Asymptotic behavior of singularly perturbed control system: non-periodic setting

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Overview

1. Introduction
2. Periodic setting: Results of Alvarez & Bardi 2003, 2010
3. Non-periodic setting: Some new results

1. Introduction

Singularly perturbed control system:

$$\begin{cases} X'(s) = f\big(X(s), Y(s), \alpha(s)\big), \;\; s > 0 \\ Y'(s) = \dfrac{1}{\varepsilon} g\big(X(s), Y(s), \alpha(s)\big), \;\; s > 0 \\ X(0) = x, \;\; Y(0) = y. \end{cases} \qquad (\mathrm{S}_\varepsilon)$$

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- $\alpha : [0, +\infty) \to A$ measurable function (control), $A \subset \mathbb{R}^d$ be a compact subset;
- $f : \mathbb{R}^n \times \mathbb{R}^m \times A \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^m \times A \to \mathbb{R}^m$ are the dynamics;
- $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ initial position, $\varepsilon > 0$ small parameter;

- trajectory:
$\big(X^\varepsilon(s), X^\varepsilon(s)\big) \equiv \big(X^\varepsilon(s; x, y, \alpha), X^\varepsilon(s; x, y, \alpha)\big)$,
$X^\varepsilon(\cdot)$ is slow trajectory, $Y^\varepsilon(\cdot)$ is fast trajectory.

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  $X^\varepsilon(\cdot)$ is slow trajectory, $Y^\varepsilon(\cdot)$ is fast trajectory.

Value function: for $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$, $t > 0$

$$v^\varepsilon(x, y, t) := \inf_\alpha \Big\{ \int_0^t \ell\big(X^\varepsilon(s), Y^\varepsilon(s), \alpha(s)\big)\, ds + h\big(X^\varepsilon(t)\big) \Big\}$$

- $\ell : \mathbb{R}^n \times \mathbb{R}^m \times A \to \mathbb{R}$ is running cost;
- $h : \mathbb{R}^n \to \mathbb{R}$ is final cost (or terminal cost).

Approaches for singular perturbation problem

Goal. Asymptotic analysis when ε goes to zero ?

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Dynamical system approach:

Asymptotic behavior of trajectory $\big(X^\varepsilon(\cdot), Y^\varepsilon(\cdot)\big)$, as $\varepsilon \to 0$.

- Order reduction method:
  - for ODE [Levinson & Tichonov];
  - for control system [Kokotovíc, Khalil, O'Reilly, Bensoussan, Dontchev, Zolezzi, Veliov, ...];
- Averaging method (limit occupational measures):
  - Artstein, Gaitsgory, Leizarowitz and orthers.

PDE approach: [Alvarez & Bardi, 2001, 2003, 2010]

Asymptotic behavior of value function v^ε, as $\varepsilon \to 0$?

Under suitable assumptions, v^ε is the unique viscosity solution of HJB equation

$$\begin{cases} v_t^\varepsilon + H\left(x, y, D_x v^\varepsilon, \frac{D_y v^\varepsilon}{\varepsilon}\right) = 0 & \text{in } \mathbb{R}^n \times \mathbb{R}^m \times (0, +\infty), \\ v^\varepsilon(x, y, 0) = h(x) & \text{in } \mathbb{R}^n \times \mathbb{R}^m, \end{cases}$$

where

$$H(x, y, p, q) = \max_{a \in A} \left\{ -p \cdot f(x, y, a) - q \cdot g(x, y, a) - \ell(x, y, a) \right\}.$$

PDE approach aims at characterizing limit of v^ε as viscosity (sub-super) solutions of appropriate effective HJB equation

$$\begin{cases} v_t + \overline{H}(x, Dv) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty), \\ v(x, 0) = h(x) & \text{in } \mathbb{R}^n. \end{cases} \tag{1}$$

$\rightarrow$ In this talk, we will follow the PDE approach.

2. Periodic setting: [Alvarez & Bardi 2003, 2010]

Standing assumptions:

- Standard assumptions on datum: f, g, ℓ, h bounded uniformly continuous; f, g Lipschitz continuous in state variables, uniformly in control variable;

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- Standard assumptions on datum: f, g, ℓ, h bounded uniformly continuous; f, g Lipschitz continuous in state variables, uniformly in control variable;
- Periodicity: f, g, ℓ are $\mathbb{Z}^m$-periodic in the fast variable;
- Bounded time controllability on the fast system

$$\begin{cases} Y'(\tau) = g\big(x, Y(\tau), \alpha(\tau)\big), & \tau > 0 \\ Y(0) = y, & x \text{ is fixed in } \mathbb{R}^n. \end{cases} \tag{FS}$$

for fixed $x \in \mathbb{R}^n$, for any $y, z \in \mathbb{R}^m$, there exist $T > 0$ and $\alpha \in \mathcal{A}$ such that

$$Y_y(\tau; \alpha, x) = z \quad \text{for some } \tau \leq T.$$

Main results: periodic setting

Cell problem: For each $(x_0, p_0) \in \mathbb{R}^n \times \mathbb{R}^n$, consider the family of equations, for a constant $\lambda \in \mathbb{R}$,

$$H\big(x_0, y, p_0, Du(y)\big) = \lambda \quad \text{in } \mathbb{R}^m. \tag{CP}$$

Find $\lambda \in \mathbb{R}$ such that (CP) has periodic sub- and supersolutions.

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Theorem 1 [Alvarez & Bardi 2010]

For each $(x_0, p_0) \in \mathbb{R}^n \times \mathbb{R}^n$, there exists a unique real value $c_0 = c_0(x_0, p_0)$ such that the cell problem (CP), with $\lambda = c_0$, admits a periodic subsolution and a periodic supersolution. The effective Hamiltonian $\overline{H}$ is then defined by setting $\overline{H}(x_0, p_0) = c_0.$

Remark 1. (Correctors)

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Remark 2. (Connection to ergodic control problem)

$$\lim_{\delta \to 0} \delta u_\delta(y; x_0, p_0) = \lim_{t \to +\infty} \frac{w(y, t; x_0, p_0)}{t} = -\overline{H}(x_0, p_0)$$

uniformly in $y \in \mathbb{R}^m$.

where

- $u_\delta(y; x_0, p_0)$ is the unique solution to the equation

$$\delta u_\delta + H\big(x_0, y, p_0, Du_\delta(y)\big) = 0 \quad \text{in } \mathbb{R}^m, \quad u_\delta \text{ periodic},$$

- $w(y, t; x_0, p_0)$ is the unique solution to the problem

$$\begin{cases} w_t + H(x_0, y, p_0, D_y w) = 0, \ (y, t) \in \mathbb{R}^m \times (0, +\infty), \\ w(y, 0) = 0, \quad w \text{ periodic in } y, \quad y \in \mathbb{R}^m. \end{cases}$$

Weak semilimits

Under standing assumptions, $\{v^\varepsilon\}$, $\varepsilon > 0$, is equibounded in $\mathbb{R}^n \times \mathbb{R}^m \times [0, +\infty)$, hence we can define upper semilimit and lower semilimit of v^ε, as $\varepsilon \to 0$, respectively, by

$$v^*(x,t) := \limsup_{\varepsilon \to 0,\,(x',t') \to (x,t)} \sup_{y \in \mathbb{R}^m} v^\varepsilon(x', y, t'),$$

$$v_*(x,t) := \liminf_{\varepsilon \to 0,\,(x',t') \to (x,t)} \inf_{y \in \mathbb{R}^m} v^\varepsilon(x', y, t').$$

Note that, $v^*(x,t)$ and $v_*(x,t)$ are, respectively, bounded upper semicontinuous and bounded lower semicontinuous in $\mathbb{R}^n \times [0, +\infty)$.

Convergence result

Theorem 2 [Alvarez & Bardi 2010]

The weak semilimits v^* and v_* are, respectively, subsolution and supersolution of the effective problem

$$\begin{cases} v_t + \overline{H}(x, Dv) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty) \\ v(x,0) = h(x) & \text{in } \mathbb{R}^n. \end{cases} \qquad (\overline{\text{HJ}})$$

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In addition, if $(\overline{\mathrm{HJ}})$ satisfies comparison principle, then v^ε locally uniformly converges on $\mathbb{R}^n \times \mathbb{R}^m \times [0, +\infty)$, as $\varepsilon \to 0$, to the unique solution $v = v^* = v_*$ of $(\overline{\mathrm{HJ}})$.

Key ideas and methods (periodic case)

- Viscosity solution theory;
- Periodic homogenization [Lions, Papanicolau, Varadhan 1986];
- Perturbed test function method [Evans 1989, 1992].

3. Non-periodic setting

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- Coercivity: ℓ is coercive in the fast variable, i.e.,

$$\min_{a \in A} \ell(x, y, a) \to +\infty, \text{ as } |y| \to +\infty, \text{ uniformly in } x;$$

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$$\min_{a \in A} \ell(x, y, a) \to +\infty, \text{ as } |y| \to +\infty, \text{ uniformly in } x;$$

- Local bounded time controllability on the fast system: Given compact subsets $C \subset \mathbb{R}^n$, $D \subset \mathbb{R}^m$. For fixed $x \in C$, for any $y, z \in D$, there exist $T = T(C, D) > 0$ and $\alpha \in \mathcal{A}$ such that

$$Y_y(\tau; \alpha, x) = z \quad \text{for some } \tau \leq T.$$

Critical value

For each $(x_0, p_0) \in \mathbb{R}^n \times \mathbb{R}^n$, consider the family of equations, for a constant $\lambda \in \mathbb{R}$,

$$H\big(x_0, y, p_0, Du(y)\big) = \lambda \quad \text{in } \mathbb{R}^m. \tag{2}$$

Remark. There is no hope to have periodic or even bounded sub- and supersolutions to (2) for some distinguished value λ.

Theorem 1

For each $(x_0, p_0) \in \mathbb{R}^n \times \mathbb{R}^m$, there exits a unique real value $c_0 = c_0(x_0, p_0)$ such that the equation

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Key ideas and methods (non-periodic case)

- Approximation by coercive, convex Hamiltonians;
- Some tools from Weak KAM theory [Fathi & Siconolfi 2005].

For each $(x_0, p_0) \in \mathbb{R}^n \times \mathbb{R}^n$, define

$$\overline{H}(x_0, p_0) := \inf \left\{ \lambda \in \mathbb{R} \,:\, (2) \text{ has a bounded subsolution} \right\}$$

We let

$$H_0(y, q) := H(x_0, y, p_0, q).$$

To approximate H_0 by a sequence of coercive, convex Hamiltonians H_k, $k \in \mathbb{N}^*$, and consider a sequence of equations

$$H_k(y, Du) = \lambda \quad \text{in } \mathbb{R}^m. \tag{3}$$

Define the critical value for H_k

$$c_k := \inf \left\{ \lambda \in \mathbb{R} \,:\, (3) \text{ has a subsolution} \right\}.$$

Claim 1. The c_k is finite for any k, and make up a non-increasing bounded sequence, and hence has a limit

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Consider the critical equation

$$H_k(y, Du) = c_k \quad \text{in } \mathbb{R}^m. \tag{HJ$_\mathrm{k}$}$$

Claim 2. There exist a sequence of equibounded subsolutions u_k and a sequence of locally equibounded, equicoercive solutions w_k to $(\mathrm{HJ_k})$.

(Using semidistances S_k and Aubry set $\mathbf{A}_k$ for H_k)

Claim 3. We set

$$u(y) = \limsup{}^{\#} u_k(y) := \sup\left\{ \limsup_{k\to+\infty} u_k(y_k) : y_k \to y \right\},$$

$$w(y) = \liminf{}_{\#} w_k(y) := \inf\left\{ \liminf_{k\to+\infty} w_k(y_k) : y_k \to y \right\},$$

then u and w are, respectively, bounded subsolution and locally bounded coercive supersolution to the equation

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Claim 4. $c_0 = \overline{H}(x_0, p_0)$.

Weak semilimits

The value functions v^ε, $\varepsilon > 0$, is locally equibounded, therefore we can define the weak sup-semilimit and weak inf-semilimit of v^ε, respectively, by

- $\limsup^{\#} v^\varepsilon(x, y, t)$
  $= \sup \left\{ \limsup_{\varepsilon \to 0} v^\varepsilon(x_\varepsilon, y_\varepsilon, t_\varepsilon) : (x_\varepsilon, y_\varepsilon, t_\varepsilon) \to (x, y, t) \right\}$
- $\liminf_{\#} v^\varepsilon(x, y, t)$
  $= \inf \left\{ \liminf_{\varepsilon \to 0} v^\varepsilon(x_\varepsilon, y_\varepsilon, t_\varepsilon) : (x_\varepsilon, y_\varepsilon, t_\varepsilon) \to (x, y, t) \right\}$.

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  $= \inf \left\{ \liminf_{\varepsilon \to 0} v^\varepsilon(x_\varepsilon, y_\varepsilon, t_\varepsilon) : (x_\varepsilon, y_\varepsilon, t_\varepsilon) \to (x, y, t) \right\}.$

Claim. $\overline{v} := \limsup{}^{\#} v^\varepsilon$ and $\underline{v} := \liminf{}_{\#} v^\varepsilon$ are, respectively, u.s.c and l.s.c, and moreover they are independent of the fast variable y.

Convergence result

Theorem 2

The weak sup-semilimit $\overline{v}$ and weak inf-semilimit $\underline{v}$ are, respectively, subsolution and supersolution of the effective Cauchy problem

$$\begin{cases} v_t + \overline{H}(x, Dv) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty) \\ v(x,0) = h(x) & \text{in } \mathbb{R}^n. \end{cases} \tag{4}$$

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In addition, if (4) satisfies comparison principle, v^ε locally uniformly converges on $\mathbb{R}^n \times \mathbb{R}^m \times [0, +\infty)$, as $\varepsilon \to 0$, to the solution $v = \overline{v} = \underline{v}$ of (4).

Work is in progress ...

- Existence of continuous viscosity solution to the critical equation

$$H\big(x_0, y, p_0, Du(y)\big) = \overline{H}(x_0, p_0) \quad \text{in } \mathbb{R}^m.$$

- Connection to ergodic control problem.

- Representation formula for $\overline{H}$ and the explicit form of limit optimal control problem via limit occupational measures.

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Thank you for your attention !