Regularity of the Hamiltonian along the Optimal Solution

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New Perspectives in Optimal Control and Games
(Joint work with R. B. Vinter)
Outline of the Talk

- Introduction.
- **Main Results** (with sketch of the proof),
- **Application 1**: Regularity of Minimizers
- **Application 2**: Nondegeneracy Conditions
- Concluding Comments
Problem Formulation

Optimal Control Problem with state constraint:

\[
(P) \quad \begin{cases} 
\text{Minimize } g(x(T)) \\
\text{over } x \in W^{1,1}([S, T]; \mathbb{R}^n) \\
\dot{x}(t) \in F(t, x(t)) \text{ a.e.} \\
x(S) = x_0, \quad x(T) \in C \\
h(x(t)) \leq 0, \quad t \in [S, T]
\end{cases}
\]

Data: \( g : \mathbb{R}^n \to \mathbb{R} \) (Lipschitz), \( F : [S, T] \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n \), \( C \subset \mathbb{R}^n \) (closed), \( h : \mathbb{R}^n \to \mathbb{R} \) (differentiable).

Solutions to \( \dot{x}(t) \in F(t, x(t)) \) called \( F\)-trajectories.

Hamiltonian: \( H(t, p, x) = \max_{v \in F(t, x)} v \cdot p \)
Standard Hypotheses (SH)

(H1): $F(.,.)$ takes values closed convex sets and $F(.,x)$ is measurable for each $x$;

(H2): There exist $k(.,.) \in L^1$, and $\varepsilon > 0$ s.t.:

a) $F(t,x) \subset F(t,x') + k(t)|x - x'|B$

b) $F(t,x) \subset c(t)B$

for all $x, x' \in \bar{x}(t) + \varepsilon B$, a.e. $t \in [0, 1]$.

(for $F$-trajectory $\bar{x}(.)$ of interest).
Hamiltonian Inclusion N. C.

Take $\bar{x}(.)$ \textit{L}^\infty -\textbf{local minimizer} for $(P)$.

There exist $\lambda \geq 0$, $p(. ) \in W^{1,1}$ and a measure $\mu$ s.t.:

- $\lambda + \|p(.)\|_{L^\infty} + \|\mu\|_{T.V.} = 1$, \textbf{supp}(\mu) $\subset \{t : h(\bar{x}(t)) = 0\}$

- $(-\dot{p}(t), \dot{\bar{x}}(t)) \in \text{co} \partial_{x,p} H(t, \bar{x}(t), q(t))$,

- $-q(T) \in \lambda \partial g(\bar{x}(T)) + NC(\bar{x}(T))$,

where

$$q(t) = \begin{cases} 
  p(t) + \int_{[S,t]} \nabla h(\bar{x}(s)) \mu(ds) & \text{if } t \in [S,T) \\
  p(T) + \int_{[S,T]} \nabla h(\bar{x}(s)) \mu(ds) & \text{if } t = T
\end{cases}.$$
.... plus

1) If $F(t,x) \equiv F(x)$ (autonomous problem):
   \[
   H(t, \bar{x}(t), q(t)) = c, \text{ for } t \in (S, T) \text{ (open)}
   \]

2) If $t \mapsto F(t,x)$ Lipschitz continuous:
   \[
   t \mapsto H(t, \bar{x}(t), q(t)) \text{ Lipschitz on } (S, T) \text{ (open)}
   \]

Not obvious because $t \mapsto q(t)$ is not continuous!!
Refined necessary conditions

Theorem: (Arutyunov-Aseev ’94)

Assume (SH) and \( t \mapsto F(t, x) \) Lipschitz. For \( \bar{x}(.) \) minimizer, then

\[
 t \mapsto H[t] := H(t, \bar{x}(t), q(t))
\]

is Lipschitz on the closed interval \([S, T]\).

Again,

\[
 q(t) = \begin{cases} 
 p(t) + \int_{[S,t]} \nabla h(\bar{x}(s))\mu(ds) & \text{if } t \in [S, T) \\
 p(T) + \int_{[S,T]} \nabla h(\bar{x}(s))\mu(ds) & \text{if } t = T.
\end{cases}
\]
When do $H(.,.,.)$ regularity issue arise?

1) higher order analysis of optimal control problems. 
   *(singular optimal solutions).*

2) non degeneracy for first order necessary conditions.

3) regularity properties of minimizers.

We will subsequently concentrate on 2) and 3).
Questions

We know:

- if $F(t, x) \equiv F(x) \implies H(t, \bar{x}(t), q(t))$ is constant.

- if $F(., x)$ Lipsch. $\implies t \mapsto H(t, \bar{x}(t), q(t))$ is Lipsch.

Question:

If $t \mapsto F(t, x)$ contin. $\implies t \mapsto H(t, \bar{x}(t), q(t))$ is contin.?

We answer to a related question...
B.V. Multifunctions

t ↦ F(t, x) is bounded variation (B.V.), uniformly along ¯x(.) if

\[ \eta(T) < \infty, \]

where

\[ \eta(t) := \sup_{\tau} \left\{ \sum_{i=0}^{N-1} \sup_{x} d_H(F(t_{i+1}, x), F(t_i, x)) : x \in G \right\}, \]

sup. over partitions \( \tau = \{t_0 = S, ..., t_N = t\} \) of \([S, t]\),

\[ G = \{x : x \in \bar{x}([S, T])\} \]

t ↦ \( \eta(t) \) is denoted as cumulative variation function.
Properties of B.V. Multifunctions

Take $F : [S, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ B.V. multifunction.

1) $\eta(.)$ cumulative variation function of $F(.,.)$. Then

$$d_H(F(t, x), F(s, x)) \leq \eta(t) - \eta(s)$$

for all $[s, t] \subset [S, T]$, $x \in G$.

2) Take $s \in [S, T)$ and $t \in (S, T]$; then

$$\lim_{s' \rightarrow s^+} d_H(F(s', x), F(s, x)) = 0, \quad \lim_{t' \rightarrow t^-} d_H(F(t', x), F(t, x)) = 0$$

(there exist limits from the left and from the right!)

3) There exists a countable set $A$ such that

$$\lim_{t' \rightarrow t} d_H(F(t', x), F(t, x)) = 0 \quad \forall t \in (S, T) \setminus A, \ x \in G.$$
Main Result

Theorem: (Palladino, Vinter)

Take $\bar{x}(.)$ minimizer for $(P)$. Assume hypotheses $(SH)$ and $t \mapsto F(t,x)$ is B.V.

Then the multipliers $(\lambda, p(\cdot), \mu)$ can be chosen to satisfy the additional condition:

$$|H(t, \bar{x}(t), q(t)) - H(s, \bar{x}(s), q(s))| \leq |q(\cdot)|_{L^\infty} \times (\eta(t) - \eta(s))$$

and $H[.]$ is right and left continuous at the respective endpoints.
Sketch of the Proof

1) Discrete Approximation

Take a partition \( \tau = \{t_0 = S, \ldots, t_N = T\} \)

\[
(P_N) \begin{cases} 
\text{Minimize } g(x(T)) dt \\
\dot{x}(t) \in \sum_{j=0}^{N-1} F(t_j, x(t)) \chi_{[t_j, t_{j+1}]}(t) \quad t \in [S, T], \\
h(x(t)) \leq 0, \quad \text{for all } t \in [S, T] \\
x(S) = x_0, \quad x(T) \in C,
\end{cases}
\]

**Convexity** implies existence of a minimizer \( x_N(.) \).

By Filippov Theorem and Compactness Arguments,

\[ x_N(.) \to \bar{x}(.) \quad \text{uniformly.} \]
Sketch of the Proof (Continued)

2) Use of the Multistage Necessary Conditions.

$H[.]$ is piecewise constant.

Jumps:

\[
\Delta_j = H(t_j^+, x_N(t_j), q(t_j)) - H(t_j^-, x_N(t_j), q(t_j)) \quad j = 1, \ldots, N-1
\]

and

\[
|\Delta_j| \leq K(\eta(t_j^+) - \eta(t_j^-)) \quad j = 1, \ldots, N-1
\]

Jumps are controlled by the cumulative variation function!
Application 1: Regularity

(CV) \[ \left\{ \begin{array}{l}
\text{Minimize } \int_{S}^{T} L(t, x(t), \dot{x}(t)) dt \\
\text{over } x(.) \in W^{1,1}([S, T]) \text{ s.t.} \\
x(S) = x_0 \text{ and, } x(T) = x_1.
\end{array} \right. \]

(CV) admits a minimizer \( \bar{x}(.) \) if:

(HE) (i): \( L(.,.,.) \) is \( \mathcal{L} \times \mathcal{B}^{n \times n} \) measurable, and \( L(t,.,.) \) is lower semicontinuous for each \( t \in [S, T] \).

(ii): \( L(t, x, .) \) is convex for each \( (t, x) \in \mathbb{R}^n \).

(iii): There exist \( \theta(.) : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) convex s.t. \( \lim_{r \uparrow \infty} \theta(r)/r = \infty \), and \( \alpha > 0 \) s.t. 
\[ L(t, x, v) \geq \theta(|v|) - \alpha |x| \quad \forall \ (t, x, v) \in [S, T] \times \mathbb{R}^n \times \mathbb{R}^n \]
Ball-Mizel Example

\[(Q) \quad \left\{ \begin{array}{l}
\text{Minimize } \int_0^1 \{r x^2(t) + (x^3(t) - t^2)^2 x^{14}(t)\} dt \\
\text{over } x(.) \in W^{1,1}(\mathbb{R}) \quad \text{s.t.} \\
x(0) = 0 \text{ and } x(1) = \bar{k}.
\end{array} \right.
\]

where \( r > 0 \) and \( \bar{k} > 0 \) are constants such that

\[
r = (2\bar{k}/3)^{12}(1 - \bar{k}^3)(13\bar{k}^3 - 7).
\]

There exists \( \varepsilon > 0 \) s.t., for all \( \bar{k} \in (1 - \varepsilon, 1) \),

\[
\bar{x}(t) = \bar{k}t^{2/3}
\]

is the unique (non-Lipschitz) minimizer for \((Q)\).
Corollary: Take $\bar{x}(.)$ minimizer. Assume $\text{(HE)}$ and

- $(x,v) \mapsto L(t,x,v)$ is loc. Lipschitz a.e. $t \in [S,T]$.

- $t \mapsto L(t,x,v)$ is $\text{B.V.}$ uniformly along $(\bar{x}(.),\dot{\bar{x}}(.))$.

Then $\bar{x}(.)$ is a Lipschitz minimizer.

(Replace Lipschitz continuity of $t \mapsto L(t,x,v)$ by $\text{B.V.}$ assumption.)
Application 2: Non degeneracy

Consider the optimal control problem

\[
(ND) \quad \begin{cases}
\text{Minimize } g(x(T)) \\
\text{over } x \in W^{1,1}([S, T]; \mathbb{R}^n) \\
\dot{x}(t) \in U(t) \text{ a.e.} \\
x(S) = x_0, \\
h(x(t)) \leq 0, \quad t \in [S, T]
\end{cases}
\]

Take any feasible trajectory \( \bar{x}(.) \) s.t. \( h(\bar{x}(S)) = 0 \). Then

\[
p(t) \equiv -\nabla h(\bar{x}(S)), \; \mu(t) = \delta_{\{S\}}(t) \text{ and } \lambda = 0,
\]

satisfies necessary conditions.

(Trivial Multiplier!)
Application 2: (Continued)

If the data is $\mathbf{B.V.}$, then

$$|H(t, \bar{x}(t), q(t)) - H(s, \bar{x}(s), q(s))| \leq K \times (\eta(t) - \eta(s))$$
on every $[s, t] \subset [S, T]$ and $H[.]$ is right and left continuous at the endpoints.

Such strenghtened conditions give existence of non trivial multipliers.

Extend (Arutyunov-Aseev) result: replace $t \mapsto F(t, x)$ Lipschitz by $t \mapsto F(t, x)$ $\mathbf{B.V.}$
**Application 2: (Continued)**

**Corollary:** Take $\bar{x}(.)$ minimizer. Assume $(\text{SH})$, $F(.,x)\text{ B.V.}$ and

(I): There exists $v \in \lim \inf_{s \downarrow S} F(s,x_0)$ such that

$$\nabla h(x_0) \cdot v < 0 .$$

Then the strengthened non triviality condition holds true:

$$\lambda + \int_{(S,T]} \mu(ds)) \neq 0$$

*(Rule out trivial multipliers!)*
Concluding Remarks

This talk has showed that $H[.]$ inherits the same regularity of $F(.,x)$. We know:

- if $F(t,x) \equiv F(x) \implies t \mapsto H(\tilde{x}(t), q(t))$ is constant.
- if $F(.,x) \text{ Lipsch.} \implies t \mapsto H(t, \tilde{x}(t), q(t))$ is Lipsch.
- if $F(.,x) \text{ B.V.} \implies t \mapsto H(t, \tilde{x}(t), q(t))$ is B.V. (New!).

If $t \mapsto F(t,x) \text{ contin.} \implies t \mapsto H(t, \tilde{x}(t), q(t))$ is contin.?

Open Question!