Local Regularity of the Minimum Time Function

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joint work with H. Frankowska

Consider the control system:

$$\begin{cases} y'(t) & = & f(y(t), u(t)) \\ u(t) & \in & U, & \quad a.e\ t > 0 \\ y(0) & = & x \in \mathbb{R}^N, \end{cases} \tag{1}$$

where the function $f : \mathbb{R}^N \times U \to \mathbb{R}^N$ and the control set U, a nonempty subset of $\mathbb{R}^M$, are given.

We often require the following assumptions

(A1) U is compact and the set $f(x, U)$ is convex for any $x \in \mathbb{R}^N$.

(A2) f is continuous, locally Lipschitz with respect to x, uniformly in u and there exists $k > 0$ such that $\sup_{u \in U} |f(x, u)| \leq k(1 + |x|)$, for all $x \in \mathbb{R}^N$.

(A3) $D_x f(x, u)$ exists for all x, u and is locally Lipschitz in x, uniformly in u.

Denote by U_{ad} the set of admissible controls i.e.,

$$U_{ad} = \{u : \mathbb{R}^+ \to U | u(\cdot) \text{ is measurable}\}.$$

Under (A2), for each $u(\cdot) \in U_{ad}$, (1) has a unique solution denoted by $y(\cdot; x, u)$.
We call $y(\cdot; x, u)$ the trajectory starting at x corresponding to the control u.

We consider a closed nonempty set $\mathcal{K} \subset \mathbb{R}^N$ which is called the target.
Let $\mathcal{T}$ be the minimum time to reach the target $\mathcal{K}$ from x, i.e.,

$$\mathcal{T}(x) = \inf\{t \geq 0 | y(t; x, u) \in \mathcal{K},\ u \in \mathcal{U}_{ad}\}. \tag{2}$$

A minimizing control in (2), say $u^*(\cdot)$, is called an optimal control for x. The trajectory $y(\cdot; x, u^*)$ corresponding to $u^*(\cdot)$ is called an optimal trajectory for x.
Under (A1) - (A2), the infimum in (2) is attained.

Cannarsa and Sinestrari (1995) proved that :
If the target satisfies an internal sphere condition i.e.,

(A4) There exists $r > 0$ such that $\forall x \in \mathcal{K}, \exists x_0 : x \in \bar{B}(x_0, r) \subset \mathcal{K}$,

and Petrov condition holds i.e.,

(A5) For any $z \in bdry\mathcal{K}$ and for any n_z a unit outward normal to $\mathcal{K}$ at z, one has

$$\min_{u \in U} \langle f(z, u), n_z \rangle < 0,$$

then $\mathcal{T}$ is locally semiconcave (semiconcavity $\sim$ quadratic perturbation of concavity).
In this case, $\mathcal{T}$ is twice differentiable a.e. in $\mathcal{R} := \{x : \mathcal{T}(x) < \infty\}$.
However, it may fail to be everywhere differentiable and its differentiability at a point x does not guarantee continuous differentiability around x. We identify hypotheses on the dynamic data and the target to ensure continuous differentiability of $\mathcal{T}$ around a given point.

To prove our result we need the following property:

(P) $\mathcal{T}$ is differentiable at a point x if and only if there exists a unique optimal trajectory sarting at x.

In [P. Cannarsa, Sinestrari, On a class of nonlinear time optimal control problems, Discrete Contin. Dynam. Systems 1 (1995), 285-300], the authors provide some conditions to ensure that the property (P) holds true: namely, they assume that

(F) *bdry* $f(x, U)$ is of class C^1 for any $x \in \mathbb{R}^N$.

Using (F), they proved:

- If $y(\cdot)$ is an optimal trajectory for x then $\mathcal{T}$ is differentiale at $y(t)$ for any $t \in (0, \mathcal{T}(x))$.

We consider the minimum time problem for the control system

$$\begin{cases} \dot{y}_1(t) &= u \\ \dot{y}_2(t) &= 0 \end{cases}, \quad u \in U := [-1, 1],$$

Define

$$\begin{aligned} \mathcal{D} &= \{x : 2x_1 - 3x_2 - 2 > 0\} \cap \{x : 2x_1 + 3x_2 - 2 > 0\} \\ &\cap \{x : 2x_1 + 3x_2 - 14 < 0\} \cap \{x : 2x_1 - 3x_2 - 14 < 0\}. \end{aligned}$$

The target is the set $\mathcal{K} = \mathbb{R}^2 \setminus \mathcal{D}$.
Let $x = (x_1, x_2) \in \mathcal{D} \cap \{(x_1, x_2) : x_1 < 4\}$. Then $u^* \equiv -1$ is the optimal control for x we can easily compute that

$$\mathcal{T}(x) = x_1 - \frac{3}{2}|x_2| - 1.$$

If $x = (x_1, x_2) \in \mathcal{D} \cap \{(x_1, x_2) : x_1 > 4\}$, then $u^* \equiv 1$ is the optimal control for x and

$$\mathcal{T}(x) = -x_1 - \frac{3}{2}|x_2| + 7.$$

We show that (P) still holds true if we repace (F) by

(A6) $b_{\mathcal{K}}(:= d_{\mathcal{K}} - d_{\mathcal{K}^c})$ is of class $C^{1,1}_{loc}$ on a neighborhood of bdry$\mathcal{K}$.

Theorem (Pontryagin's Maximum Principle)

Assume (A1) - (A6). Let $x \in \mathcal{R} \setminus \mathcal{K}$, $u(\cdot)$ *be an optimal control for* x *and* $y(\cdot) := y(\cdot; x, u)$ *be the corresponding optimal trajectory. Set* $z = y(\mathcal{T}(x))$ *and let* ζ *be the outer unit normal to* $\mathcal{K}$ *at* z. *Then for any* $\mu > 0$, *the solution of the system*

$$\begin{cases} p'(t) & = & -D_x f(y(t), u(t))^T p(t) \\ p(\mathcal{T}(x)) & = & \mu\zeta \end{cases} \tag{3}$$

satisfies

$$-\langle f(y(t), u(t)), p(t)\rangle = H(y(t), p(t)), \tag{4}$$

for a.e. $t \in [0, \mathcal{T}(x)]$.

A nonzero absolutely continuous function $p(\cdot)$ satisfying (3) for some $\mu > 0$ is called a dual arc associated to the optimal trajectory $y(\cdot; x, u)$.

Let $H : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be the Hamiltonian associated to (1), i.e.,

$$H(x,p) := \max_{u \in U} \{\langle -f(x,u), p\rangle\}, \qquad (x,p) \in \mathbb{R}^N \times \mathbb{R}^N. \quad (5)$$

If

(H1) $H \in C^{1,1}_{loc}(\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}))$,

Maximum Principle can be written in the form of the Hamiltionian system i.e.,
Let $y(\cdot)$ be an optimal trajectory for some $x \in \mathcal{R} \setminus \mathcal{K}$ and let $p(\cdot)$ be an associated dual arc. Then the pair $(y(\cdot), p(\cdot))$ solves the Hamiltonian system

$$\begin{cases} y'(t) &= -H_p(y(t), p(t)) \\ p'(t) &= H_x(y(t), p(t)) \end{cases} \quad (6)$$

in $[0, \mathcal{T}(x)]$.

Let $\Omega \subset \mathbb{R}^N$ be open and let $f : \Omega \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous.

(i) The *proximal subdifferential* $\partial^P f(x)$ of f at a point $x \in dom(f)$ is the set

$$\partial^P f(x) = \Big\{ v \in \mathbb{R}^N : \exists\ c, \rho > 0 \text{ such that } \ \forall y \in B(x, \rho) \\ f(y) - f(x) - \langle v, y - x \rangle \geq -c|y - x|^2, \Big\}.$$

(ii) The *Fréchet subdifferential* $D^- f(x)$ of f at a point $x \in dom(f)$ is the set

$$D^- f(x) = \left\{ p \in \mathbb{R}^N : \liminf_{y \to x} \frac{f(y) - f(x) - \langle p, y - x \rangle}{|y - x|} \geq 0 \right\}.$$

(iii) The *Fréchet superdifferential* $D^+ f(x)$ of f at a point $x \in dom(f)$ is the set

$$D^+ f(x) = \left\{ p \in \mathbb{R}^N : \limsup_{y \to x} \frac{f(y) - f(x) - \langle p, y - x \rangle}{|y - x|} \leq 0 \right\}.$$

Theorem (P. Cannarsa, H. Frankowska, C. Sinestrari, 2000)

Under the same assumptions, in the Maximum Principle, if μ is so that $H(z, \mu\zeta) = 1$, then the dual arc $p(\cdot)$ satisfies

$$p(t) \in D^{+}\mathcal{T}(y(t)), \quad \forall t \in [0, \mathcal{T}(x)).$$

Theorem (H. Frankowska, L., JOTA)

Assume (A1) - (A3). Let $x \in \mathcal{R} \setminus \mathcal{K}$ *and* $(\bar{y}(\cdot), \bar{u}(\cdot))$ *be an optimal pair for* x. *Assume that* $\partial^P \mathcal{T}(x) \neq \emptyset$ *and let* $p : [0, \mathcal{T}(x)] \to \mathbb{R}^n$ *be a solution of*

$$p'(t) = -D_x f(\bar{y}(t), \bar{u}(t))^T p(t) \tag{7}$$

satisfying $p(0) \in \partial^P \mathcal{T}(x)$. *Then for some* $c > 0$ *and for all* $t \in [0, \mathcal{T}(x_0))$, *there exists* $r > 0$ *such that, for every* $y \in B(\bar{y}(t), r)$,

$$\mathcal{T}(y) - \mathcal{T}(\bar{y}(t)) \geq \langle p(t), y - \bar{y}(t) \rangle - c|y - \bar{y}(t)|^2. \tag{8}$$

Consequently, $p(t) \in \partial^P \mathcal{T}(\bar{y}(t))$ *for all* $t \in [0, \mathcal{T}(x))$.

Theorem

Assume (A1) - (A3). Let $x \in \mathcal{R} \setminus \mathcal{K}$ *and* $(\bar{y}(\cdot), \bar{u}(\cdot))$ *be an optimal pair for* x*. Assume that* $D^{-}\mathcal{T}(x) \neq \emptyset$ *and let* $p : [0, \mathcal{T}(x)] \to \mathbb{R}^n$ *be a solution of*

$$p'(t) = -D_x f(\bar{y}(t), \bar{u}(t))^T p(t) \tag{9}$$

satisfying $p(0) \in D^{-}\mathcal{T}(x)$*. Then one has* $p(t) \in D^{-}\mathcal{T}(\bar{y}(t))$ *for all* $t \in [0, \mathcal{T}(x))$*.*

Theorem (H. Frankowska, L., JOTA)

Assume (A1) - (A6) and (H1). The minimum time function is differentiable at a point $x \in \mathcal{R} \setminus \mathcal{K}$ *if and only if there exists a unique optimal trajectory starting at* x.

We consider the minimum time problem for the control system

$$\begin{pmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad |u_i| \leq 1, \ i = 1, 2,$$

with $y_1(0) = x_1, y_2(0) = x_2$. The target is the set

$$\begin{aligned} \mathcal{K} &= \{x : x_1 \leq 0\} \cap \left\{x : x_2 \leq 4 + \sqrt{-x_1^2 - 4x_1}\right\} \\ &\cap \{x : x_1 \geq -4\} \cap \left\{x : x_2 \geq -4 - \sqrt{-x_1^2 - 4x_1}\right\} \end{aligned}$$

The Hamiltonian is defined by
$H(x,p) = |p_1| + |p_2| - 1, \forall x \in \mathbb{R}^2, p = (p_1, p_2) \in \mathbb{R}^2$.
(A1)- (A6) are satisfied anh (H1) is not satisfied. $\mathcal{T}$ is of class $C^{1,1}_{loc}(\mathcal{R} \setminus \mathcal{K})$.
There are multiple optimal trajectories starting at $x = (1,0)$.
Indeed, the trajectories corresponding to the controls $u_1 \equiv (-1,0)$, $u_2 \equiv (-1,1)$ and $u_3 \equiv (-1,-1)$ are optimal for x.

We assume that

(A7) $\mathrm{bdry}\mathcal{K}$ is of class C^2.

(H2) $H \in C^2(\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}))$

Consider the Hamiltonian system

$$\begin{cases} -x'(t) & = & H_p(x(t), p(t)) \\ p'(t) & = & H_x(x(t), p(t)), \end{cases} \tag{10}$$

on $[0, T]$ for some $T > 0$, with the final conditions

$$\begin{cases} x(T) & = & z \\ p(T) & = & \varphi(z), \end{cases} \tag{11}$$

where z is in a neighborhood of $\mathrm{bdry}\,\mathcal{K}$ and $\varphi(z) = \mu(z)\nabla b_{\mathcal{K}}(z)$ with

$$\mu(z) = \frac{1}{H(z, \nabla b_{\mathcal{K}}(z))}. \tag{12}$$

For a given z in a neighborhood of bdry $\mathcal{K}$, let $(x(\cdot;z), p(\cdot;z))$ be the solution of (10) - (11) defined on a time interval $[0, T]$ with $T > 0$.
By differentiating the solution map of (10)- (11) with respect to z, we obtain that $(D_z x(\cdot;z), D_z p(\cdot;z))$ satisfies the variational system

$$\begin{cases} -X' &= H_{xp}(x(t,z),p(t,z))X + H_{pp}(x(t,z),p(t,z))P, \\ P' &= H_{xx}(x(t,z),p(t,z))X + H_{px}(x(t,z),p(t,z))P, \\ X(T) &= I, \\ P(T) &= D\varphi(z). \end{cases} \tag{13}$$

The solution (X, P) of (13) is defined in $[0, T]$ and depends on z. Note that $X(t)$ is invertible for t sufficiently close to T.

Definition (Conjugate times)

For $z \in \operatorname{bdry} \mathcal{K}$, the time

$$t_c(z) := \inf\{t \in [0, T] : X(s)\theta \neq 0, \forall 0 \neq \theta \in T_{\operatorname{bdry} \mathcal{K}}(z), \forall s \in [t, T]\}$$

is said to be conjugate for z iff there exists $0 \neq \theta \in T_{\operatorname{bdry} \mathcal{K}}(z)$ such that

$$X(t_c(z))\theta = 0.$$

In this case, the point $x(t_c(z))$ is called conjugate for z.

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Theorem (H. Frankowska, L., JOTA)

Assume (A1) - (A7) and (H2). Let $\bar{x} \in \mathcal{R} \setminus \mathcal{K}$ be such that $\mathcal{T}$ is differentiable at $\bar{x}$ and $x(\cdot)$ be the optimal trajectory for $\bar{x}$. Set $\bar{z} = x(\mathcal{T}(\bar{x}))$. If there is no conjugate time in $[0, \mathcal{T}(\bar{x})]$ for $\bar{z}$ then $\mathcal{T}$ is of class C^1 in a neighborhood of $\bar{x}$.

Theorem (H. Frankowska, L., JOTA)

Assume (A1) - (A7), (H2) and that the kernel of $H_{pp}(x,p)$ has the dimension equal to 1 for every $(x,p) \in \mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\})$. Let $\bar{x} \in \mathcal{R} \setminus \mathcal{K}$. If $\partial^P \mathcal{T}(\bar{x}) \neq \emptyset$, then $\mathcal{T}$ is of class C^1 in a neighborhood of $\bar{x}$.

Theorem (H. Frankowska, L., JOTA)

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The idea of proof.

- P. Cannarsa, H. Frankowska, From pointwise to local regularity for solutions of Hamilton-Jacobi equations. Calc. Var. Partial Differential Equations. 49 (2014), 1061 - 1074.
- P. Cannarsa, H. Frankowska, Local regularity of the value function in optimal control, Systems Control Lett. 62 (2013), 791-794.

Basing on the following facts:

- $p(t) \in \partial^P \mathcal{T}(x(t))$, for all $t \in [0, \mathcal{T}(\bar{x}))$.
- $0 \neq p \in \ker H_{pp}(x,p), \forall x \in \mathbb{R}^N$.
- $H(x(t), p(t)) = 1$, for all $t \in [0, \mathcal{T}(\bar{x})]$.

$\Rightarrow$ there is no conjugate time.

Example

Consider the control system with the dynamics given by

$$f(x,u) = h(x) + g(x)u,$$

where $h : \mathbb{R}^N \to \mathbb{R}^N$, $g : \mathbb{R}^N \to \mathcal{L}(\mathbb{R}^N; \mathbb{R}^M)$ and the control set U is the closed ball in $\mathbb{R}^M$ of center zero and radius $R > 0$.
We assume that h, g are of class C^2, g is surjective and there exists $k \geq 0$ such that

$$|h(x)| + ||g(x)|| \leq k(1 + |x|), \ \ \forall x \in \mathbb{R}^N,$$

If the target $\mathcal{K}$ is of class C^2 and for any $z \in$bdry $\mathcal{K}$, the classical inward pointing condition

$$\min_{u \in U} \langle n_z, h(z) + g(z)u \rangle < 0$$

hold true then all assumptions in Theorem are satisfied.

Thank you for your attention!