Higher order discrete controllability and the approximation of the minimum time function

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November 11, 2014

Introduction

Numerical schemes designed for approximating the minimum time problem, following the dynamic programming approach, are fully justified in the literature only when the minimum time function T is Lipschitz.

However, non Lipschitz behavior shows up in simple- even linear in the plane- examples.

In this talk, we provide a numerical scheme suitable for the case where T is Hölder continuous (i.e, a higher order controllability condition holds) and justify an approximate numerical feedback, together with its convergence. To this aim, a higher order robust discrete controllability result is also needed.

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Consider the affine control system in $\mathbb{R}^n$

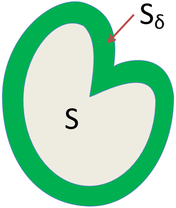
$$\begin{cases} \dot{x} = f(x) + \sum_{i=1}^{M} g_i(x) u_i =: F(x,u), \\ x(0) = \xi, \end{cases} \tag{1}$$

where $u := (u_1, ..., u_M) \in [-1,1]^M$. Let

$$U := \{(u_1, ..., u_M) \colon \ u_i \colon [0,\infty) \to [-1,1], i = 1, ..., M \ measurable\}.$$

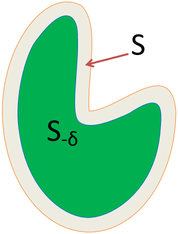
Let S_δ, $S_{-\delta}$ be an enlargement and a shrinking of S respectively defined as follow

$$S_\delta = \{x \in \mathbb{R}^n : d_S(x) \leq \delta\},$$



$$S_{-\delta} = \left\{x \in \mathbb{R}^n : d_{\overline{S^c}}(x) \geq \delta\right\},$$

where $S^c = \mathbb{R}^n \setminus S$.



The standard assumptions on F and the target set S we need are the following:

Assumptions 1

(1) f, g_i are $\mathcal{C}^\infty$ and all partial derivatives are Lipschitz with Lipschitz constant $L > 0$, $i = 1, \ldots, M$; moreover,

$$\|f(y)\|, \|g_i(y)\| \leq K_0(1 + \|y\|)$$

for all $y \in \mathbb{R}^n$, where K_0 is a positive constant.
(2) S is compact

Such assumptions will be always supposed to be satisfied in the sequel

If ∂S is smooth enough, and the so called *Petrov condition* holds, i.e.

$$\min_{u \in U} \langle \nabla d_S(x), F(x,u) \rangle \leq -\mu < 0, \qquad (2)$$

for x in a neighborhood S_δ of S, then one can steer any point of S_δ to S in finite time $T(x)$ and $T(x)$ is Lipschitz continuous.

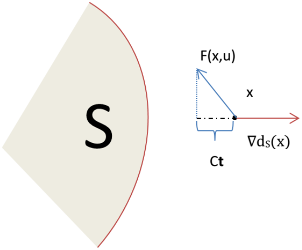


Figure 1: Petrov case

What can we do when (2) fails and S is less regular?

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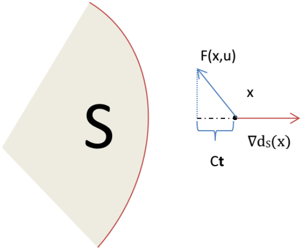

Figure 1: Petrov case

What can we do when (2) fails and S is less regular?

For the sake of simplicity, we consider second order controllability of the driftless system with two controls, namely

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2, \qquad x(0) = \xi. \tag{3}$$

By subsequently following the flows of $g_1(\xi), g_2(\xi), -g_1(\xi), -g_2(\xi)$, each one for time t, it is well known that

$$x(4t) = \xi + t^2[g_2, g_1](\xi) + O(t^3),$$

where $[g_2, g_1](x) = \nabla g_2(x)g_1(x) - \nabla g_1(x)g_2(x)$.

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where $[g_2, g_1](x) = \nabla g_2(x)g_1(x) - \nabla g_1(x)g_2(x)$.

Let ∂S be of class C^2 and assume

$$\langle \nabla d_S(\xi), [g_2, g_1](\xi) \rangle \leq -\mu,$$

then $d_S(\cdot)$ is of class $C^{1,1}$ in $S_\delta \setminus S$ and so

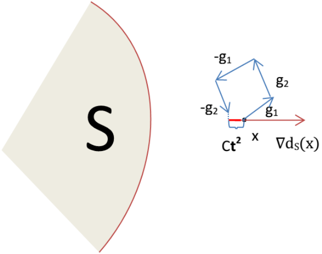


Figure 2: Second order Lie Bracket case

$$\begin{aligned} d_S(x(4t)) &= d_S(\xi) + \langle \nabla d_S(\xi), x(4t) - \xi \rangle + O(\|x(4t) - \xi\|^2) \\ &= d_S(\xi) + t^2 \langle \nabla d_S(\xi), [g_2, g_1](\xi) \rangle + O(t^3) \\ &\leq d_S(\xi) - t^2 \mu + O(t^3) \leq d_S(\xi) - t^2 \frac{\mu}{2}. \end{aligned} \tag{4}$$

In this case, $T(x)$ is (finite and) $\frac{1}{2}$- Hölder continuous on S_δ.

We will need the robustness of controllability condition w. r. t. a suitable shrinking $S_{-\sigma}$ of S for discrete controllability and approximate feedback control construction.

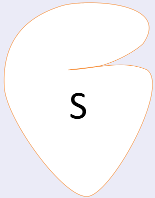
Theorem 2 (a particular case of Marigonda (2006), Marigonda and Rigo(2014), Marigonda and Th.(2014))

Let S be compact and let the following be valid

IS.1 *let S be satisfying a ρ–internal sphere condition,*

IS.2 *there exist $\delta > 0$, $\mu > 0$, such that for every $\xi \in S_\delta \setminus S$, there exists $\zeta_\xi \in \partial^P d_S(\xi)$ with the property:*

$$\langle \zeta_\xi, [g_2, g_1](\xi) \rangle \leq -\mu < 0.$$



Then the minimum time function to reach S from ξ subject to the dynamics (3), $T(\xi)$, is (finite and) Hölder continuous with exponent $\frac{1}{2}$ on S_δ.

We will need the robustness of controllability condition w. r. t. a suitable shrinking $S_{-\sigma}$ of S for discrete controllability and approximate feedback control construction. A result in this direction:

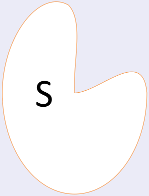
Theorem 3 (C., Th.)

Let S be compact and let the following be valid

IS.1' *let S be satisfying a ρ–internal sphere condition, and external cone condition*

IS.2 *there exist $\delta > 0$, $\mu > 0$, such that for every $\xi \in S_\delta \setminus S$, there exists $\zeta_\xi \in \partial^P d_S(\xi)$ with the property :*

$$\langle \zeta_\xi, [g_2, g_1](\xi) \rangle \leq -\mu < 0. \quad (5)$$



Then the minimum time function to reach $S_{-\sigma}$ from ξ subject to the dynamics (3), $T(\xi)$, is (finite and) Hölder continuous with exponent $\frac{1}{2}$ on S_δ.

Approximation scheme

We see from (4) (i.e. $d_S(x(4t)) \le d_S(\xi) - t^2\frac{\mu}{2}$) that one can design a trajectory which reaches the target through successive steps in which the gain of the distance is of order t^2 (or more in general it is of order higher than 1). Therefore, we need a sufficiently high order approximation in time, in order to preserve the gain of the distance.

Recall the control system (1):

$$\begin{cases} \dot{x} = f(x) + \sum_{i=1}^{M} g_i(x)u_i := F(x,u), \\ x(0) = \xi, \end{cases} \tag{6}$$

where $u = (u_1, ..., u_M) \in [-1,1]^M$.

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where $u = (u_1, ..., u_M) \in [-1, 1]^M$.

Given a fixed step $h > 0$ small enough, we approximate (6) by a one step $(q+1)$-th order scheme which has the form

$$\begin{cases} y_{n+1} & = y_n + h\Phi(y_n, A_n, h) \\ y_0 & = \xi \end{cases} \tag{7}$$

where A_n is a $M \times l$ matrix, $A_n = (u_n^1, ..., u_n^l)$ with $u_n^i \in [-1,1]^M$. Here $l > 0$ depends on the specific method.

For instance, if $q = 0$ or $q = 1$, we can simply take (7) as Euler or Heun's method, respectively,i.e.

$$\Phi(x, u, h) = F(x, u),$$

$$\Phi(x, u_1, u_2, h) = \frac{F(x, u_1) + F(x + hF(x, u_1), u_2)}{2},$$

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Now consider, for example, the driftless control system (3), namely

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2, \quad x(0) = \xi, \tag{8}$$

where $u = (u_1, u_2) \in [-1, 1]^2$.

In the second order controllability case, we take (7) as a one step third order method, namely

$$\|x_\upsilon(h, \xi) - (\xi + h\Phi(\xi, \upsilon, ..., \upsilon, h))\| \leq C_\Phi h^4, \tag{9}$$

for some $\upsilon \in [-1, 1]^2$

We define the function

$$n_h(\{A_i\}, \xi) = \min\{n \in \mathbb{N} : y_n \in S\} \leq +\infty, \tag{10}$$

$$N_h(\xi) = \min_{\{A_i\} \in [-1,1]^{MI}} \{n_h(\{A_i\}, \xi)\}. \tag{11}$$

The discrete minimum time function is now defined by setting

$$T_h(\xi) = hN_h(\xi). \tag{12}$$

Discrete controllability (a second order case)

Theorem 4 (C., Th.)

Let S be satisfying ρ-internal sphere condition and external cone condition, together with

$$\langle \zeta_\xi, [g_2, g_1](\xi) \rangle \leq -\mu < 0, \text{ where } \xi \in S_\delta \setminus S, \zeta_\xi \in \partial^P d_S(\xi).$$

Consider the discrete dynamics generated by a one step third order method. Then there exist $\bar{h}$, $C > 0$ such that for every $0 < \delta$, $0 < h \leq \bar{h}$, $\xi \in S_\delta \setminus S$, we have

$$T_h(\xi) \leq C\sqrt{d_S(\xi)}. \tag{13}$$

Hamilton-Jacobi-Bellman equation
Semidiscrete scheme

Consider the Kružkov transformation, namely, we define

$$v(x) = 1 - e^{-T(x)}, \tag{14}$$

and recall that v is the unique bounded viscosity solution of the boundary value problem

$$\begin{cases} v(x) + \sup_{u \in [-1,1]^M} \{ \langle -F(x,u), \nabla v(x) \rangle \} = 1 & \text{in } \mathbb{R}^n \setminus S, \\ v(x) = 0 & \text{on } S \end{cases} \tag{15}$$

For a given stepsize $h > 0$, define

$$v_h(x) = 1 - e^{-T_h(x)}, \tag{16}$$

Recall that v_h is the unique bounded solution of the following problem, see [Bardi, Falcone (1990)]:

$$\begin{cases} v_h(x) = \inf_{A \in [-1,1]^{Ml}} \left\{ e^{-h} v_h(x + h\Phi(x, A, h)) \right\} + 1 - e^{-h} & \text{on } \mathbb{R}^n \setminus S \\ v_h(x) = 0 & \text{on } S. \end{cases} \tag{17}$$

Under our assumptions, for proving the convergence of $v_h(\cdot)$ and $T_h(\cdot)$,

one can follow the same techniques as in [Bardi, Falcone (1990)].

Fully discrete scheme

To preserve the order of the considered method, we assume, furthermore,

A.1 For any $x \in \mathbb{R}^n$ and any measurable $u\colon [0,h) \to [-1,1]^M$ there exists a $M \times l$ (where l depends on the chosen method) matrix $A \in [-1,1]^{Ml}$ such that

$$\|y(h,x,u) - y_h(h,x,A)\| \leq Ch^{q+2}, \tag{18}$$

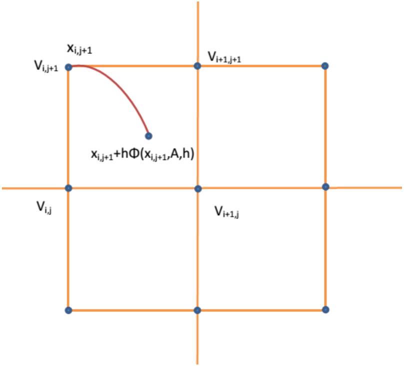
A.2 for any matrix $A \in [-1,1]^{Ml}$, there exists a measurable control $u\colon [0,h) \to [-1,1]^M$ such that (18) holds.

Higher order one step methods satisfying (18) are constructed for control affine systems in [Ferretti (1997), Grüne and Kloeden (2001)]

For the fully discrete scheme, the idea is to use the first order interpolation, which described briefly as follows.

Let $\Gamma = \{x_{i,j} : i, j = 1, ..., I\}$ be a space grid for the domain $\Omega \subset S_\delta$.

Now we construct a fully discrete version of (17) by substituting $v_h(x_i + h\Phi(x_i, A, h))$ with $I^1_\Gamma[v_h](x_i + h\Phi(x_i, A, h))$,



where

$$I^1_\Gamma[v_h](x_i + h\Phi(x_i, A, h)) = \sum_j^I \lambda_j(A) v_h(x_j),$$

if $x_i + h\Phi(x_i, A, h) = \sum_j^I \lambda_j(A) x_j, \quad \lambda_j(A) \in [0, 1], \quad \sum_{j=1}^I \lambda_j(A) = 1.$

More precisely,

$$\Gamma^\star := \{x \in \Gamma : \exists A \text{ such that } x + h\Phi(x, A, h) \in \Omega\}$$

and the fully discrete problem reads as

$$\begin{cases} v_h^{\Delta x}(x) = \min_{A\in[-1,1]^{Ml}} & \left\{e^{-h} I_\Gamma^1[v_h^{\Delta x}](x + h\Phi(x, A, h))\right\} \\ \qquad\qquad +1 - e^{-h} & \text{if } x \in \Gamma^\star \setminus S, \\ v_h^{\Delta x}(x) = 0 & \text{if } x \in \Gamma^\star \cap S, \\ v_h^{\Delta x}(x) = 1 & \text{if } x \in \Gamma \setminus \Gamma^\star \\ v_h^{\Delta x}(x) = I_\Gamma^1[v_h^{\Delta x}](x) & \text{if } x \in \Omega \setminus \Gamma. \end{cases} \tag{19}$$

Under our assumptions, the minimum time T is Hölder continuous on S_δ and

$$T(x) \le C \sqrt[k]{d_S(x)}, \quad \forall x \in S_\delta \setminus S. \tag{20}$$

This inequality implies that $v(x) \in C^{0,1/k}(S_\delta)$ Moreover, the discrete minimum time function T_h satisfies

$$T_h(x) \le C \sqrt[k]{d_S(x)}, \quad \forall x \in S_\delta \setminus S, \tag{21}$$

provided $h > 0$ is small enough.

Error estimate

Theorem 5 (C.,Th.)

Assume that (20), (21) hold in a neighborhood S_δ of the target S , together with the preserving-order assumptions on the scheme (A.1) and (A.2). Then there exist $\bar{h}$ and C, C_1, $C_2 > 0$ such that

$$\|v - v_h\|_{\infty,\Omega} \leq Ch^{\frac{q+1}{k}},$$

$$\left\|v - v_h^{\Delta x}\right\|_{\infty,\Omega} \leq C_1 h^{\frac{q+1}{k}-1} + C_2 \frac{(\Delta x)^{1/k}}{h}.$$

for any $x \in S_\delta$, $0 < h \leq \bar{h}$.

Approximate feedback control

For synthesis of approximate feedback controls for infinite horizon problem, we refer to [Falcone (1997)].

Why are approximate feedback controls relevant in practice?

From the theoretical point of view, assume that $H(x,p)=0$, let $u(x)\in \mathrm{argmin} H(x,p)$. By plugging into the dynamics, we obtain

$$\dot{x} = f(x, u(x))$$

where $f(\cdot,\cdot)$ is generally discontinuous. One can try to regularize the right hand side w. r. t. $f(\cdot,\cdot)$, for instance, $\dot{x}\in G(x)$. However, there are many solutions which are not the needed ones. On the other hand, an approximate feedback control is able to overcome this issue and maybe is enough for practical purposes.

Construction of semidiscrete feedback controls

Now we construct semidiscrete feedback controls with respect to the shrinking $S_{-\sigma}$ of S, in particular, let

$$A_h(x) \in \operatorname{argmin}_{A\in[-1,1]^{Ml}} \left\{ e^{-h} v_{h,\sigma}(x + h\Phi(x, A, h)) \right\}, \qquad (22)$$

Define a sequence of semidiscrete feedback control matrices $A_h(y_m)$, where y_m is the solution of the discrete dynamical system

$$\begin{cases} y_{m+1} &= y_m + h\Phi(y_m, A_h(y_m), h) \\ y_0 &= x. \end{cases}$$

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Construction of fully discrete feedback controls

The strategy for constructing fully discrete feedback control is similar to the semidiscrete case. The only difference is $I^1_\Gamma[v^{\Delta x}_{h,\sigma}](x + h\Phi(x, A, h))$ in place of $v_{h,\sigma}(x + h\Phi(x, A, h))$ in (22), i.e.

$$A_{\Delta x,h}(x) \in \operatorname{argmin}_{A\in[-1,1]^{Ml}} \left\{ e^{-h} I^1_\Gamma[v^{\Delta x}_{h,\sigma}](x + h\Phi(x, A, h)) \right\}. \tag{23}$$

Under our assumptions, there exist measurable feedback controls $u_h(\cdot)$, $u_{\Delta x,h}(\cdot)$ corresponding to $A_h(\cdot)$, $A_{\Delta x,h}(\cdot)$ such that the local error of the method is preserved.

Let $\Delta x = h^{q+1}$ and set $\gamma := \frac{q+1}{k} - 1\,(> 0)$.

Theorem 6 (C., Th.)

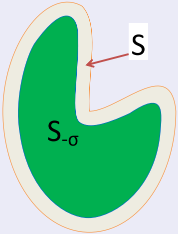
Under our assumptions on controllability and on the numerical scheme, there exist nonnegative functions $\epsilon(\sigma, h)$, $\epsilon'(\sigma, h)$ such that

$\epsilon(\sigma, h)$, $\epsilon'(\sigma, h) \to 0$, *as* $\sigma, h \to 0$ *and*

$$t(u_h(\cdot), x) \leq T(x) + \epsilon(\sigma, h),$$

$$t(u_{\Delta x,h}(\cdot), x) \leq T(x) + \epsilon'(\sigma^{\frac{1}{k}}, h^{\gamma-1}),$$

for suitably small $\sigma > 0$ *and every* $x \in \Omega$,



Example 1: double integrator $\ddot{x} = u$, $|u| \leq 1$

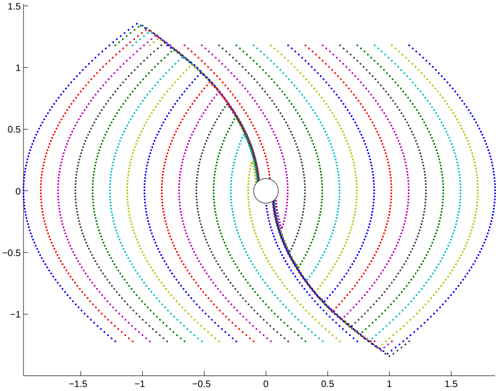


Figure 3: Computed discrete trajectories following discrete feedback controls

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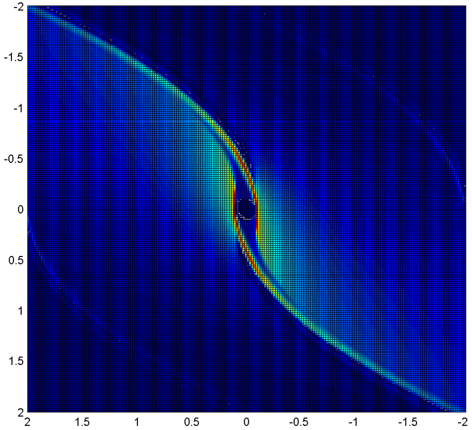

Figure 4: Error location of the value function

Example 2: bilinear system

$$\dot{x}_1 = -\frac{x_2}{8} - x_2 u, \;\; \dot{x}_2 = \frac{x_1}{8} + 2x_1 u, \;\; |u| \leq 1$$

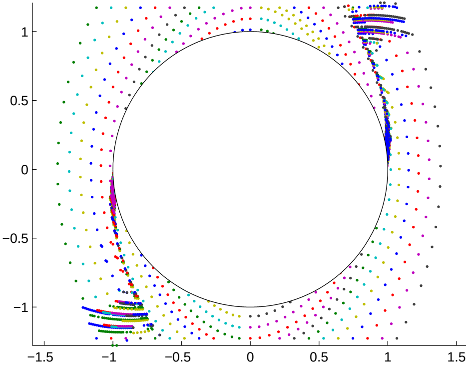

Figure 5: Computed discrete trajectories following discrete feedback controls

Thanks for your attention!