Singular Perturbations of Stochastic Control Problems with Unbounded Fast Variables

Joao Meireles  
joint work with  
Martino Bardi and Guy Barles

University of Padua, Italy

Workshop "New Perspectives in Optimal Control and Games"  
Nov. 10-12, 2014, Rome  
SADCO Meeting
Plan

- The Two-scale system
- The Optimal Control Problem
- The HJB equation
- The PDE approach to the singular limit $\epsilon \to 0$
- The effective Hamiltonian $\tilde{H}$
- The Ergodic Problem (EP)
- Convergence Result
  - Tools
  - Approximation of (EP) by truncation or state constraints
  - Sketch of the proof
Two-scale systems

$(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ complete filtered probability space.

We consider stochastic control systems with small parameter $\epsilon > 0$ of the form:

\[
\begin{cases}
    dX_s = F(X_s, Y_s, u_s)\,ds + \sqrt{2}\sigma(X_s, Y_s, u_s)\,dW_s, & X_{s_0} = x \in \mathbb{R}^n \\
    dY_s = -\frac{1}{\epsilon}\xi_s\,ds + \sqrt{\frac{1}{\epsilon}}\tau(Y_s)\,d\hat{W}_s, & Y_{s_0} = y \in \mathbb{R}^m.
\end{cases}
\]

Basic assumptions

- $F$ and $\sigma$ Lipschitz functions in $(x, y)$ uniformly w.r.t. $u$,

\[
|F(x, y, u)| + \|\sigma(x, y, u)\| \leq C(1 + |x|)
\]

+ conditions (later)

- $\tau\tau^T = 1$

- $u \in U$ (compact), $\xi$ takes values in $\mathbb{R}^m$
The Optimal Control Problem

For $\theta^* > 1$, we consider Payoff Functionals for $t \in [0, T]$ of the form

$$J^\varepsilon(t, x, y, u, \xi) = \mathbb{E}^{x, y} \left[ \int_t^T (l(X_s, Y_s, u_s) + \frac{1}{\theta^*} |\xi_s|^{\theta^*}) ds + g(X_T) \right].$$

$\alpha > 1$

- $g$ continuous, $g(x) \leq C(1 + |x|^\alpha)$ and $g$ bounded below
- $l$ continuous and

$$l_0 |y|^\alpha - l_0^{-1} \leq |l(x, y, u)| \leq l_0^{-1}(1 + |y|^\alpha)$$

+ conditions (later)

Value Function is:

$$V^\varepsilon(t, x, y) = \inf_{u, \xi} J^\varepsilon(t, x, y, u, \xi), \quad 0 \leq t \leq T$$
Admissible control $\xi$

Definition
For $T > 0$, we say that $\xi$ is admissible if

$$E^y \left[ \int_0^T |\xi_s|^{\theta^*} \, ds \right] < +\infty.$$ 

It is possible to see that

- $E^x \left[ \int_0^T |X_s|^\alpha \, ds \right] < +\infty$ \textit{(standard)}
- $E^y \left[ \int_0^T |Y_s|^\alpha \, ds \right] < +\infty$ \textit{(less standard, uses the admissibility of $\xi$ and $\alpha \leq \theta^*$)}
- $V^\epsilon$ is bounded by below
- $|V^\epsilon(t, x, y)| \leq C(1 + |x|^\alpha + |y|^\alpha)$ \textit{(equi boundedness)}
The HJB equation

The HJB equation associated via Dynamic Programming to the value function $V^\epsilon$ is

$$- V^\epsilon_t + H(x, y, D_x V^\epsilon, D_{xx}^2 V^\epsilon, \frac{D_{xy} V^\epsilon}{\sqrt{\epsilon}}) - \frac{1}{2\epsilon} \Delta_y V^\epsilon + \frac{1}{\theta} |\frac{D_y V^\epsilon}{\epsilon}|^\theta = 0$$

with

$$H(x, y, p, M, Z) := \sup_u \{-\text{trace}(\sigma \sigma^T M) - F \cdot p - \sqrt{2}\text{trace}(\sigma \tau^T Z^T) - l\}$$

in $(0, T) \times \mathbb{R}^n \times \mathbb{R}^m$ complemented with the terminal condition

$$V^\epsilon(T, x, y) = g(x, y)$$

This is a fully nonlinear degenerate parabolic equation.
Well Posedness

Theorem

Suppose \( \theta \leq \alpha^* \) where \( \alpha^* \) is the conjugate number of \( \alpha \), that is, \( \alpha^* = \frac{\alpha}{\alpha - 1} \). For any \( \epsilon > 0 \), the function \( V^\epsilon \) is the unique continuous viscosity solution to the Cauchy problem with at most \( \alpha \)-growth in \( x \) and \( y \). Moreover the functions \( V^\epsilon \) are locally equibounded.

Uses Comparison principle between sub and super solution to parabolic problems super linear growth conditions (see [Da Lio - Ley 2011]),
Search **effective Hamiltonian** $\overline{H}$ s. t.

$$V^\varepsilon(t, x, y) \to V(t, x) \quad \text{as} \quad \varepsilon \to 0,$$

$V$ solution of

\begin{equation}
\begin{aligned}
&-\frac{\partial V}{\partial t} + \overline{H}(x, D_x V, D_{xx}^2 V) = 0 \quad \text{in} \quad (0, T) \times \mathbb{R}^n, \\
&V(T, x) = g(x) \quad \text{in} \quad \mathbb{R}^n
\end{aligned}
\end{equation}
The effective Hamiltonian $\overline{H}$

Finding the candidate limit Cauchy problem of the singularly perturbed problem as $\epsilon \to 0$...

**Ansatz:** $V^\epsilon(t, x, y) = V(t, x) + \epsilon\chi(y)$, with $\chi(y) \in C^2(\mathbb{R}^m)$.

We get

$$-V_t + H(x, y, D_x V, D_{xx}^2 V, 0) - \frac{1}{2}\Delta_y \chi + \frac{1}{\theta} |D_y \chi|^\theta = 0$$

with $H(x, y, p, M, 0) = \sup_u \{-\text{trace}(\sigma \sigma^T M) - F \cdot p - l\}$.

We wish that

$$\overline{H}(x, D_x V, D_{xx}^2 V) = H(x, y, D_x V, D_{xx}^2 V, 0) - \frac{1}{2}\Delta_y \chi + \frac{1}{\theta} |D_y \chi|^\theta.$$ 

Idea is to frozen $(\bar{t}, \bar{x})$, set $\bar{p} := D_x V(\bar{t}, \bar{x})$ and $\bar{M} := D_{xx}^2 V(\bar{t}, \bar{x})$ and let only $y$ varie.
The effective Hamiltonian $\bar{H}$

$\bar{H}(\bar{x}, \bar{p}, \bar{M})$ is a constant that we will denote by $-\lambda$. Thus

$$-\lambda = H(\bar{x}, y, \bar{p}, \bar{M}, 0) - \frac{1}{2} \Delta y \chi + \frac{1}{\theta} |D_y \chi|^\theta$$

$$\Leftrightarrow \lambda - \frac{1}{2} \Delta y \chi + \frac{1}{\theta} |D_y \chi|^\theta = -H(\bar{x}, y, \bar{p}, \bar{M}, 0).$$

If we call $f(y) := -H(\bar{x}, y, \bar{p}, \bar{M}, 0)$ and impose $\chi(0) = 0$, to avoid the ambiguity of additive constant, we are lead to the following ergodic problem:

$$(EP) \quad \begin{cases} 
\lambda - \frac{1}{2} \Delta y \chi + \frac{1}{\theta} |D_y \chi|^\theta = f(y) \quad \text{in} \quad \mathbb{R}^m, \\
\chi(0) = 0,
\end{cases}$$

where unknown is the pair $(\lambda, \chi) \in \mathbb{R} \times C^2(\mathbb{R}^m)$. 
The (EP)

Ergodic Problem

\[
\begin{aligned}
\lambda - \frac{1}{2} \Delta y \chi + \frac{1}{\theta} |Dy\chi|^{\theta} &= f(y) \quad \text{in} \quad \mathbb{R}^{m}, \\
\chi(0) &= 0.
\end{aligned}
\]

Unknown is the pair \((\lambda, \chi) \in \mathbb{R} \times C^{2}(\mathbb{R}^{m})\).

Such type of ergodic problems were studied by Naoyuki Ichihara in [Ichihara 2012].

Assumptions in [Ichihara 2012]

- \(f \in C^{2}(\mathbb{R}^{m})\)
- \(\exists f_{0} > 0\) s.t. \(f_{0}|y|^{\alpha} - f_{0}^{-1} \leq f(y) \leq f_{0}^{-1}(1 + |y|^{\alpha})\)

(comes from my \(l\))
(local gradient bounds)
For any $r > 0$, there exists a constant $C > 0$ depending only on $r$, $m$ and $\theta$ such that for any solution $(\lambda, \chi)$ of (EP),

$$\sup_{B_r} |D\chi| \leq C(1 + \sup_{B_{r+1}} |f - \lambda|^{\frac{1}{\theta}} + \sup_{B_{r+1}} |Df|^{\frac{1}{2\theta-1}})$$

$$|\chi(y)| \leq C(1 + |y|^\gamma) \ (\gamma = \frac{\alpha}{\theta} + 1)$$

(Uniqueness) There exists a unique solution $(\lambda, \chi)$ of (EP) such that $\chi$ belongs to

$$\Phi_\gamma := \{ \nu \in C^2(\mathbb{R}^m) \cap C_p(\mathbb{R}^m) | \liminf_{|y| \to \infty} \frac{\nu(y)}{|y|^\gamma} > 0 \}.$$

$\overline{H} = -\lambda$ is the minimum of the constants for which (EP) has a solution $\phi \in C^2(\mathbb{R}^m)$.
The (EP)- From Ichihara

Theorem

Let \((\lambda, \chi)\) be a solution of (EP) such that \(\chi\) is bounded by below. Then,

\[
\varepsilon \chi(y) + \lambda(T - t) = \inf_{\xi \in \mathcal{A}} \mathbb{E}^y \left[ \int_t^T \left( \frac{1}{\theta^*} |\xi_s|^{\theta^*} + f(Y^\xi_s) \right) ds + \varepsilon \chi(Y^\xi_T) \right], \quad T > t.
\]

equality is reach for the optimal feedback control \(\xi^*\).

From this result, you can deduce FORMULA (F)

\[
\varepsilon(\chi_1 - \chi_2)(y) + (\lambda_1 - \lambda_2)(T - t) \leq \mathbb{E}^y \left[ \int_t^T (f_1 - f_2)(Y^{\xi^*_s}) ds \right]
+ \varepsilon \mathbb{E}^y [(\chi_1 - \chi_2)(Y^{\xi^*_T})]
\]

\((\lambda_i, \chi_i)\) solution of (EP) with \(\chi_i\) bounded by below and \(f = f_i\) \((i = 1, 2)\). \(\xi^*\) optimal feedback for (EP) with \(f = f_2\).
The Convergence Theorem

Theorem

\[ \lim_{\epsilon \to 0} V^\epsilon(t, x, y) = V(t, x) \text{ locally uniformly, } V \text{ being the unique solution of} \]

\[
\begin{aligned}
-\frac{\partial V}{\partial t} + \overline{H}(x, D_x V, D_{xx} V) &= 0 \quad \text{in } (0, T) \times \mathbb{R}^n, \\
V(T, x) &= g(x) \quad \text{in } \mathbb{R}^n
\end{aligned}
\]

satisfying

\[ |V(t, x)| \leq C(1 + |x|^\alpha). \]
- $V^\epsilon$ equibounded uniformly in $\epsilon$
  \[ |V^\epsilon(t, x, y)| \leq C(1 + |x|^{\alpha} + |y|^{\alpha}) \]

- $-\infty < u^\epsilon \leq V^\epsilon$

  Exists $\rho \in (0, 1)$ s.t
  \[ u^\epsilon(t, x, y) = (T - t)[\epsilon \rho (1 + |y|^2)^{\gamma/2} - 2f_0^{-1}] + \inf g \]

  satisfies
  \[ -\infty < u^\epsilon \leq V^\epsilon \]

- Then the relaxed semilimits
  \[ V(t, x) = \liminf_{\epsilon \to 0, (t', x') \to (t, x)} \inf_{y \in \mathbb{R}^m} V^\epsilon(t', x', y), \]
  \[ \bar{V}_R(t, x) = \limsup_{\epsilon \to 0, (t', x') \to (t, x)} \sup_{y \in B_R(0)} V^\epsilon(t', x', y). \]

  are finite!
$V$ is a super solution of the limit PDE

**Tools**

- **Perturbed test function method**, evolving from Evans (periodic homogenisation) and Alvarez-M.Bardi (singular perturbations with bounded fast variables).

- **Approximation of (EP) by truncation**
$V$ is a super solution of the limit PDE

**Tools**

- **Perturbed test function method**, evolving from Evans (periodic homogenisation) and Alvarez-M. Bardi (singular perturbations with bounded fast variables).

- **approximation** of (EP) by truncation
Approximation of \((EP)\) by truncation

\[ (EP)_R \begin{cases} \lambda_R - \frac{1}{2} \Delta y \chi_R + \frac{1}{\theta} |D_y \chi_R|^\theta = f(y) \wedge (f_0 |y|^{\alpha - \frac{1}{R}} + R) \quad \text{in} \quad \mathbb{R}^m, \\ \chi_R(0) = 0. \end{cases} \]

By Ichihara results,

- it has a unique pair of solutions \((\lambda_R, \chi_R) \in \mathbb{R} \times C^2(\mathbb{R}^m)\) such that \(\chi_R \in \Phi_{\gamma - \frac{1}{R \theta}}\)
- \(|\chi_R(y)| \leq C(1 + |y|^{\gamma - \frac{1}{R \theta}})\)
Approximation of \((EP)\) by truncation

\[
(EP)_R \quad \begin{cases} 
\lambda_R - \frac{1}{2} \Delta y \chi_R + \frac{1}{\theta} |Dy \chi_R|^\theta = f(y) \wedge (f_0|y|^\alpha - \frac{1}{R} + R) \quad \text{in} \quad \mathbb{R}^m, \\
\chi_R(0) = 0.
\end{cases}
\]

By Ichihara results,

- it has a unique pair of solutions \((\lambda_R, \chi_R) \in \mathbb{R} \times C^2(\mathbb{R}^m)\) such that \(\chi_R \in \Phi_{\gamma - \frac{1}{R\theta}}\)
- \(|\chi_R(y)| \leq C(1 + |y|^{\gamma - \frac{1}{R\theta}})\)
Approximation of $(EP)$ by truncation

$$(EP)_R \quad \begin{cases} 
\lambda_R - \frac{1}{2} \Delta y \chi_R + \frac{1}{\theta} |D_y \chi_R|^\theta = f(y) \wedge (f_0 |y|^{\alpha - \frac{1}{R}} + R) & \text{in } \mathbb{R}^m, \\
\chi_R(0) = 0.
\end{cases}$$

By Ichihara results,

- it has a unique pair of solutions $(\lambda_R, \chi_R) \in \mathbb{R} \times C^2(\mathbb{R}^m)$ such that
  \[ \chi_R \in \Phi_{\gamma - \frac{1}{R\theta}} \]
- \[ |\chi_R(y)| \leq C(1 + |y|^{\gamma - \frac{1}{R\theta}}) \]
Approximation of \((EP)\) by truncation

**Theorem**

There exists a sequence \(R_j \to \infty\) as \(j \to \infty\) s.t. the pair of solutions \((\lambda_{R_j}, \chi_{R_j})\) of \((EP)_{R_j}\) with \(\chi_{R_j} \in \Phi_{\gamma - \frac{1}{R \theta}}\) converges to the unique solution \((\lambda \chi)\) of \((EP)\) such that \(\chi \in \Phi_{\gamma}\).

**Sketch of the proof**

- \(\forall 0 < R' < R\)
  \[
  \sup_{B'_{R}} |D\chi_{R}| \leq C
  \]
  \(C\) not depending on \(R\)

- Classical theory for quasilinear elliptic equations + Schauder’s Theory \(\implies |\chi_{R}|_{2+\Gamma, B'_{R}}\) is bounded by a constant not depending on \(R > R'\).
In particular, \( \{\chi_R\}_{R>R'} \) is pre-compact. Namely, 
\[ \exists R_j \to \infty \text{ as } j \to \infty \text{ and } \nu \in C^2(\mathbb{R}^m) \text{ s.t.} \]

\[ \chi_{R_j} \to \nu, D\chi_{R_j} \to D\nu, D^2\chi_{R_j} \to D^2\nu \text{ in } C^2(\mathbb{R}^m) \text{ as } j \to \infty \]

Formula (F) gives

- \( \lambda_{R_j} \leq \lambda \)
- \( \lambda_{R_j} \leq \lambda_{R_{j+1}} \)

IMPLIES there exists a convergence subsequence, \( \lambda_{R_j} \to c \in \mathbb{R} \)

Conclusion: \( (\lambda_{R_j}, \chi_{R_j}) \to (c, \nu) \). BUT \( \chi_{R_j} \in \Phi_{\gamma-\frac{1}{R\theta}} \) and \( \chi_{R_j}(0) = 0 \), so \( \lim \chi_{R_j} = \nu \in \Phi_{\gamma} \) and \( \nu(0) = 0 \). We are in the right class for which there is uniqueness for (EP), \( \nu = \chi, c = \lambda \).
V is a super solution of the limit PDE \((0, T) \times \mathbb{R}^n\)?

Fix an arbitrary \((\bar{t}, \bar{x}) \in (0, T) \times \mathbb{R}^n\). This means, if \(\psi\) is a smooth function such that \(\psi(\bar{t}, \bar{x}) = V(\bar{t}, \bar{x})\) and \(V - \psi\) has a strict minimum at \((\bar{t}, \bar{x})\) then

\[-\psi_t(\bar{t}, \bar{x}) + \bar{H}(\bar{x}, D_x \psi(\bar{t}, \bar{x}), D_{xx}^2 \psi(\bar{t}, \bar{x})) \geq 0.\]

WE ARGUE BY CONTRADICTION. Assume that there exists \(\eta > 0\) such that

\[-\psi_t(\bar{t}, \bar{x}) + \bar{H}(\bar{x}, D_x \psi(\bar{t}, \bar{x}), D_{xx}^2 \psi(\bar{t}, \bar{x})) < -2\eta < 0.\]
Perturbed test function method

- Perturbed test function: \( \psi^\epsilon(t, x, y) = \psi(t, x) + \epsilon \chi_R(y) \).

\( \chi_R \), with \( R = R_j \), in the conditions of last Theorem with \( f \) substituted by \( l(y) = \inf_{|x-\bar{x}|,|t-\bar{t}|<\frac{1}{R}} f_{t,x}(y) \)

\( (f_{t,x}(y) = -H(x, y, D_x \psi(t, x), D^2_{xx} \psi(t, x), 0)) \)

Then \( \psi^\epsilon \) satisfies

\[
- \psi^\epsilon_t + H(x, y, D_x \psi^\epsilon, D^2_{xx} \psi^\epsilon, 0) - \frac{1}{2\epsilon} \Delta_y \psi^\epsilon + \frac{1}{\theta} \left| \frac{D_y \psi^\epsilon}{\epsilon} \right|^{\theta} < 0
\]

in

\[
Q_R = [\bar{t} - \frac{1}{R}, \bar{t} + \frac{1}{R}] \times B_{\frac{1}{R}}(\bar{x}) \times \mathbb{R}^m.
\]
Perturbed test function method

\[ V^\epsilon(t, x, y) - \psi^\epsilon(t, x, y) \geq u^\epsilon(t, x, y) - \psi(t, x) - \epsilon C(1 + |y|^{\gamma - \frac{1}{\theta R}}) > -\infty. \]

Hence it exists

\[
\liminf_{\epsilon \to 0, (t', x') \to (t, x)} \inf_{y \in \mathbb{R}^m} (V^\epsilon - \psi^\epsilon)(t', x', y) > -\infty.
\]

Since \( \psi^\epsilon \) is bounded by below, we can conclude that

\[
\liminf_{\epsilon \to 0, (t', x') \to (t, x)} \inf_{y \in \mathbb{R}^m} (V^\epsilon - \psi^\epsilon)(t', x', y) = (V - \psi)(t, x).
\]

But \((\bar{t}, \bar{x})\) is a strict minimum point of \( V - \psi \) so the above relaxed lower limit is \( > 0 \) on \( \partial Q_R \). Hence we can find

\[ \zeta > 0 : V^\epsilon - \zeta \geq \psi^\epsilon \) on \( \partial Q_R \) for \( \epsilon \) small. \]

Claim \( V^\epsilon - \zeta \geq \psi^\epsilon \) in \( Q_R \) and **Contradiction** on the black board.
\(\tilde{V}_R\) is a sub solution of a perturbed PDE

\[
\begin{aligned}
\begin{cases}
-\frac{\partial V}{\partial t} + \overline{H}(x, D_x V, D_{xx} V) - \frac{1}{R} = 0 & \text{in } (0, T) \times \mathbb{R}^n \\
V(T, x) = g(x) & \text{in } \mathbb{R}^n
\end{cases}
\end{aligned}
\]
$
abla R$ is a sub solution of a perturbed PDE

**Tools**
- Perturbed test function method
- Approximation of (EP) by state constraints

\[ S(y) = \sup_{|x-x|,|t-t|< \frac{1}{R}} f_{t,x}(y) \]

- Sub quadratic case ($1 < \theta \leq 2$)
  \[ \begin{cases} 
  \lambda_R - \frac{1}{2} \Delta_y \chi_R + \frac{1}{\theta} |D_y \chi_R|^\theta = S(y) & \text{in } B_R(0), \\
  \chi_R \to +\infty & \text{as } y \to \partial B_R(0), \\
  \chi_R(0) = 0. 
  \end{cases} \]

- Super quadratic ($\theta > 2$)
  \[ \begin{cases} 
  \lambda_R - \frac{1}{2} \Delta_y \chi_R + \frac{1}{\theta} |D_y \chi_R|^\theta = S(y) & \text{in } B_R(0), \\
  \lambda_R - \frac{1}{2} \Delta_y \chi_R + \frac{1}{\theta} |D_y \chi_R|^\theta \geq S(y) & \text{on } \partial B_R(0), \\
  \chi_R(0) = 0. 
  \end{cases} \]
$V_R$ is a sub solution of a perturbed PDE

- **Comparison Results** for such problems (T. Tchamba’s Phd thesis [super quadratic case] + [Barles-Da Lio, 2006] [sub quadratic case])

- $H$ is the minimum of the constants for which $(EP)$ has a solution $\phi \in C^2(\mathbb{R}^m)$
sup \( \bar{V}_R \) is a sub solution of the limit PDE

\[ \bar{V}_R \text{ is a sub solution of } \]
\[ \begin{cases}
- \frac{\partial V}{\partial t} + \overline{H}(x, D_x V, D_{xx}^2 V) - \frac{1}{R} = 0 & \text{in } (0, T) \times \mathbb{R}^n \\
V(T, x) = g(x) & \text{in } \mathbb{R}^n
\end{cases} \]

for all \( R \) large enough

Since the supremum of sub solutions is a sub solution, we see that sup\( \bar{V}_R \) is a sub solution of the limit PDE

\[ \begin{cases}
- \frac{\partial V}{\partial t} + \overline{H}(x, D_x V, D_{xx}^2 V) = 0 & \text{in } (0, T) \times \mathbb{R}^n \\
V(T, x) = g(x) & \text{in } \mathbb{R}^n
\end{cases} \]
\[
\sup_R \bar{V}_R \text{ is a sub solution of the limit PDE}
\]

\[
\bar{V}_R \text{ is a sub solution of}
\begin{cases}
-\frac{\partial V}{\partial t} + \overline{H}(x, D_x V, D_{xx}^2 V) - \frac{1}{R} = 0 \quad \text{in} \quad (0, T) \times \mathbb{R}^n \\
V(T, x) = g(x) \quad \text{in} \quad \mathbb{R}^n
\end{cases}
\]

for all \( R \) large enough

Since the supremum of sub solutions is a sub solution, we see that \( \sup_R \bar{V}_R \) is a sub solution of the limit PDE

\[
\begin{cases}
-\frac{\partial V}{\partial t} + \overline{H}(x, D_x V, D_{xx}^2 V) = 0 \quad \text{in} \quad (0, T) \times \mathbb{R}^n \\
V(T, x) = g(x) \quad \text{in} \quad \mathbb{R}^n
\end{cases}
\]
Comparison principle
between sub and super solution to parabolic problems satisfying

\[ |V(t, x)| \leq K(1 + |x|^{\alpha}) \]

(see [Da Lio - Ley 2011]), gives

- uniqueness of solution \( V \) of CP,
- \( \underline{V}(t, x) \geq \sup_{R} \bar{V}_{R}(t, x) \), then \( \underline{V} = \sup_{R} \bar{V}_{R} = V \) and, as \( \epsilon \to 0 \),

\[ V^\epsilon(t, x, y) \to V(t, x) \quad \text{locally uniformly.} \]
Work in progress and close perspectives

- associate to the limit PDE a "limit control problem";
- list examples;
- re-obtain and extend Naoyuki Ichihara results using only PDE methods (partially done);
- simpler formula "(F)"?
Work in progress and close perspectives

- associate to the limit PDE a "limit control problem";
- list examples;
- re-obtain and extend Naoyuki Ichihara results using only PDE methods (partially done);
- simpler formula "(F)"?
Work in progress and close perspectives

- associate to the limit PDE a "limit control problem";
- list examples;
- re-obtain and extend Naoyuki Ichihara results using only PDE methods (partially done);
- simpler formula "(F)"?
Work in progress and close perspectives

- associate to the limit PDE a "limit control problem";
- list examples;
- re-obtain and extend Naoyuki Ichihara results using only PDE methods (partially done);
- simpler formula "(F)"?
Grazie !
"Sadko in the Underwater Kingdom" by Ilya Repin