))'\tilde{\mu}'(t) \\
&\quad +z'\tilde{Q}(t)z + z'\tilde{S}(t)\tilde{K}(t)z + z'\tilde{K}'(t)\tilde{S}'(t)z \\
&\quad +z'\tilde{K}'(t)\tilde{R}'(t)\tilde{K}(t)z \\
0 &= \frac{\partial \tilde{l}^{[3]}}{\partial v}(t,z,\tilde{K}(t,z)) \\
&\quad +\tilde{\rho}(t+1))\tilde{g}^{[2]}(t+1,(\tilde{F}(t)+\tilde{G}(t)\tilde{K}(t))z) \\
&\quad +z'(\tilde{F}(t)+\tilde{G}(t)\tilde{K}(t))'\tilde{P}(t+1)\tilde{g}^{[1]}(t+1,(\tilde{F}(t) \\
&\quad +\tilde{G}(t)\tilde{K}(t))z) + \frac{\partial \tilde{\pi}^{[3]}}{\partial z}(t+1,(\tilde{F}(t)+\tilde{G}(t)\tilde{K}(t))z)\tilde{G} \\
&\quad +(\tilde{\kappa}^{[2]}(t,z)))'\tilde{R}^{(}t)
\end{aligned}
$$

Al'brekht's Method around an Optimal Trajectory

These reduce to a Riccati difference equation for $\tilde{P}(t)$ and three linear difference equations for $\tilde{\rho}(t),\ \tilde{K}(t),\ \tilde{\kappa}^{[2]}(t,z)$

These equations run backward in time from the terminal conditions

$$
\begin{aligned}
\tilde{\rho}(t_f) &= \tilde{\rho}_f \\
\tilde{P}(t_f) &= \tilde{P}_f \\
\tilde{K}(t_f) &= \tilde{K}_f \\
\tilde{\kappa}^{[2]}(t,z) &= \tilde{\kappa}^{[2]}_f(z)
\end{aligned}
$$

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\end{aligned}
$$

Notice that we need the terminal cost $\tilde{\kappa}_f(z)$ as well as the terminal cost $\tilde{\pi}_f(z)$.

Al'brekht's Method around an Optimal Trajectory

At the next degree we get more difference equations

$$\tilde{\pi}^{[3]}(t,z) = \tilde{l}^{[3]}(t,z,\tilde{K}(t)) + z'\tilde{S}(t)\tilde{\kappa}^{[2]}(t,z) + (\tilde{\kappa}^{[2]}(t,z))'\tilde{S}'(t)z$$
$$+\tilde{\pi}^{[3]}(t+1,z) + z'(\tilde{F}(t)+\tilde{G}(t)\tilde{K}(t))'\tilde{P}(t+1)\tilde{f}^{[2]}(t,z)$$
$$+\tilde{g}^{[1]}(t,z)\tilde{K}(t)z + \tilde{G}(t)\tilde{\kappa}^{[2]}(t,z)$$
$$+\rho(t+1)(\tilde{f}^{[3]}(t,z) + \tilde{g}^{[2]}(t,z)\tilde{K}(t)z$$
$$+\tilde{g}^{[1]}(t,z)\tilde{\kappa}^{[2]}(t,z) + \tilde{G}(t)\tilde{\kappa}^{[3]}(t,z)$$

$$0 = \frac{\partial l^{[4]}}{\partial v}(t,z,\tilde{K}(t)z) + \left(\frac{\partial l^{[3]}}{\partial v}(t,z,\tilde{\kappa}^{[2]}(t,z)\right)^{[3]}$$
$$+\tilde{\rho}(t+1)\tilde{g}^{[3]}(t,z) + z'(\tilde{F}(t)+\tilde{G}(t)\tilde{K}(t))'\tilde{P}(t+1)\tilde{g}^{[2]}(t,z)$$
$$+\frac{\partial\tilde{\pi}^{[3]}}{\partial z}(t+1,z)\tilde{g}^{[1]}(t,z) + \frac{\partial\tilde{\pi}^{[4]}}{\partial z}(t+1,z)\tilde{G}(t) + (\tilde{\kappa}^{[3]}(t,z))'\tilde{R}(t)$$

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$$+\tilde{\rho}(t+1)\tilde{g}^{[3]}(t,z) + z'(\tilde{F}(t) + \tilde{G}(t)\tilde{K}(t))'\tilde{P}(t+1)\tilde{g}^{[2]}(t,z)$$
$$+\frac{\partial \tilde{\pi}^{[3]}}{\partial z}(t+1,z)\tilde{g}^{[1]}(t,z) + \frac{\partial \tilde{\pi}^{[4]}}{\partial z}(t+1,z)\tilde{G}(t) + (\tilde{\kappa}^{[3]}(t,z))'\tilde{R}(t)$$

Notice the linear triangular structure and the presence of $\tilde{\pi}^{[4]}(t+1,z)$ in the second equation. If we stop at degree three then this is set to zero.

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&+\tilde{\pi}^{[3]}(t+1,z) + z'(\tilde{F}(t)+\tilde{G}(t)\tilde{K}(t))'\tilde{P}(t+1)\tilde{f}^{[2]}(t,z) \\
&+\tilde{g}^{[1]}(t,z)\tilde{K}(t)z + \tilde{G}(t)\tilde{\kappa}^{[2]}(t,z) \\
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\end{aligned}$$

$$\begin{aligned}
0 &= \frac{\partial l^{[4]}}{\partial v}(t,z,\tilde{K}(t)z) + \left(\frac{\partial l^{[3]}}{\partial v}(t,z,\tilde{\kappa}^{[2]}(t,z)\right)^{[3]} \\
&+\tilde{\rho}(t+1)\tilde{g}^{[3]}(t,z) + z'(\tilde{F}(t)+\tilde{G}(t)\tilde{K}(t))'\tilde{P}(t+1)\tilde{g}^{[2]}(t,z) \\
&+\frac{\partial\tilde{\pi}^{[3]}}{\partial z}(t+1,z)\tilde{g}^{[1]}(t,z) + \frac{\partial\tilde{\pi}^{[4]}}{\partial z}(t+1,z)\tilde{G}(t) + (\tilde{\kappa}^{[3]}(t,z))'\tilde{R}(t)
\end{aligned}$$

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The higher degree terms are found in a similar fashion.

Patchy Method

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Patchy Methods

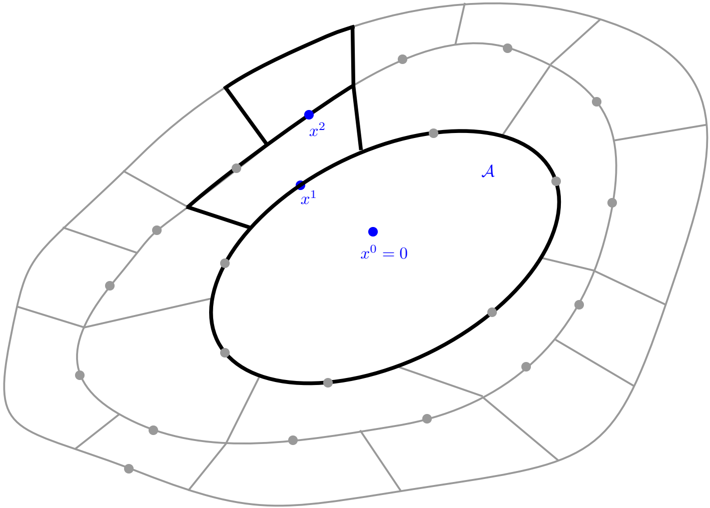

Figure : Sequence of Patches

Patch Calculation

The HJB equations are not singular away from the origin. The map

$$\pi^{[d+1]}(x) \mapsto \frac{\partial \pi^{[d+1]}}{\partial x}(x) f(x,u)$$

takes a polynomial of degree $d+1$ to a polynomial of degree d.

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Under suitable assumptions there is one positive root and one negative root. We take the positive root.

Patchy Methods

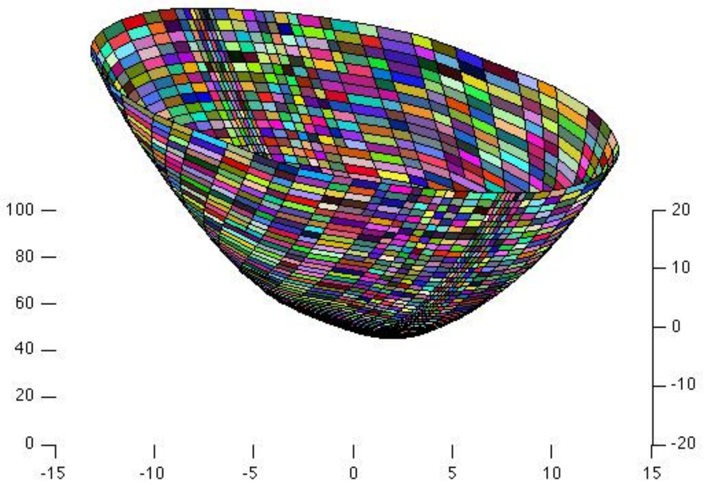

Figure : Optimal Cost of Inverting a Pendulum by a Torque at its Axis

Invert a Pendulum

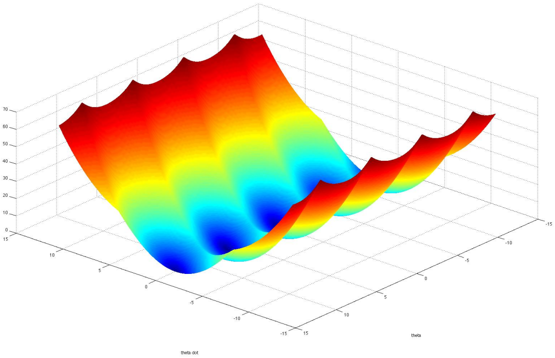

Figure : Periodicity of the Optimal Cost

The left axis is $-15 \leq \dot{\theta} \leq 15$ and the right axis is $-15 \leq \theta \leq 15$. From points on the ridges there are two optimal trajectories, one going to the left well and the other going to the right well.

Adaptive Algorithm

The algorithm is adaptive. It splits a patch in two when the relative residue of the first HJB equation is too high at the lower corners of a patch. It also lowers the upper level of a ring of patches if the relative residue is too high on it.

Ring	1	2	3	4
Initial Patch Level	0.64	1.21	1.96	2.89
Final Patch Level	0.36	0.63	1.38	2.23
Initial No. Patches	1	24	26	26
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$$(0.8)^2 \quad (1.1)^2 \quad (1.4)^2 \quad \ldots \quad (10.7)^2$$

Only the first ten patch levels were adjusted down.

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The last ring (34) contains 78 patches.

Error Comparison

A nonlinear change of state coordinates on an LQR problem yields a nonlinear optimal control problem.

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Here are the errors between the true optimal cost and the computed optimal cost which is of degree $d+1$.

	Max Error	Max Rel Error	Error Factor
LQR $(d=1)$	0.3543	0.8860	54.56
Al'brecht $(d=3)$	0.1636	0.4101	25.16
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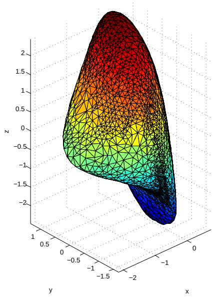
The patchy method can also be used when there are constraints. The constraint may be active or inactive on a patch.

Three Dimensional Example

Patchy method applied to a three dimensional problem



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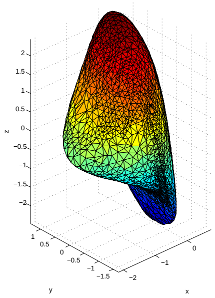

The complexity of keeping track of the patches makes the patchy method infeasible in higher dimesions.

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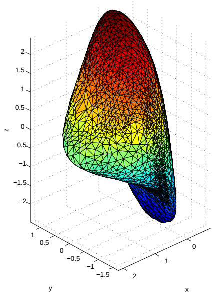
Patchy method applied to a three dimensional problem



The complexity of keeping track of the patches makes the patchy method infeasible in higher dimesions.

But adding one or two shells of patches to Al'brekht is feasible in moderate dimensions. The feedback can be linear on these shells which reduces the possibility of finite escape by the closed loop dynamics.

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But Al'brekht only yields a local solution about the origin.

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To make the solution global we combine Al'brekht with Model Predictive Control (MPC).

Model Predictive Control

Consider the infinite horizon problem of minimizing

$$\sum_{t=0}^{\infty} l(x(t), u(t))$$

subject to

$$\begin{aligned} x^+ &= f(x, u) \\ x(0) &= x^0 \\ 0 &\leq g(x, u) \end{aligned}$$

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The decision variables are $u(0), \ldots, u(T-1)$.

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Use the control $u^1(1)$ to get the state to $x^2 = x(2)$, etc.

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- **The initial guess of $u^0(0), \dots, u^0(T-1)$ that is fed to the solver must be close to optimal else the solver may fail to converge to the true solution.**

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- Al'brekht alone or with a shell or two of patches can furnish $\pi_T(x)$ and $\kappa_T(x)$ on a reasonably large $\mathcal{X}_T$!

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Al'brekht around an optimal trajectory also can be used to increase the available computational time which we now explain.

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So we compute the variational $\tilde{\pi}(s,z),\ \tilde{\kappa}(s,z)$ for $s = t, \ldots, t+T-1$ by Al'brekht around the optimal trajectory generated by $u^t(t), \ldots, u^t(t+T-1)$ and then at time t when $x(t)$ becomes known we use the control $u(t) = u^t(t) + \tilde{\kappa}(t, x(t) - \hat{x}(t))$.

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- **For a copy of these slides contact** `ajkrener@ucdavis.edu`