Singular Perturbations of Stochastic Control Problems with Unbounded Fast Variables

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Plan

- The Two-scale system
- The Optimal Control Problem
- The HJB equation
- The PDE approach to the singular limit $\epsilon \to 0$
- The effective Hamiltonian $\bar{H}$
- The Ergodic Problem (EP)
- Convergence Result
  - Tools
  - Approximation of (EP) by truncation or state constraints
  - Sketch of the proof

Two-scale systems

$(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ complete filtered probability space.

We consider stochastic control systems with small parameter $\epsilon > 0$ of the form:

$$\begin{cases} dX_s = F(X_s, Y_s, u_s)ds + \sqrt{2}\sigma(X_s, Y_s, u_s)dW_s, & X_{s_0} = x \in \mathbb{R}^n \\ dY_s = -\frac{1}{\epsilon}\xi_s ds + \sqrt{\frac{1}{\epsilon}}\tau(Y_s)d\hat{W}_s, & Y_{s_0} = y \in \mathbb{R}^m. \end{cases}$$

Basic assumptions

- F and σ Lipschitz functions in (x, y) uniformly w.r.t. u,

$$|F(x,y,u)| + \|\sigma(x,y,u)\| \leq C(1 + |x|)$$

  + conditions (later)
- $\tau\tau^T = 1$
- $u \in U$ (compact), ξ takes values in $\mathbb{R}^m$

The Optimal Control Problem

For $\theta^* > 1$, we consider Payoff Functionals for $t \in [0, T]$ of the form

$$J^\epsilon(t, x, y, u, \xi) = \mathbb{E}^{x,y}[\int_t^T (l(X_s, Y_s, u_s) + \frac{1}{\theta^*}|\xi_s|^{\theta^*})ds + g(X_T)].$$

$\alpha > 1$

- g continuous, $g(x) \leq C(1 + |x|^\alpha)$ and g bounded below
- l contiuous and

$$l_0|y|^\alpha - l_0^{-1} \leq |l(x, y, u)| \leq l_0^{-1}(1 + |y|^\alpha)$$

+ conditions (later)

Value Function is:

$$V^\epsilon(t, x, y) = \inf_{u,\xi} J^\epsilon(t, x, y, u, \xi), \quad 0 \leq t \leq T$$

Admissible control ξ

Definition

For $T > 0$, we say that ξ is admissible if

$$\mathbb{E}^y \Big[\int_0^T |\xi_s|^{\theta^*} ds \Big] < +\infty.$$

It is possible to see that

- $\mathbb{E}^x \big[\int_0^T |X_s|^\alpha ds \big] < +\infty$ (standard)
- $\mathbb{E}^y \big[\int_0^T |Y_s|^\alpha ds \big] < +\infty$ (less standard, uses the admissibility of ξ and $\alpha \leq \theta^*$)
- V^ϵ is bounded by below
- $|V^\epsilon(t,x,y)| \leq C(1 + |x|^\alpha + |y|^\alpha)$ (equi boundedness)

The HJB equation

The HJB equation associated via Dynamic Programming to the value function V^ϵ is

$$-V_t^\epsilon + H(x, y, D_x V^\epsilon, D_{xx}^2 V^\epsilon, \frac{D_{xy}^2 V^\epsilon}{\sqrt{\epsilon}}) - \frac{1}{2\epsilon}\Delta_y V^\epsilon + \frac{1}{\theta}|\frac{D_y V^\epsilon}{\epsilon}|^\theta = 0$$

with

$$H(x, y, p, M, Z) := \sup_u\{-\mathrm{trace}(\sigma\sigma^T M) - F \cdot p - \sqrt{2}\mathrm{trace}(\sigma\tau^T Z^T) - l\}$$

in $(0, T) \times \mathbb{R}^n \times \mathbb{R}^m$ complemented with the terminal condition

$$V^\epsilon(T, x, y) = g(x, y)$$

This is a fully nonlinear degenerate parabolic equation.

Well Posedness

Theorem

Suppose $\theta \leq \alpha^*$ where α^* is the conjugate number of α, that is, $\alpha^* = \frac{\alpha}{\alpha-1}$. For any $\epsilon > 0$, the function V^ϵ is the unique continuous viscosity solution to the Cauchy problem with at most α-growth in x and y. Moreover the functions V^ϵ are locally equibounded.

Uses Comparison principle between sub and super solution to parabolic problems super linear growth conditions (see [Da Lio - Ley 2011]),

PDE approach to the singular limit $\epsilon \to 0$

Search *effective Hamiltonian* $\overline{H}$ s. t.

$$V^{\varepsilon}(t,x,y) \to V(t,x) \quad \text{as} \quad \varepsilon \to 0,$$

V solution of

$$(\overline{\text{CP}}) \qquad \begin{cases} -\frac{\partial V}{\partial t} + \overline{H}\left(x, D_x V, D^2_{xx} V\right) = 0 & \text{in} \quad (0,T) \times \mathbf{R}^n, \\ V(T,x) = g(x) \quad \text{in} \quad \mathbf{R}^n \end{cases}$$

The effective Hamiltonian $\overline{H}$

Finding the candidate limit Cauchy problem of the singularly perturbed problem as $\epsilon \to 0$...

Ansatz: $V^\epsilon(t,x,y) = V(t,x) + \epsilon\chi(y)$, with $\chi(y) \in C^2(\mathbb{R}^m)$.

We get

$$-V_t + H(x,y,D_xV,D^2_{xx}V,0) - \frac{1}{2}\Delta_y\chi + \frac{1}{\theta}|D_y\chi|^\theta = 0$$

with $H(x,y,p,M,0) = \sup_u\{-\text{trace}(\sigma\sigma^T M) - F\cdot p - l\}$.

We wish that

$$\overline{H}(x,D_xV,D^2_{xx}V) = H(x,y,D_xV,D^2_{xx}V,0) - \frac{1}{2}\Delta_y\chi + \frac{1}{\theta}|D_y\chi|^\theta.$$

Idea is to frozen $(\bar{t},\bar{x})$, set $\bar{p} := D_xV(\bar{t},\bar{x})$ and $\bar{M} := D^2_{xx}V(\bar{t},\bar{x})$ and let only y varie.

The effective Hamiltonian $\overline{H}$

$\bar{H}(\bar{x}, \bar{p}, \bar{M})$ is a constant that we will denote by $-\lambda$. Thus

$$-\lambda = H(\bar{x}, y, \bar{p}, \bar{M}, 0) - \frac{1}{2}\Delta_y \chi + \frac{1}{\theta}|D_y \chi|^\theta$$
$$\Leftrightarrow \lambda - \frac{1}{2}\Delta_y \chi + \frac{1}{\theta}|D_y \chi|^\theta = -H(\bar{x}, y, \bar{p}, \bar{M}, 0).$$

If we call $f(y) := -H(\bar{x}, y, \bar{p}, \bar{M}, 0)$ and impose $\chi(0) = 0$, to avoid the ambiguity of additive constant, we are lead to the following ergodic problem:

$$(EP) \quad \begin{cases} \lambda - \frac{1}{2}\Delta_y \chi + \frac{1}{\theta}|D_y \chi|^\theta = f(y) & \text{in} \quad \mathbb{R}^m, \\ \chi(0) = 0, \end{cases}$$

where unknown is the pair $(\lambda, \chi) \in \mathbb{R} \times C^2(\mathbb{R}^m)$.

The (EP)

Ergodic Problem

$$\begin{cases} \lambda - \frac{1}{2}\Delta_y \chi + \frac{1}{\theta}|D_y \chi|^\theta = f(y) \quad \text{in} \quad \mathbb{R}^m, \\ \chi(0) = 0. \end{cases}$$

Unknown is the pair $(\lambda, \chi) \in \mathbb{R} \times C^2(\mathbb{R}^m)$.

Such type of ergodic problems were studied by Naoyuki Ichihara in [Ichihara 2012].

Assumptions in [Ichihara 2012]

- $f \in C^2(\mathbb{R}^m)$
- $\exists f_0 > 0$ s.t.

$$f_0|y|^\alpha - f_0^{-1} \leq f(y) \leq f_0^{-1}(1 + |y|^\alpha)$$

(comes from my l)

The (EP) - Results from Ichihara

- (**local gradient bounds**)
  For any $r > 0$, there exists a constant $C > 0$ depending only on r, m and θ such that for any solution (λ, χ) of (EP),

$$\sup_{B_r} |D\chi| \leq C(1 + \sup_{B_{r+1}} |f - \lambda|^{\frac{1}{\theta}} + \sup_{B_{r+1}} |Df|^{\frac{1}{2\theta - 1}})$$

- $|\chi(y)| \leq C(1 + |y|^\gamma)$ $(\gamma = \frac{\alpha}{\theta} + 1)$
- (**Uniqueness**) There exists a unique solution (λ, χ) of (EP) such that χ belongs to

$$\Phi_\gamma := \{v \in C^2(\mathbb{R}^m) \cap C_p(\mathbb{R}^m) | \liminf_{|y| \to \infty} \frac{v(y)}{|y|^\gamma} > 0\}.$$

- $\overline{H} = -\lambda$ is the minimum of the constants for which (EP) has a solution $\phi \in C^2(\mathbb{R}^m)$

The (EP)- From Ichihara

Theorem

Let (λ, χ) be a solution of (EP) such that χ is bounded by below. Then,

$$\epsilon\chi(y) + \lambda(T-t) = \inf_{\xi \in \mathcal{A}} \mathbb{E}^y\Big[\int_t^T \Big(\frac{1}{\theta^*}|\xi_s|^{\theta^*} + f(Y_s^{\xi})\Big)ds + \epsilon\chi(Y_T^{\xi})\Big], \quad T > t.$$

equality is reach for the optimal feedback control ξ^*.

From this result, you can deduce FORMULA (F)

$$\epsilon(\chi_1 - \chi_2)(y) + (\lambda_1 - \lambda_2)(T-t) \le \mathbb{E}^y\Big[\int_t^T \big(f_1 - f_2)(Y_s^{\xi^*})\big)ds\Big] + \epsilon\mathbb{E}^y[(\chi_1 - \chi_2)(Y_T^{\xi^*})]$$

(λ_i, χ_i) solution of (EP) with χ_i bounded by below and $f = f_i$ $(i = 1, 2)$.
ξ^* optimal feedback for (EP) with $f = f_2$.

The Convergence Theorem

Theorem

$\lim_{\epsilon\to 0} V^{\epsilon}(t,x,y) = V(t,x)$ *locally uniformly, V being the unique solution of*

$$\begin{cases} -\frac{\partial V}{\partial t} + \overline{H}\left(x, D_x V, D^2_{xx} V\right) = 0 \qquad in \quad (0,T)\times \mathbf{R}^n, \\ V(T,x) = g(x) \qquad in \qquad \mathbf{R}^n \end{cases}$$

satisfying

$$|V(t,x)| \leq C(1+|x|^{\alpha}).$$

Tools

- V^ϵ equibounded uniformly in ϵ

$$|V^\epsilon(t,x,y)| \leq C(1+|x|^\alpha+|y|^\alpha)$$

- $-\infty < u^\epsilon \leq V^\epsilon$

  Exists $\rho \in (0,1)$ s.t

$$u^\epsilon(t,x,y) = (T-t)[\epsilon\rho(1+|y|^2)^{\frac{\gamma}{2}} - 2f_0^{-1}] + \inf g$$

  satisfies

$$-\infty < u^\epsilon \leq V^\epsilon$$

- Then the relaxed semilimits

$$\underline{V}(t,x) = \liminf_{\epsilon\to 0,(t',x')\to(t,x)} \inf_{y\in\mathbb{R}^m} V^\epsilon(t',x',y),$$

$$\bar{V}_R(t,x) = \limsup_{\epsilon\to 0,(t',x')\to(t,x)} \sup_{y\in B_R(0)} V^\epsilon(t',x',y).$$

  are finite!

$\underline{V}$ is a super solution of the limit PDE

Tools

- Perturbed test function method, evolving from Evans (periodic homogenisation) and Alvarez-M.Bardi (singular perturbations with bounded fast variables).
- approximation of (EP) by truncation

$\underline{V}$ is a super solution of the limit PDE

Tools

- Perturbed test function method, evolving from Evans (periodic homogenisation) and Alvarez-M.Bardi (singular perturbations with bounded fast variables).
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Approximation of (EP) by truncation

$$(EP)_R \quad \begin{cases} \lambda_R - \frac{1}{2}\Delta_y \chi_R + \frac{1}{\theta}|D_y \chi_R|^\theta = f(y) \wedge (f_0|y|^{\alpha - \frac{1}{R}} + R) \quad \text{in} \quad \mathbb{R}^m, \\ \chi_R(0) = 0. \end{cases}$$

By Ichihara results,

- it has a unique pair of solutions $(\lambda_R, \chi_R) \in \mathbb{R} \times C^2(\mathbb{R}^m)$ such that $\chi_R \in \Phi_{\gamma - \frac{1}{R\theta}}$
- $|\chi_R(y)| \leq C(1 + |y|^{\gamma - \frac{1}{R\theta}})$

Approximation of (EP) by truncation

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Approximation of (EP) by truncation

Theorem

There exists a sequence $R_j \to \infty$ as $j \to \infty$ s.t. the pair of solutions (λ_R, χ_R) of $(EP)_R$ with $\chi_R \in \Phi_{\gamma - \frac{1}{R\theta}}$ converges to the unique solution $(\lambda\chi)$ of (EP) such that $\chi \in \Phi_\gamma$

Sketch of the proof

- $\forall 0 < R' < R$

$$\sup_{B'_R} |D\chi_R| \leq C$$

  C not depending on R

- Classical theory for quasilinear elliptic equations + Schauder's Theory $\Longrightarrow |\chi_R|_{2+\Gamma, B'_R}$ is bounded by a constant not depending on $R > R'$.

- In particular, $\{\chi_R\}_{R>R'}$ is pre-compact. Namely, $\exists R_j \to \infty$ as $j \to \infty$ and $v \in C^2(\mathbb{R}^m)$ s.t.

$$\chi_{R_j} \to v, D\chi_{R_j} \to Dv, D^2\chi_{R_j} \to D^2 v \text{ in } C^2(\mathbb{R}^m) \text{ as } j \to \infty$$

- Formula (F) gives
  - $\lambda_{R_j} \leq \lambda$
  - $\lambda_{R_j} \leq \lambda_{R_{j+1}}$

  IMPLIES there exists a convergence subsequence, $\lambda_{R_j} \to c \in \mathbb{R}$
- Conclusion: $(\lambda_{R_j}, \chi_{R_j}) \to (c, v)$. BUT $\chi_{R_j} \in \Phi_{\gamma - \frac{1}{R\theta}}$ and $\chi_{R_j}(0) = 0$, so $\lim \chi_{R_j} = v \in \Phi_\gamma$ and $v(0) = 0$. We are in the right class for which there is uniqueness for (EP), $v = \chi$, $c = \lambda$.

Perturbed test function method

$\underline{V}$ is a super solution of the limit PDE () in $(0, T) \times \mathbb{R}^n$?

Fix an arbitrary $(\bar{t}, \bar{x}) \in (0, T) \times \mathbb{R}^n$. This means, if ψ is a smooth function such that $\psi(\bar{t}, \bar{x}) = \underline{V}(\bar{t}, \bar{x})$ and $\underline{V} - \psi$ has a strict minimum at $(\bar{t}, \bar{x})$ then

$$-\psi_t(\bar{t}, \bar{x}) + \bar{H}(\bar{x}, D_x\psi(\bar{t}, \bar{x}), D^2_{xx}\psi(\bar{t}, \bar{x})) \geq 0.$$

WE ARGUE BY CONTRADICTION. Assume that there exists $\eta > 0$ such that

$$-\psi_t(\bar{t}, \bar{x}) + \bar{H}(\bar{x}, D_x\psi(\bar{t}, \bar{x}), D^2_{xx}\psi(\bar{t}, \bar{x})) < -2\eta < 0.$$

Perturbed test function method

- Perturbed test function: $\psi^\epsilon(t,x,y) = \psi(t,x) + \epsilon\chi_R(y)$. χ_R, with $R = R_j$, in the conditions of last Theorem with f substituted by $l(y) = \inf_{|x-\bar{x}|,|t-\bar{t}|<\frac{1}{R}} f_{t,x}(y)$ $(f_{t,x}(y) = -H(x,y,D_x\psi(t,x),D^2_{xx}\psi(t,x),0))$
- Then ψ^ϵ satisfies

$$-\psi^\epsilon_t + H(x,y,D_x\psi^\epsilon,D^2_{xx}\psi^\epsilon,0) - \frac{1}{2\epsilon}\Delta_y\psi^\epsilon + \frac{1}{\theta}\left|\frac{D_y\psi^\epsilon}{\epsilon}\right|^\theta < 0$$

in

$$Q_R =]\bar{t} - \frac{1}{R}, \bar{t} + \frac{1}{R}[\times B_{\frac{1}{R}}(\bar{x}) \times \mathbb{R}^m.$$

Perturbed test function method

$$V^\epsilon(t,x,y) - \psi^\epsilon(t,x,y) \geq u^\epsilon(t,x,y) - \psi(t,x) - \epsilon C\left(1 + |y|^{\gamma - \frac{1}{\theta R}}\right) > -\infty.$$

Hence it exists

$$\liminf_{\epsilon \to 0, (t',x') \to (t,x)} \inf_{y \in \mathbb{R}^m} (V^\epsilon - \psi^\epsilon)(t',x',y) > -\infty.$$

Since ψ^ϵ is bounded by below, we can conclude that

$$\liminf_{\epsilon \to 0, (t',x') \to (t,x)} \inf_{y \in \mathbb{R}^m} (V^\epsilon - \psi^\epsilon)(t',x',y) = (\underline{V} - \psi)(t,x).$$

But $(\bar{t}, \bar{x})$ is a strict minimum point of $\underline{V} - \psi$ so the above relaxed lower limit is > 0 on ∂Q_R. Hence we can find

$$\zeta > 0 \quad : \quad V^\epsilon - \zeta \geq \psi^\epsilon \text{ on } \partial Q_R \text{ for } \epsilon \text{ small.}$$

Claim $V^\epsilon - \zeta \geq \psi^\epsilon$ in Q_R and **Contradiction** on the black board.

$\bar{V}_R$ is a sub solution of a perturbed PDE

$$\begin{cases} -\frac{\partial V}{\partial t} + \overline{H}\left(x, D_x V, D^2_{xx} V\right) - \frac{1}{R} = 0 & \text{in} \quad (0,T) \times \mathbf{R}^n \\ V(T,x) = g(x) \quad \text{in} \quad \mathbf{R}^n \end{cases}$$

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Tools

- Perturbed test function method
- approximation of (EP) by state constraints

  $S(y) = \sup_{|x-\bar{x}|,|t-\bar{t}|<\frac{1}{R}} f_{t,x}(y)$

  - sub quadratic case ($1 < \theta \leq 2$)

$$\begin{cases} \lambda_R - \frac{1}{2}\Delta_y \chi_R + \frac{1}{\theta}|D_y\chi_R|^\theta = S(y) \quad \text{in} \quad B_R(0), \\ \chi_R \to +\infty \text{ as } y \to \partial B_R(0), \\ \chi_R(0) = 0. \end{cases}$$

  - super quadratic ($\theta > 2$)

$$\begin{cases} \lambda_R - \frac{1}{2}\Delta_y \chi_R + \frac{1}{\theta}|D_y\chi_R|^\theta = S(y) \quad \text{in} \quad B_R(0), \\ \lambda_R - \frac{1}{2}\Delta_y \chi_R + \frac{1}{\theta}|D_y\chi_R|^\theta \geq S(y) \quad \text{on } \partial B_R(0), \\ \chi_R(0) = 0. \end{cases}$$

$\bar{V}_R$ is a sub solution of a perturbed PDE

- Comparison Results for such problems (T. Tchamba's Phd thesis [super quadratic case] + [Barles-Da Lio, 2006] [sub quadratic case]
- $\overline{H}$ is the minimum of the constants for which (EP) has a solution $\phi \in C^2(\mathbb{R}^m)$

$\sup_R \bar{V}_R$ is a sub solution of the limit PDE

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  for all R large enough

- Since the supremum of sub solutions is a sub solution, we see that $\sup_R \bar{V}_R$ is a sub solution of the limit PDE

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Comparison Principle and Uniform Convergence

Comparison principle
between sub and super solution to parabolic problems satisfying

$$|V(t,x)| \leq K(1 + |x|^{\alpha})$$

(see [Da Lio - Ley 2011]), gives

- uniqueness of solution V of $\overline{\text{CP}}$,
- $\underline{V}(t,x) \geq \sup_R \bar{V}_R(t,x)$, then $\underline{V} = \sup_R \bar{V}_R = V$ and, as $\epsilon \to 0$,

$$V^{\epsilon}(t,x,y) \to V(t,x) \quad \text{locally uniformly.}$$

Work in progress and close perspectives

- associate to the limit PDE a "limit control problem";
- list examples;
- re-obtain and extend Naoyuki Ichihara results using only PDE methods (partially done);
- simpler formula "**(F)**"?

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Grazie !

"Sadko in the Underwater Kingdom" by Ilya Repin