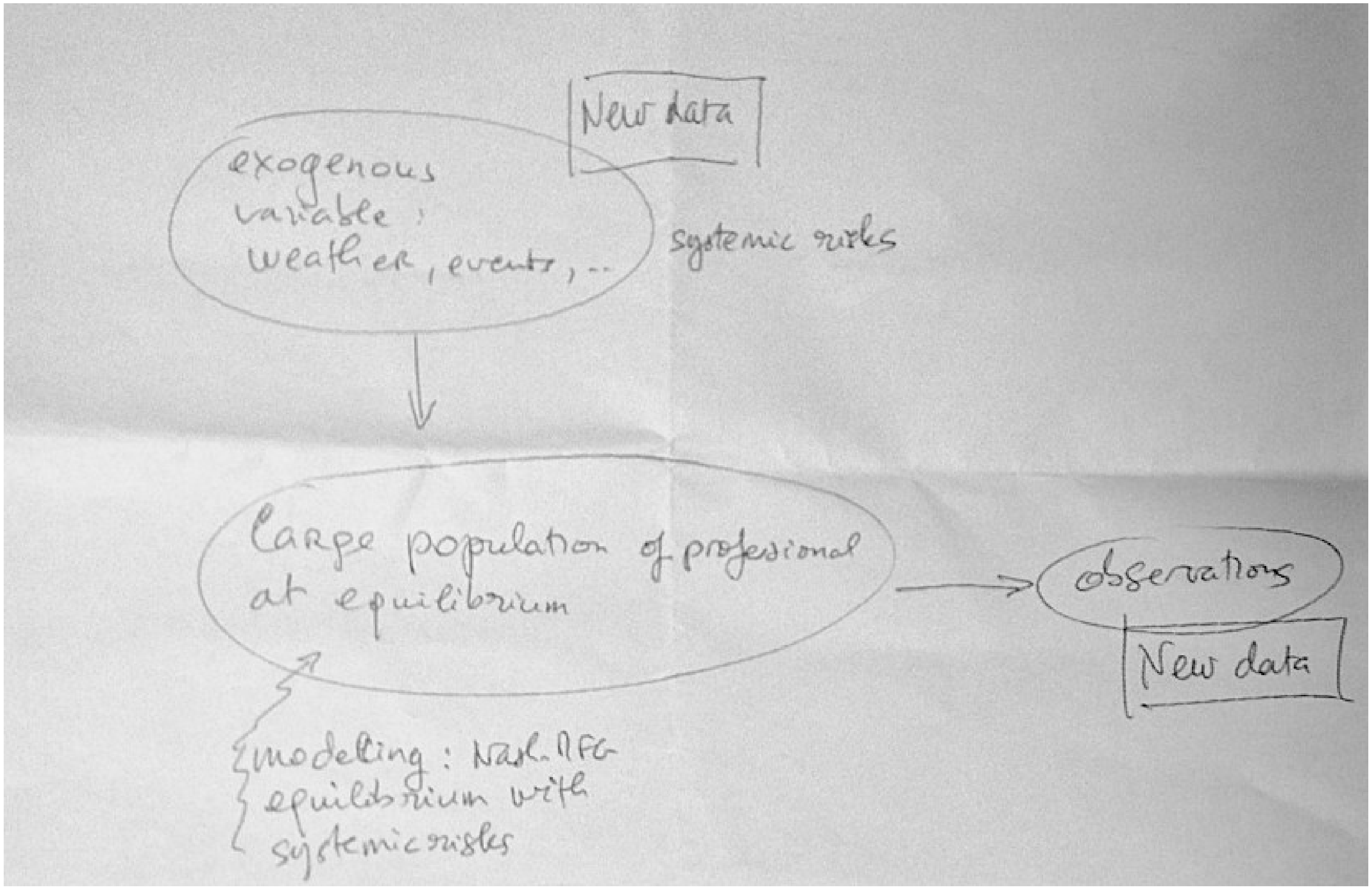
Modeling with Mean Field Games: MFG and new data

Jean Michel Lasry, november 10th, 2014
New perspectives in control and games
Roma, La Sapienza

« New perspectives in control and games »: a claim

- Mean Field Games together with new data
- will play a key role in the design of new socio-economics equilibrium models
- Ex: for activities with regulated pricing policy

- This will generate needs : new math results and new machine learning methods


New data
exogenous variable : weather, events, ..
systemic risks
Large population of professional at equilibrium
observations
New data
modeling : Nash MFG equilibrium with systemic risks

Insight?

- These works (mixing MFG and new data)
- Might produce some new machine learning algorithmic ideas

agenda

- 0. Mean field games : a fast track overview
- I. Forward/backward systems
- II. MFG-d : finite state space

0. Mean Field Games : Fast track overview

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Mean Field games

- Theory

  stochastic differential games

  with a continuum of players

- Applications

  a new modelling technology

A huge powerful inheritance

- Game theory : agents, strategies, cross expectations, equilibrium, common knowledge,..
  - Implies: coherence and inheritance with concepts, tools,.. of other class of games
- Continuous time and/or state space, use of differential and stochastic calculus..
  - Implies coherence and inheritance with deterministic and stochastic control theory

MFG = stochastic differential games with a continuum of players

- « Continuum of players »
  - is the key feature
  - that frame MFG theory
- The mix of
  - continuum of players
  - and continuous time stochastic control theory
  - is extremly powerful

Continuum of players

- This means that the community of agents is described by its density **m** on the state space of agents
- A tractable approximation of large class of differential games with N players, N large
- good news : while not so easy, the limit, i-e: MFG, is orders of magnitude more simpler than N players games (even with N small)
- Why? : Because agents are « atomized », anonymous, hence have no strategic power. MFG are halfway between optimisation and game theory

Other approach of the same concept

- Mean field methodology in physics:
  – Replace particles by agents, i-e players,
  – Meaning optimisation and cross expectations on other strategies
- Economics with incomplete markets, equilibrium under constraints and regulations, can often be viewed as MFG

MFG classification based on risk structure

- Each agent i faces risk $Z=(X_i,Y)$
  - X_i is the idiosyncratic (individual) risk
  - Y is the systemic risk
  - X_i , X_j , Y : are all pairwise independent

Classification :

- General case : infinite dimension PDE
- No systemic risk Y : forward/backward systems
- Finite agents state space : non linear hyperbolic PDE systems

forward/backward systems

- Each agent i faces risk $Z=(X_i,Y)$
- No systemic risk (no Y) :
- forward/backward systems
  - Forward FP equation of the deterministic dynamic of the density m of agents in the state space
  - Backward generic agent HJB equation
- Since seminal papers (2006), most MFG papers are about forward/backward systems

General case : Master Equation

- Each agent strategy depend on
  - x : his current state
  - m : the state of the community of agents
- The value function V depends both on the position x of the generic agent and on the density m of agents
- Hence the Master equation is a PDE in infinite dimension
- Must be worked out for macro economic theory:
  - JML&PLL introduced Master Equation to build sound mathematical answers to Robert Lucas insights and questions about Krussell Smith model

MFG-p: Master equation

$$(\text{MFG.P}) \begin{cases} \frac{\partial U}{\partial t} + (\nu+\alpha)\Delta_x U + H(x, \nabla_x U, m) + \\ +\langle \frac{\partial U}{\partial m}, +(\nu+\alpha)\Delta m - \text{ div } (\frac{\partial H}{\partial p} m)\rangle + \\ +\alpha \frac{\partial U}{\partial m^2}(\nabla m, \nabla m) + 2\alpha \langle \frac{\partial}{\partial m} \nabla_x U, \nabla m \rangle = 0 \end{cases}$$

and $U\big|_{t=T} = g(x,m)$

MFG-d and hyperbolic systems

- Each agent i faces risk $Z=(X_i,Y)$
- Finite state space : MFG hyperbolic systems
- Monotone case: positive results on hyperbolic systems contrasting with negative results in fluid dynamics
- Amazingly large and efficient modelling power
- Yet few papers using monotone systems

MFG on graphs

- Agents state space is a (finite, large) graph
- Specific features linked to large graphs case
- Potential applications to the new large sets of data (« big data »)

Family I:
forward/backward systems

(classic) stochastic control: agent 's problem and HJB framework

$$Max\ \mathbb{E}\int_0^T [f(X_t) + g(a_t)]dt + h(X_T)$$

$$with\ X_0 = x\ and\ dX = a_t dt + \sigma dW$$

$$the\ value\ function\ defined\ as$$

$$u(x,s) = Max\ \mathbb{E}\int_s^T [f(X_t) + g(a_t)]dt$$

$$with\ X_s = x\ and\ dX = a_t dt + \sigma dW$$

$$is\ the\ viscosity\ solution\ of\ (backward)\ HJB\ equation\ (s.t.hyp..):$$

$$(HJB) \quad 0 = \partial_t u + f(x) + g^*(\nabla u) + ½\sigma^2 \Delta u, \quad and \quad u(T,x) = h(x)$$

$$g^*(p) = Max\{p.y + g(y)\}; \quad a^*(x,t) = ArgMax\{\nabla u(x).y + g(y)\}$$

MFG agent 's problem

$$Max\ \mathbb{E}\int_0^T [f(X_t) + g(a_t)]dt + h(X_T)$$
$$with\ X_0 = x\ and\ dX = a_t dt + \sigma dW$$

$$\text{MFG:}\quad \mathrm{f(x) = f(m, x)},\quad h(x) = h(m_T, x)$$
$$\text{where } \mathrm{m(x,t)} \text{ is the density of agents in the state space}$$

$$example: \qquad Max\ \int_0^T ½(\dot{x}_t)^2 dt + m(T, X_T)$$

$$f = 0,\ \ \sigma = 0, ..$$

MFG agent 's optimal control

$$the\ value\ function$$

$$is\ the\ viscosity\ solution\ of\ (backward)\ HJB\ equation\ (s.t.hyp..):$$

$$(HJB) \quad 0 = \partial_t u + f(m,x) + g^*(\nabla u) + ½\sigma^2 \Delta u,$$

$$with\ \ u(T,x) = h(m_T,x)$$

$$g^*(p) = Max\{p.y + g(y)\}$$

$$a^*(x,t) = ArgMax\{\nabla u(t,x).y + g(y)\}$$

$$a^*(x,t) = \nabla g^*(\nabla u(t,x))$$

Density dynamics: the forward FP equation

$$each\ agent's dynamics\ is \quad dX = a^*(x,t)\,dt + \sigma dW$$

$$all\ agents\ have\ the\ same\ contol\ problem:$$
$$the\ optimal\ strategy\ of\ an\ agent\ depends\ where\ he\ is: x\ at\ time\ t$$
$$and\ does\ not\ depend\ on\ who\ he\ is$$

$$identical\ optimal\ feedback\ \ control\ a^*(x,t) = \nabla g^*\big(\nabla u(t,x)\big)$$

$$all\ agents\ sources\ of\ risk\ (all\ Brownian\ W)\ are\ independent\ \ (iid)$$

$$hence\ the\ density\ function\ dynamics\ is\ defined$$
$$by\ the\ deterministic\ forward\ FP\ equation$$

$$0 = \partial_t m + div(\,ma^*) - \tfrac{1}{2}\sigma^2 \Delta m, \ \ with\ m(0,x) = m_0(x)$$

MFG Equilibrium

$$the\ deterministic\ forward\ FP\ equation$$

$$0 = \partial_t m + div(\, m a^*) - ½\sigma^2 \Delta m, \quad with\ m(0, x) = m_0(x)$$

$$together\ with\ the\ previous\ HJB:$$

$$0 = \partial_t u + f(m, x) + g^*(\nabla u) + ½\sigma^2 \Delta u, \quad and\ \ u(T, x) = h(m, x)$$

$$and\ the\ definition\ of\ optimal\ feedback\ a^* = \nabla g^* \circ \nabla u$$

$$define\ the\ forward/backward\ MFG\ equilibrium$$

MFG stationnary Equilibrium

$$Max\ \mathbb{E}\int_0^{\infty} e^{-rt}[f(m_t, X_t) + g(a_t)]dt$$

$$with\ X_0 = x\ and\ dX = a_t dt + \sigma dW$$

$$then\ the\ forward\ FP\ equation\ writes$$

$$0 = div(\ ma^*) - ½\sigma^2 \Delta m$$

$$and\ the\ HJB\ writes$$

$$0 = -ru + f(m, x) + g^*(\nabla u) + ½\sigma^2 \Delta u, \quad and\ \ u(T, x) = h(m, x)$$

$$and\ the\ optimal\ feedback\ a^* = \nabla g^* \circ \nabla u$$

Eductive dynamics to equilibrium

$define\ u\ and\ m\ as\ a\ function\ of\ x\ and\ a\ virtual\ time\ \theta$
$satisfying\ the\ PDE\ system:$

$$\partial_\theta m = -div(\, m\, \nabla g^* \circ \nabla u_\theta) + ½\sigma^2 \Delta m$$

$$\partial_\theta u = -ru + f(m,x) + g^*(\nabla u) + ½\sigma^2 \Delta u$$

$This\ approximation\ process\ refers\ to\ the\ mind\ time$
$of\ an\ external\ observer\ trying\ to\ correct\ a\ mistaken\ equilibrium$

Entrants, in and out flow of agents

$in\ and\ out\ \ flow\ of\ players: a\ source\ term\ \varphi\ forward\ in\ the\ FP\ equation$

$$0 = \partial_t m + div(\ ma^*) - ½\sigma^2 \Delta m - \varphi$$

$no\ change\ in\ the\ HJB$

$$0 = \partial_t u + f(m,x) + g^*(\nabla u) + ½\sigma^2 \Delta u,$$

$in\ many\ applications\ the\ flow\ depends\ on\ u\ and\ m$

$example: \varphi = u - \varepsilon m,$

$entrant\ flow\ proportionnal\ to\ utility\ u\ to\ be\ in,$

$and\ death\ process\ proportional\ to\ m$

MFG agent ’s problem : congestion

$$Max\ \mathbb{E}\int_0^T [f(X_t) + g(a_t)]dt + h(X_T)$$
$$with\ X_0 = x\ and\ dX = a_t dt + \sigma dW$$

MFG: $\mathrm{f(x) = f(m, x)}$, $h(x) = h(m_T, x)$
where $\mathrm{m(x, t)}$is the density of agents in the state space

$$g(a_t) = g(a_t, m)$$

$$example:\ g(a, m) = (1/\alpha)\, a^{\alpha}(\gamma + m^{\beta})$$

MFG agent 's problem : n populations

$$each\ agent\ in\ population\ i\ \ (i = 1, \dots, n) maximise$$

$$Max\ \mathbb{E} \int_0^T [f_i(m_t, X_t) + g_i(a_t)]dt + h_i(m_T, X_T)$$

$$with\ X_0 = x\ and\ dX = a_t dt + \sigma dW$$

$$f_i(\mathrm{m}, \mathrm{x}) = f_i(m_1, \dots, m_n, \mathrm{x}),$$

where $m_i(\mathrm{x}, \mathrm{t})$ is the density of population i

N populations MFG equilibrium

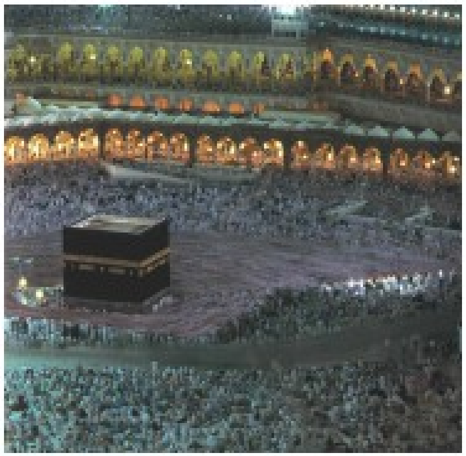
$$Hence, if\ there\ are\ n\ populations\ the\ MFG\ equilibrium, is\ defined\ by\ the\ following\ system\ forward - backward\ equations$$

$$0 = \partial_t m_i + div(\, m_i a_i^*) - ½\sigma^2 \Delta m_i, \;\; with\ m_i(0, x) = m_{i_0}(x)$$

$$0 = \partial_t u_i + f_i(m, x) + g_i^*(\nabla u_i) + ½\sigma^2 \Delta u_i,$$

$$and\ the\ definition\ of\ optimal\ feedback\ a_i^* = \nabla g_i^* \circ \nabla u_i$$

Crowd dynamics



Motivations

- Today half of the human population lives in urban areas, in 1950 $\sim$ 30%, prediction for 2050 $\sim$ 70%.
- Fatal accidents in the last decades increased, e.g. Hadj in Mekka, Love Parade in Duisburg, Water Festival in Phnom Penh . . ..
- Empirical studies of human crowd started about 50 years ago, based on observations, photographs and video data.
- Mathematical modeling and simulations have been used successfully to secure dangerous

Crowd dynamics

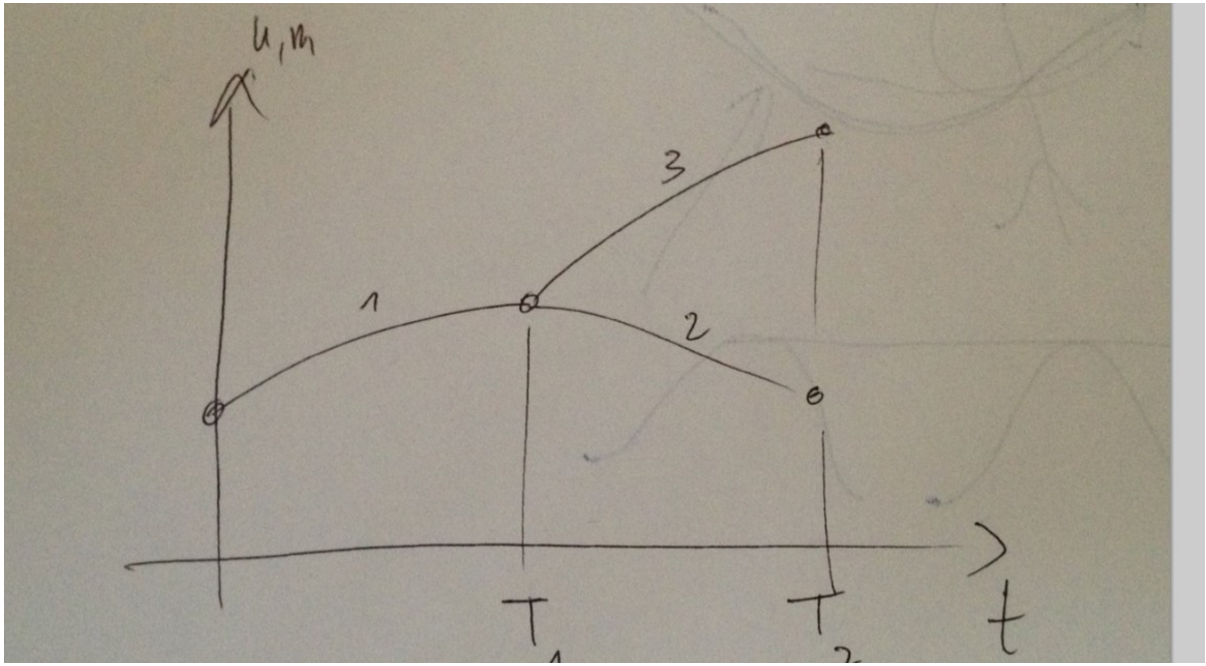
- Is crowd dynamics a game ?
- Is it common knowledge between people in the crowd that crowd dynamics is a game ?

- Depends on the context
- Also depends on modeling objectives

Crowd dynamics: MFG modeling

- MT Wolfram, A Lachapelle; Y Achdou;..
- Sophisticated forward looking crowd behaviours can be easily explained by MFG models
- Including some systemic information shocks

A quantum of systemic risk



A quantum of systemic risk: agent's problem

- $$Max\ \mathbb{E} \int_0^{T_1} F_1(X_t, a_t, m_t)\, dt + \int_{T_1}^{T_2} F_k(X_t, a_t, m_t)\, dt$$

$$with\ X_0 = x\ and\ dX = a_t dt + \sigma dW$$

$$where\ expectation\ \mathbb{E}\ refers\ to\ the\ two\ stochastic\ variables: W\ and\ k$$

$$k = 2\ with\ probability\ p,\ \ k = 3\ with\ probability\ 1 - p$$

$$It\ is\ a\ common\ knowledge\ at\ initial\ time\ 0$$
$$that\ the\ value\ of\ k\ will\ be\ brodcasted\ at\ time\ T_1$$

A quantum of systemic risk: MFG equilibrium

$$Three\ forward - backward\ systems\ (one\ on\ each\ branch)$$

$$0 = \partial_t m_i + div\left(m_i a_t^i \right) - ½\sigma^2 \Delta m_i \quad i = 1,2,3$$

$$0 = \partial_t u_i + F_i^*\left(X_t, a_t^i, m_t\right) + ½\sigma^2 \Delta u_i \quad i = 1,2,3$$

$$where \quad F_i^*(x,p,m) = Max\ \{pa + F_i(x,a,m)\}$$
$$and \quad a_t^i(t,x) = ArgMax\ \{pa + F_i(x,a,m)\}$$

$$initial\ condition\ \ m_1(0,x) = m_0(x)$$
$$and\ final\ condition\ u_k(T_2,x) = 0\ \ for\ k = 2,3$$

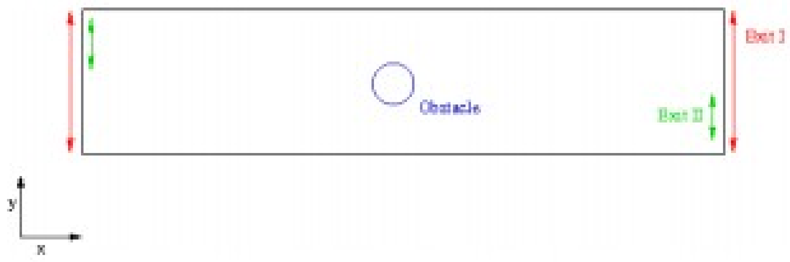
and « systemic risk » matching conditions at time T_1 :

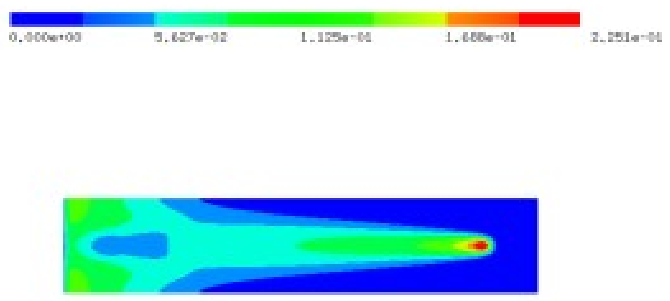
$$m_k(T_1,x) = m_1(T_1,x)\ \ for\ k = 2,3$$

$$u_k(T_1,x) = p\, u_2(T_1,x) + (1-p)\, u_3(T_1,x)$$

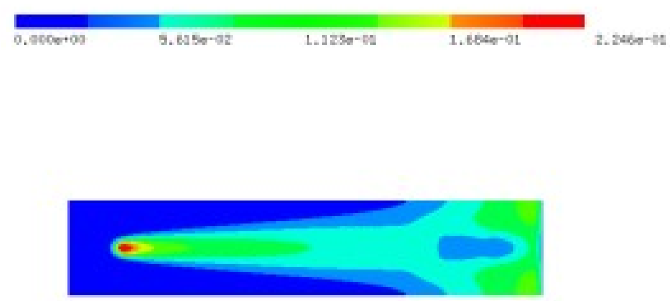
A Lachapelle MT Wolfram
Sophisticated two populations dynamics

Same parameters as in the previous examples but the exits are different





(a) Population m_1

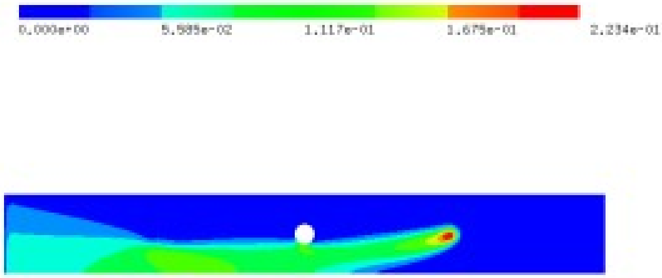


(b) Population m_2

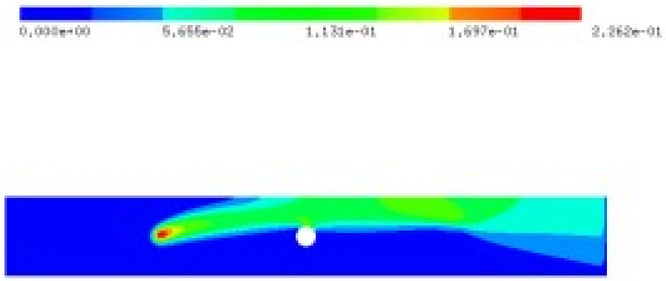
A Lachapelle MT Wolfram
Sophisticated two populations dynamics

- Computational domain $\Omega = [-1.5, 1.5] \times [-0.2, 0.2]$
- Single source of people for every species, i.e. $f(x) = 50 \times \exp(-\frac{(x \pm 0.75)^2 + y^2}{10^{-3}})$
- The parameters are

$$a = 0.25,\ \tilde{a} = 0.75,\ q = 2,\ ,\nu = 0.05,\ k = 1,\ r = 1.$$



(c) Population m_1

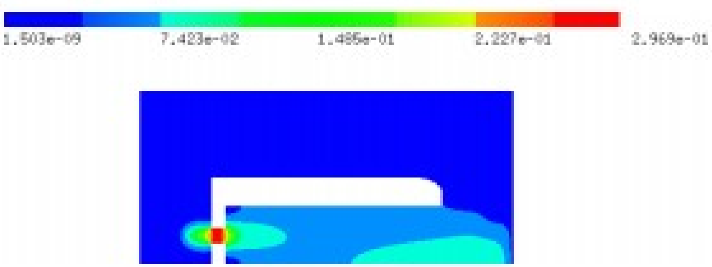


(d) Population m_2

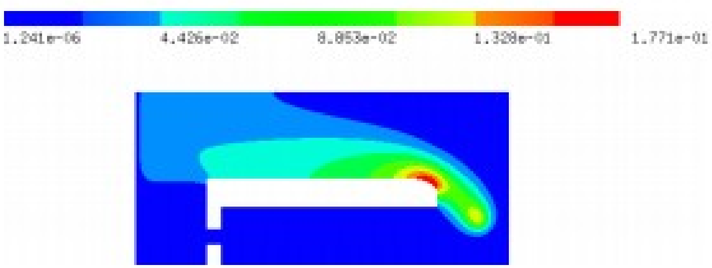
A Lachapelle MT Wolfram
Sophisticated two populations dynamics

- Rectangular domain with two corridors and a small door (bottleneck).
- Two sources placed in the lower left and lower right corner
- The parameters are

$$a = 0.25,\ \tilde{a} = 2,\ q = 2,\ ,\nu = 0.1,\ k = 1,\ r = 1.$$



(e) Population m_1



(f) Population m_2

Family II.
Finite state space:
MFG-d systems

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MFG-d systems

- State of the population : $m(t)=(m_1(t),\ldots, m_n(t))$
- Where $m_j(t)$ is the number of agents in state j
- In/out flows of players from state to state, and from outside world
- flows depend on the values $u_i(t,m)$ and are impacted by exogenous shocks
- Agent’s gain per unit of time depends on m
- As the dynamic of m is random, all agents

state of the population of agents is $m = (m_1, \dots, m_n)$,

where m_i *is the quantity of agents in state* i *(a real number)*

the previous density function m *on some domain in* $\mathbb{R}^n$ *is replaced by an histogram*

$u_j(m)$ *value function of an agent in state* j

$i - e$: *discounted expectations of net gains*

$$u_j(m) = \mathrm{E} \int_0^{\infty} e^{-rt} \beta_j(m_t) dt$$

with $dm_t = \alpha(u)dt + \varepsilon dB$

where α *is tentatively exogenous,* $i - e$:

we postpone the explanation of how α *is endogeneously defined by the equilibrium*

Hence, for $j = 1, \dots, n$:

$$0 = -ru_j + \Sigma_1^n \alpha_k(u) \frac{\partial u_j}{\partial m_k} + \beta_j(m) + \tfrac{1}{2}\varepsilon^2 \Delta u_j$$

Some MFG-d systems are « HJB gradients »

- $$Start\ by\ an\ HJB\ eqution:$$

$$(HJB) \quad 0 = -r\varphi(m) + G\big(\nabla\varphi(m)\big) + F(m) + \varepsilon\Delta\varphi$$

$$denote \quad u_j(m) = \partial\varphi(m)/\partial m_j$$

$$assuming\ smoothness,\ \ hence: \ \frac{\partial^2\varphi(m)}{\partial m_j \partial m_k} = \frac{\partial^2\varphi(m)}{\partial m_k \partial m_j}$$

$$compute\ the\ jth\ derivative\ of\ (HJB), one\ finds:$$

$$0 = -ru_j + \ \Sigma_1^n\, \alpha_k(u)\, \frac{\partial u_j}{\partial m_k} + \beta_j(m) + \varepsilon\Delta u_j\,, \quad with\ \alpha = \nabla G\ and\ \beta = \nabla F:$$

$$\alpha_k(u,m) = \partial G(u_{1,\dots,}u_n)/\partial u_k \quad and \quad \beta_j(u,m) = \partial F(m_{1,\dots,}m_n)/\partial m_j$$

«HJB gradients» and cross derivatives:

-

$$if\ \alpha = \nabla G\ and\ \beta = \nabla F, then\ (assuming\ smoothness):$$

$$(CE1) \quad \partial \alpha_k(u)/\partial u_j = \partial \alpha_j(u)/\partial u_k$$

$$(CE2) \quad \partial \beta_k(m)/\partial m_j = \partial \beta_j(m)/\partial m_k$$

MFG-d/g : the equilibrium framework

MFG-g: agent's optimisation problem

each vertex $i \in G$, denote V_i the subset of G of vertex s.t. (i, j) is an hedge of G. Suppose that each agent can switch from i to $j \in V_i$ according to a Poisson process of intensity λ_{ij}. Each agent in state i controls his own $\lambda_i = (\lambda_{ij})_{j \in V_i}$ and $g_i(\lambda_i)$ is the cost of choosing λ_i. Each agent solves the stochastic control problem (stationary case)

$$\max_{\lambda^s} E \int_0^\infty e^{-rt} \left[f(i_s) + g(\lambda^s) + \varphi(m_s) \right] ds$$

where $(m_s) = (m_{s,i})_{i \in G}$ is the number of agents at each vertex of the graph.

The value functions

Defining the value function $u_i(m_i)$ as the value function of a generic agent in state i, i.e. as the above max. expected value when $i_0 = k$ and $m_0 = m$, one has the following MFG.D system.

$$0 = -ru_i(m) + f(k) + \varphi_i(m) + \max_{\lambda_i}\left[\sum_{j \in V_i} \lambda_{ij}\left[u_j - u_i\right] + g_i(\lambda_i)\right] + \sum_{j \in G} \alpha_j^* \frac{\partial u_i}{\partial m_j}$$

Indeed, each agent computes his optimal strategy

$$\tilde{\lambda}_i(u) = Arg\max_{\lambda_i}\left[\sum_{j \in V_i} \lambda_{ij}(u_j - u_i) + g_i(\lambda_i)\right]$$

The population dynamics and the equilibibium condition

$$\dot{m}_k = \alpha_k^* \text{ where } \alpha_k^*(u,m) = \sum_{k \in V_j} \lambda^*_{jk} m_j - \left[\sum_{i \in V_k} \lambda^*_{ki} \right] mk.$$

At equilibrium $\tilde{\lambda}_i = \lambda *$

Modelling with MFG-d

- Three examples of models using MFG-d framework:
  - A. Taxi equilibrium: a stylized mini-model
  - B. Dynamics of industrial capacities in a time to build context
  - C. Dynamics of order books

A. Taxi equilibrium: a stylized mini-model

The town-airport stylized equilibrium

$two\ locations : town\ , airport$

$m\ \ (resp.: 1-m)\ is\ the\ proportion\ of\ taxi\ in\ town\ (resp. at\ the\ airport)$

$y\ is\ the\ random\ flow\ of\ airport\ clients$
$(ex.: y\ is\ a\ reflected\ brownian\ in\ a\ range\ (y_0, y_1))$

$u_1(m,y)\ \ (resp.: u_2(m,y))\ \ is\ \ the\ expected\ gain\ of\ a\ taxi\ in\ town\ (resp. at\ the\ airport)$

$a_1 dt\ \ \ is\ \ the\ probability\ of\ a\ taxi\ in\ town\ to\ move\ to\ the\ airport\ during\ the\ next\ period\ dt$
$a_2 dt\ \ \ is\ \ the\ probability\ of\ a\ taxi\ at\ the\ airport\ to\ move\ to\ town\ during\ the\ next\ period\ dt$

$b\ is\ the\ net\ flow\ of\ gain\ in\ town\ per\ unit\ of\ time,$
$c\ is\ the\ waiting\ cost\ at\ the\ airport\ per\ unit\ of\ time$
$R\ is\ the\ net\ gain\ in\ the\ moves\ between\ town\ and\ airport\ (both\ ways)$

$Each\ taxi\ control\ his\ own\ a_1\ subject\ to\ a\ cost\ C(a_1)$

$$a_2 = \ y/(1-m)$$

$let\ us\ write\ the\ stationary\ equilibrium\ equations\ satisfied\ by\ functions\ u_1\ and\ u_2$

The town-airport stylized equilibrium

$$\text{If the dynamic of the population is given by } dm = f(m,y)dt$$
$$\text{then the optimal strategy of an agent is given by}$$
$$a^*(m,y) = ArgMax\left\{a\left(u_2(m,y) + R - u_1(m,y)\right) - C(a)\right\}$$

$$\text{for example, if the cost function is}: \quad C(a) = ½\, a^2$$
$$\text{then } a^*(m,y) = u_2(m,y) + R - u_1(m,y)$$

$$\text{At equilibrium, the optimal strategy of each agent should create}$$
$$\text{the expected dynamic of the population.}$$
$$\text{Hence the function } f \text{ should be equal to}:$$
$$f(m,y) = -a^*(m,y)m + y$$

$$\text{Hence functions } u_1 \text{ and } u_2 \text{ satisfy}:$$

$$u_1(m,y) = (1 - a^*dt)\{u_1(m + dm, y + dy) + bdt\} + a^*dt\{R + u_2(m,y)\}$$
$$u_2(m,y) = (1 - a_2dt)\{u_2(m + dm, y + dy)\} + a_2dt\{R + u_1(m,y)\}$$
$$\text{with } dm = fdt$$

The town-airport stylized equilibrium

$Hence, by\ Taylor\ expansion\ and\ Ito's\ lemma, functions\ u_1\ and\ u_2\ satisfy:$

$$0 = ½[u_2(m,y) + R - u_1(m,y)]^2 + f(m,y)\partial_m u_1(m,y) + ½\partial_{yy} u_1(m,y)$$
$$0 = a_2[u_1(m,y) + R - u_2(m,y)] + f(m,y)\partial_m u_2(m,y) + ½\partial_{yy} u_2(m,y)$$

$$with\ \ f(m,y) = -u_2(m,y) - R + u_1(m,y) + y$$

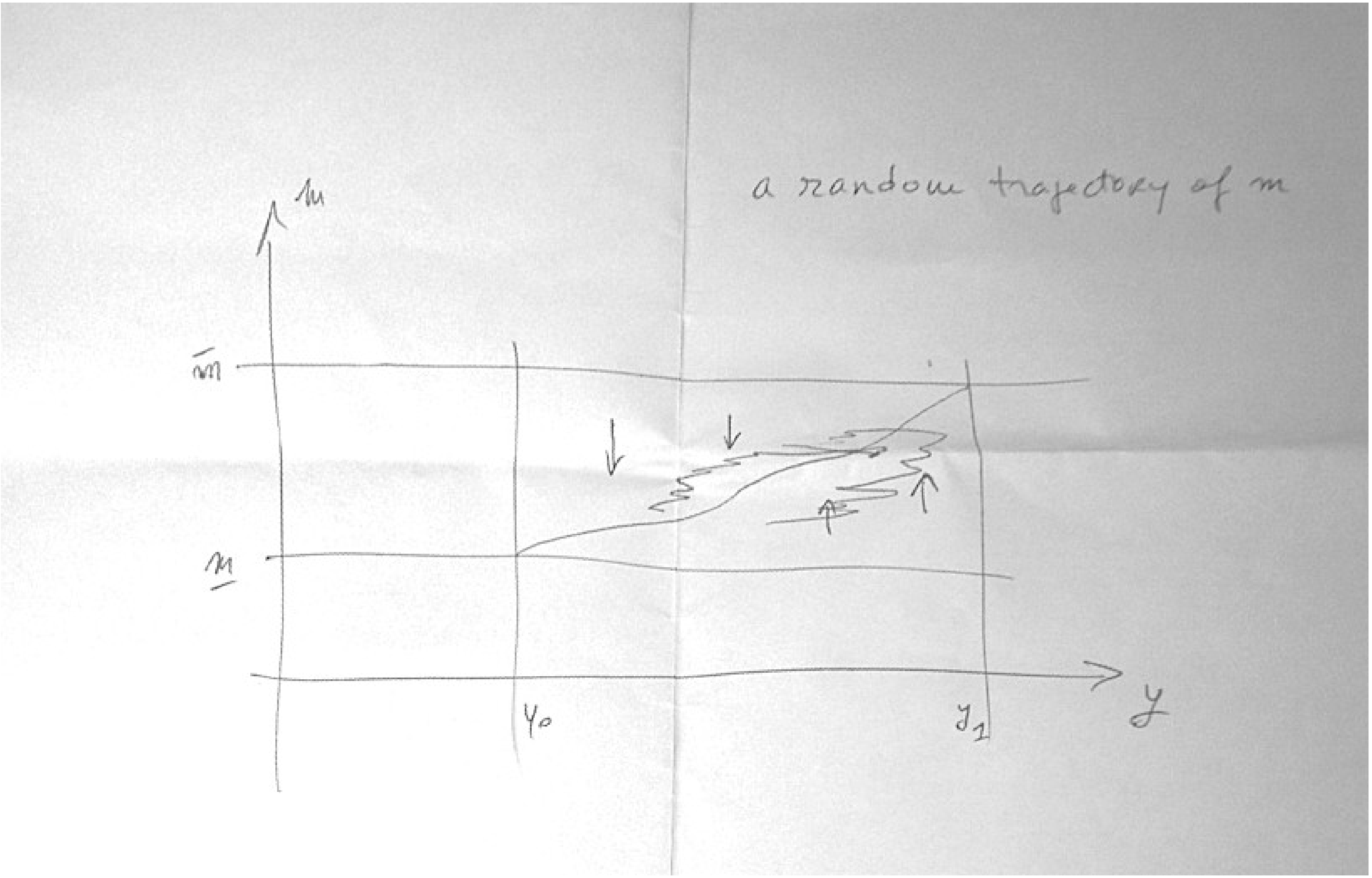
$Introducing\ w = u_2 - u_1, one\ find\ the\ non\ linear\ PDE:$

$$0 = a_2[R - w] - ½[w + R]^2 + (y - w - R)\partial_m w + ½\partial_{yy} w$$

$with\ Neumann\ boundary\ conditions\ on\ lines\ y = y_0\ and\ y = y_1$

$The\ challenge\ here\ is\ to\ show\ that\ the\ Burger's\ non\ linearity\ \ w\partial_m w$
$does\ not\ create\ shocks$

Sample of a random trajectory of m



B. Dynamics of industrial capacities in a time to build context (from a joint work with Pierre Louis Lions and Pierre Noël Giraud)

- Capacity industries, ex. : power plants
- Time to build (stylized): pay to change the amount of existing capacities
- **Attention ! we will call x_i what was called before m_i : the number of agents in state i.**

- Ex.: Lucas-Prescott model
  – Main goal of Lucas-Prescott : link with Benevolent planner framework and math tools

Model with identical power plants

Stationary equilibrium : value function

- Focus on stationary equilibrium
- Value function u(x) = discounted pay off per one unit of capacity of production
- x is the production capacity = number of production units = size of the population
- *x was previously named m*
- Agents (owners of production units) are

Flow of entrants : the time to build issue

- q^* is the flow of entrants = number of new units
- Cost of a new units is exogenous : $C(q)$
- C is the cost function of the industry which produce new power plants
- Competitive equilibrium of entrants :
- $C'(q^*) = u(x)$
- Ex: $C(q) = \frac{1}{2} q^2$ for $q>0$, hence $q^* = u(t,x)$ for $u>0$
- Convexity of the cost function C embodies (in this model) the « time to build » issue
- This framework might be compared to Lucas-Prescott model where the the costly effort x to improve existing production units has comparable effects on the size of the productive capital (see last part of this document)

Dynamics of the production capacity

- The dynamics of x is

$$dx = (q^* - a)\, dt$$

where $q^*=\max(u,0)$ is the flow of entrants and a is a constant aging rate

Demand and pay off

- The demand is exogenous : $y=D(p)$, where y is the production and p the price
- Examples :
  - $D(p)= 1/p^{\alpha}$
  - $D(p) = b-cp$ for $p < b/c$
- Constant cost e per unit of produced energy
- Power plants are identical, demand/offer competitive equilibrium : $x = D(p)$
- Pay off per one production unit : $D^{-1}(x) - e$

Recursive (mfg) equation

- We look for a stationary equilibrium:
- $u(x) = (1\text{-}rdt\text{-}adt)\ u(x+dx) + (D^{-1}(x) - e)\ dt$
- $dx = (q^*\text{-}a)\ dt$, with $q^* = \max(u,0)$

$$0 = - c u + g(u) u_x + f(x)$$

$$g(u) = \max(u,0) - a; \quad f(x) = D^{-1}(x) - e; \quad c=r+a$$

- g and –f are increasing functions

HJB and BP

- Define G,F and U by : G’=g, F’=f , U’=u
- then U satisfies the HJB equation:

$$0 = - c U + G(U') + F(x)$$

- Hence U is the Bellman value function of the control problem :

$$U(x_0) = Max \int_0^{\infty} e^{-ct}\left(F(x_t) - G^*(z_t)\right)dt$$

$$dx_t = z_t dt$$

MFG - BP / Monopolist

The BP optimization problem is not the Monopolist optimization problem

Example:

$$D(p) = 1 - p \qquad D^{-1}(x) = 1 - p$$

$$f(x) = 1 - x - e = 1 - x, \qquad F(x) = x - x^2/2$$

$$g(u) = u - a, \qquad G(u) = (u-a)^2/2, \qquad G^*(z) = az + z^2/2$$

$$U(x_0) = Max \int_0^\infty e^{-ct}\left(x_t - x_t^2/2 - az_t - z_t^2/2\right)dt$$

$$dx_t = z_t dt$$

Monopolist vs Benevolent Planner:

$$F_{mon}(x) = x(1-x)$$

Model with two types of power plants and two markets

oveview (1/3)

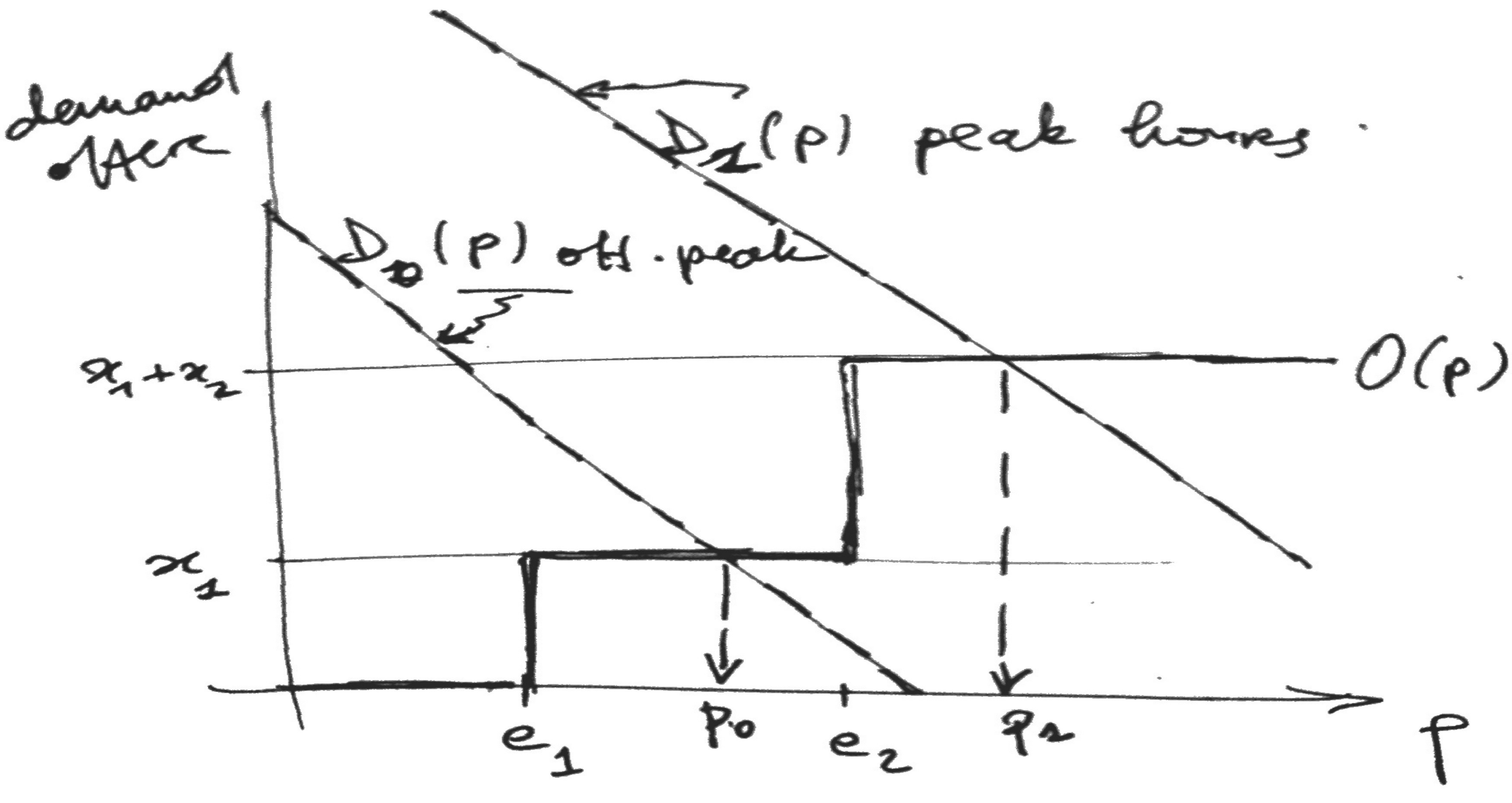
- Two types of power plants
  - Type 1: expensive to build, produce unexpensive energy
  - Type 2: unexpensive to build, produce expensive energy
- Two markets for energy
  - Peak hours: high demand for energy, both unexpensive and expensive energy can be sold
  - Off peak hours: low demand for energy, only unexpensive energy can be sold

oveview (2/3)

- Type 2 power plants
  - Receive only earnings from peak hours market,
  - but are less expensive to build

- Model will tackle interaction of :
  - Time to build with
  - Competition of two populations of producers on two markets

oveview (3/3)


demand offer
D1(p) peak hours
D0(p) off-peak
x1+x2
x1
O(p)
e1
p0
e2
q1
p

Value functions

- State of the world in this model is $x=(x_1,x_2)$ where is x_i is the existing number of units of type i
- The values functions $u_1(x_1,x_2)$ and $u_2(x_1,x_2)$ are defined as (expected) discounted pay off for the owner of one unit of type i
- We look for a stationnary competitive equilibrium, i-e: producers are price takers

Two flows of entrants

- For i=1,2, q_i = flow of entrants of type i = number of new units of type i
- $C_i(q_i)$ = cost to build one new unit of type i
- Convexity of C_i will express the « time to build issue » in this model
- Cost to build $C_i(q)$ is assumed to be greater for type 1 units : $C_1(q) > C_2(q)$
- Cost to produce one unit of energy e_i is greater for type 2 units : $e_1 < e_2$
- (NB = notations imply a adequate choice of units)

Two flows of entrants

- For the sake of analytical simplicity, we assume

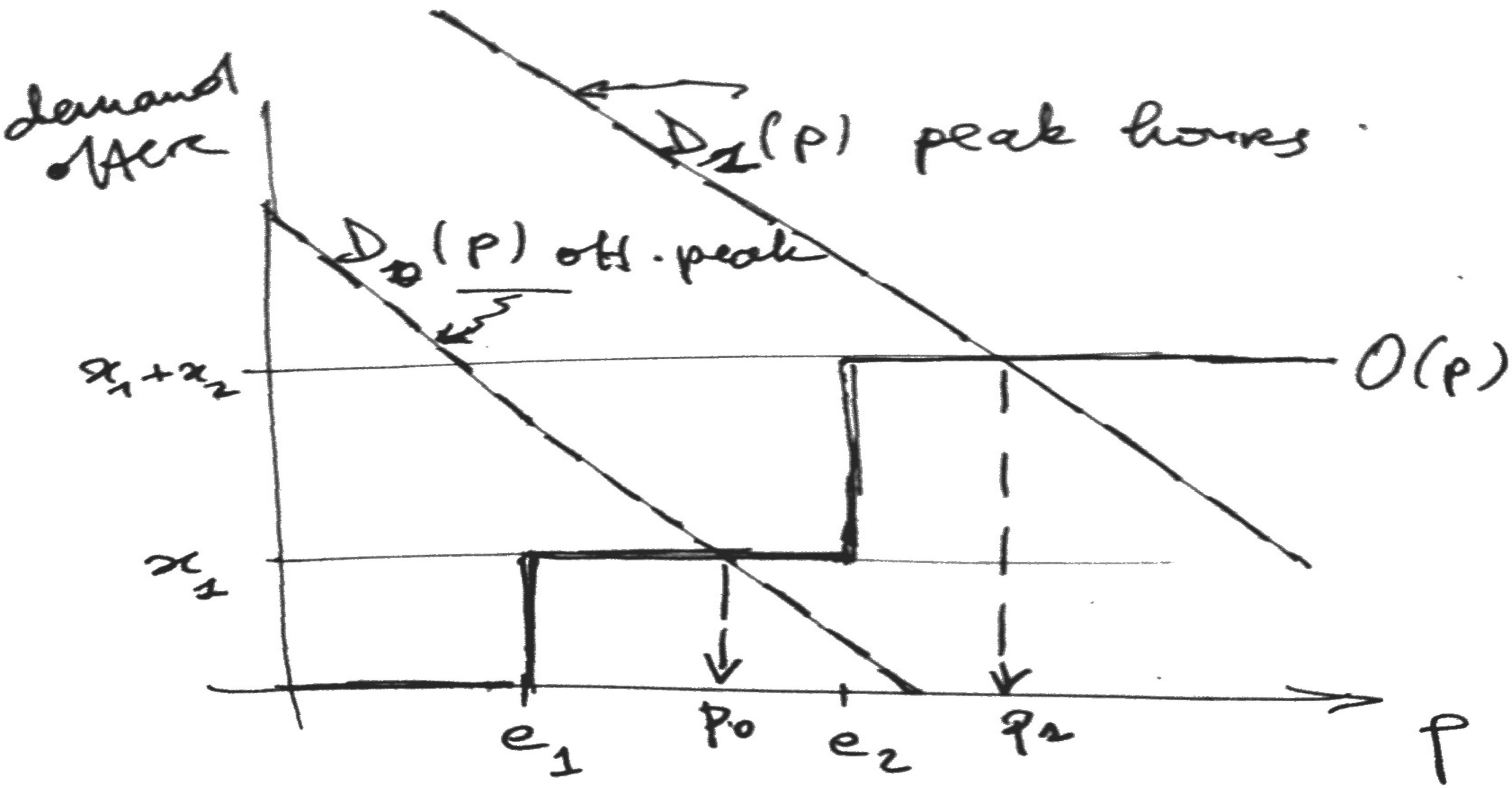
$$C(q) = ½\, c_i q^2 \text{ for } q>0$$

- $C_1(q) > C_2(q)$, hence : $c_1 > c_2$
- At equilibrium entrants flows satifies
  - $C_i'(qi^*) = u_i\,(x_1,x_2)$ $i=1,2$
  - $c_i qi^* = u_i\,(x_1,x_2)$ $i=1,2$

Peak and off-peak demand

- $D_j(p_j)$ is the demand function and p_j the energy price
  - Off peak hours j=0
  - Peak hours j=1
- Linear case : $D_j(p) = a_j – b_j\ p$, for $p < a_j/b_j$
- Example
  - $a_0 << a_1$ $b_0=b_1$
  - hence off-peak demand D_0 lower than peak demand D_1
- We assume no uncertainity in this model: each day is splitted in two part: peak hours and off

Demand and offer :
peak hours equilibrium (p_0,x_1) and
off-peak hours equilibrium (p_0, x_1+x_2)



Pay off

- The net pay off for one unit of type 2 is $f_2(x_1,x_2)=p_1(x_1,x_2)-e_2$ as this unit produce only for the peak hours market
- The pay off for one unit of type 1 is $f_2(x_1,x_2)=p_0(x_1,x_2)+p_1(x_1,x_2)-e_1$ as this unit produces both for the peak and off-peak market
- For the sake of simplicity, we will restrict (here) to the case $e_1 < p_0 < e_2 < p_1$
- Hence $p_0(x_1,x_2)=p_0(x_1)$ and $p_1(x_1,x_2) = p_1(x_1+x_2)$

The stationary equilibrium equations

- $u_i(x_1,x_2) = (1\text{-}rdt\text{-}kdt)\ u_i(x_1+dx_1,x_2+dx_2) + f_i(x_1,x_2)dt$
- $dx_i = g_i(x_1,x_2)\ dt$, with
- $g_i(u_1,u_2) = q_i^*\text{-}k = u_i(x_1,x_2)/c_i - k$

(where k is the rate of aging of all units)

Hence :

$$0 = (r+k)\,u_1 + g_1(u_1,u_2)\,\partial_1 u_1(x_1,x_2) + g_2(u_1,u_2)\,\partial_2 u_1(x_1,x_2) + f_1(x_1,x_2)$$

$$0 = (r+k)\,u_2 + g_1(u_1,u_2)\,\partial_1 u_2(x_1,x_2) + g_2(u_1,u_2)\,\partial_2 u_2(x_1,x_2) + f_2(x_1,x_2)$$

This MFG monotone system is the « gradient » of an HJB equation

$$for \quad i = 1,2: \quad 0 = (r+k)u_i + g_1(u_1,u_2)\partial_1 u_i(x_1,x_2) + g_2(u_1,u_2)\partial_2 u_i(x_1,x_2) + f_i(x_1,x_2)$$

$$with \quad \partial_2 g_1(y_1,y_2) = \partial_1 g_2(y_1,y_2) \quad and \quad \partial_2 f_1(x_1,x_2) = \partial_1 f_2(x_1,x_2)$$

$$\partial_2 g_1 = \partial_1 g_2 \quad and \quad \partial_2 f_1 = \partial_1 f_2 \quad imply \quad (g_1,g_2) = \nabla G \quad and \quad (f_1,f_2) = \nabla F$$

$$and \quad (u_1,u_2) = \nabla U \quad where \quad U \quad is \quad the \quad solution \quad of \quad the \quad HJB \quad equation:$$

$$0 = (r+k)U + G(\nabla U) + F$$

Ex. : Tax impact on equilibrium

$Let's\ \ add\ \ a\ \ tax\ \ scheme\ \ \theta = \left(\theta_1, \theta_2\right)\ \ to\ \ the\ \ previous\ \ system:$

$$0 = \left(r + k\right) u_i + g_1\left(u_1, u_2\right) \partial_1 u_i\left(x_1, x_2\right) + g_2\left(u_1, u_2\right) \partial_2 u_i\left(x_1, x_2\right) + f_i\left(x_1, x_2\right) + \theta_i\left(x_1, x_2\right)$$

$unless\quad \partial_2 \theta_1\left(x_1, x_2\right) = \partial_1 \theta_2\left(x_1, x_2\right)$

$this\ \ will\ \ not\ \ be\ \ the\ \ gradient\ \ of\ \ some\ \ HJB\ \ equation$

Remark: analytic solution in the linear case

$$If \quad all \quad functions \quad g_i, f_i, \theta_i \quad in \quad the \quad equations$$

$$0 = (r + k) u_i + g_1(u_1, u_2) \partial_1 u_i(x_1, x_2) + g_2(u_1, u_2) \partial_2 u_i(x_1, x_2) + f_i(x_1, x_2) + \theta_i(x_1, x_2)$$

$$are \quad linear \quad then \quad so \quad is \quad u$$

$$and \quad solving \quad the \quad system \quad means \quad solving \quad a \quad Riccati \quad equation$$

The Lucas-Prescott model

- One type of firms, and one aggregate risk
- Two states dynamic, hence : a two dimensional model
- But as there are no individual risks there will be a competitive equilibrium without insurance
- Agents are producers : each agent own a production unit

Lucas-Prescott framework

- The inverse demand function is $p_t = P(q_t, u_t)$ where q_t is the total production
- u_t is a stochastic demand shifter that follows a diffusion process
- $du_t = \mu(u_t)dt + \sigma(u_t)dW_t$
- Hence, agents (producers) share the same collective risk : le level of demand, hence the level of prices

Lucas-Prescott framework

- Agents can improve their own production unit
- $dk_t = h(x_t/k_t)k_t dt$
- where k_t is the size of the unit (i-e: capital own by the producer)
- and x_t is the cost of improvement, and h a technical function measuring the impact of adjustment (i-e: improvement) costs

Lucas-Prescott framework

- Agents are identical, share the same risk,, have same initial conditions,..
- Hence agents behave identically, and $q_t=x_t$, $K_t=k_t$ where K_t is the total amount of productive capital in the economy (= : up to suitable choice of units)
- Lucas-Prescott focus on the stationary equilibrium

Lucas-Prescott framework

- Each individual firm solves the individual optimization problem below, in which prices p_t are the given « mean field » (i-e: producers are atomized price takers) with $p_t = D(q_t, u_t)$

$$V_0 = \max_{\{x_t\}} \mathbb{E}_0 \int_0^{\infty} e^{-rt}[p_t k_t - x_t]dt$$

MFG equilibrium equations

- In order to write the MFG system of this problem, we write the PDEs satisfied by the value function of a producer $w(k_0,u_0)=V_0 /k_0$ (i-e, the value of one production unit) when the state of the economy is (k_0,u_0).
- Hence w will satisfies:

$$rw\,(k,u) = Max_x \; \mathrm{E}\{(1-rdt)w(k+dk,u+du)(1+h(x/k))dt) + P(k,u)dt - (x/k)dt\}$$

MFG equilibrium equations

- Hence by expansion (Ito's lemma) the "first" MFG equation is

$$rw\,(k,u) = kh(x^*/k)\,\partial_k w + \mu(u)\partial_u w + \left(\frac{1}{2}\right)\sigma(u)^2\partial_{uu} w + w\,h(x^*/k) + P(k,u) - (x^*/k)$$

- Where x* is the optimal strategy of all (identical) agents

$$x^* = x^*(w_1,k) = k.\,ArgMax_x[w_1 h(x/k) - x/k]$$

MFG equilibrium equations

- As there are two state variables k and u, the MFG monotone system should have a second unknown value function v(k,u) satisfying a similar PDE.
- In this specific case the most straightforward path to this second PDE is indirect, as we will see
- Note that the above PDE can be solved independently of the “second” MFG equation

From one MFG equation to the MFG « gradient » system…

- $$Hence\ the\ two\ questions\ are:$$
  $$can\ we\ write\ a\ MFG\ system\ of\ two\ equations$$

$$rw_i = g_1(w_1,w_2,k,u)\partial_k w_i + g_2(w_1,w_2,k,u)\partial_u w_i + a\partial_{uu} w_i + f_i(w_1,w_2,k,u)$$

$$with\ w = w_1 \quad g_1(w_1,w_2,k,u) = kh(x^*/k) \quad g_2(w_1,w_2,k,u) = \mu(u) \quad a = \sigma^2(u)/2$$

$$f_1(w_1,w_2,k,u) = w\,h(x^*/k) + D(k,u) - (x^*/k)$$

$$x^* = x^*(w_1,k) = k.ArgMax_x[w_1 h(x/k) - x/k]$$

$$Can\ this\ system\ be\ the\ \text{gradient}\ of\ an\ HJB\ equation$$

From one MFG equation to the MFG « gradient » system…

- *The previous system is the* gradient *of the HJB equation* :

$$r\varphi(k,u) = H(k,u,\varphi,\partial_k\varphi,\partial_u\varphi) + a\partial_{uu}\varphi$$

with:

$$\partial_y H(k,u,\varphi,y,z) = g_1(y,z,k,u) \quad \partial_z H(k,u,\varphi,y,z) = g_2(y,z,k,u)$$

$$\text{with } w = w_1 \quad g_1(w_1,w_2,k,u) = kh(x^*/k) \quad g_2(w_1,w_2,k,u) = \mu(u) \quad a = \sigma^2(u)/2$$

…from the MFG « gradient » system to the HJB equation of Lucas-Prescott..

- Using the previous relationship one recover the Lucas-Prescott HJB equation

$$r\varphi(k,u) = Max_x\,[-x + h(x/k)k\partial_k\varphi] + \mu(u)\partial_u\varphi + \left(\frac{1}{2}\right)\sigma(u)^2\partial_{uu}\varphi + s(k,u)$$

Where s is the surplus defined by (D=P in Lucas-Prescott notations):

$$\partial_k s(k,u) = D(k,u)$$

…from the HJB equation to the Lucas-Prescott Benevolent Planner

- The solution φ of the HJB equation :

$$r\varphi(k,u) = Max_x\,[-x + h(x/k)k\partial_k\varphi] + \mu(u)\partial_u\varphi + \left(\frac{1}{2}\right)\sigma(u)^2\partial_{uu}\varphi + s(k,u)$$

- Is the Bellman value function of the optimal control problem :

$$\varphi(k_0,u_0) = \max_{\{x_t\}} \mathbb{E}_0 \int_0^\infty e^{-rt}[s(k_t,u_t) - x_t]dt \qquad \begin{aligned} du_t &= \mu(u_t)dt + \sigma(u_t)dW_t \\ dk_t &= h(x_t/k_t)k_t dt \end{aligned}$$

- This optimal control problem is the BP problem introduced by Lucas-Prescott
- Actually, Lucas-Prescott did it the reverse way : the designed directly the previous BP problem from their economics insights, then deduced the HJB equation from the BP pb.

To summarize..

- Starting with a framework of competitive market model, and writing the MFG monotone system (or part of it), we found a gradient like system, hence an HJB equation. This HJB equation HJB defines an optimal control problem which is the Benevolent planner problem
- the competitive equilibrium is identical to the solution of this BP problem : agents behave as if they were driven by the “invisible hand” of a BP

First and second welfare theorems

First and second welfare theorems state that:

- any competitive equilibrium leads to a Pareto efficient allocation of resources,
- such a Pareto allocation solves an optimization problem
- Hence the competitive equilibrium is equal to a solution designed by some Benevolent Planner

Lucas-Prescott breakthrough

Lucas and Prescott breakthrough :

- Given that the competitive market equilibrium is also the optimal solution defined by some Benevolent Planer
- In order to compute the competitive market equilibrium can proceed in two steps :
  - found the right BP, i-e: the right optimization problem
  - solve this optimization problem using all classic tools of optimal control : namely HJB equation when the context is optimal control of stochastic diffusion

Mfg vs BP

- While Lucas and Prescott found the right BP using there economical insights, MFG approach give an analytical process requiring no insight:
  - Write the mfg system
  - Check if equality of cross derivatives ECD1 and ECD2 holds
  - If yes, compute unknown function F and G

$$\alpha_k(u,m) = \partial G(u_{1,\ldots,}u_n)/\partial u_k \quad et \quad \beta_j(u,m) = \partial F(m_{1,\ldots,}m_n)/\partial m_j$$

Mfg vs BP

- Of course, as soon as there are « non market interactions » between agents, the equivalence of equilibrium and BP optimization cancels (most of the time, unless ECD still holds)
- « non market interactions » might be:
  - tax,
  - frictions,
  - Externalities,
  - …

C. Dynamics of order books
joint work with Aimé Lachapelle, Charles Albert Lehalle et Pierre Louis Lions

- Articles initiaux
  - Mean field games, JM Lasry, PL Lions, Japanese Journal of Mathematics 2 (1), 229-260
  - Jeux à champ moyen. I–Le cas stationnaire, JM Lasry, PL Lions, Comptes Rendus Mathématique 343 (9), 619-625
  - Jeux à champ moyen. II–Horizon fini et contrôle optimal, JM Lasry, PL Lions, Comptes Rendus Mathématique 343 (10), 679-684
- Et aussi,
  - Mean field games and applications, O Guéant, JM Lasry, PL Lions, Paris-Princeton Lectures on Mathematical Finance 2010, 205-266
  - Notes on Mean Field Games, Pierre Cardaliaguet, January 15, 2012 (et la bibliographiede ce texte)
  - PL Lions, cours aux Collège de France 2006-2012, http://www.college-de-france.fr/site/en-pierre-louis-lions/course-2011-2