# Regularity of the Hamiltonian along the Optimal Solution

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Roma, 11 November 2014

New Perspectives in Optimal Control and Games

(Joint work with R. B. Vinter)

ITN SADCO
Initial Training Network
Sensitivity Analysis for Deterministic Controller Design

Imperial College London

# Outline of the Talk

- Introduction.

- Main Results (with sketch of the proof),

- **Application 1**: Regularity of Minimizers

- **Application 2**: Nondegeneracy Conditions

- Concluding Comments

# Problem Formulation

Optimal Control Problem with state constraint:

$$(P) \quad \begin{cases} \text{Minimize } g(x(T)) \\ \text{over } x \in W^{1,1}([S,T];\mathbb{R}^n) \\ \dot{x}(t) \in F(t,x(t)) \text{ a.e.} \\ x(S) = x_0, \quad x(T) \in C \\ h(x(t)) \leq 0, \quad t \in [S,T] \end{cases}$$

Data: $g : \mathbb{R}^n \to \mathbb{R}$ (Lipschitz), $F : [S,T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$, $C \subset \mathbb{R}^n$ (closed), $h : \mathbb{R}^n \to \mathbb{R}$ (differentiable).

Solutions to $\dot{x}(t) \in F(t,x(t))$ called *$F$-trajectories.*

Hamiltonian: $H(t,p,x) = \max_{v \in F(t,x)} v \cdot p$

# Standard Hypotheses (SH)

$(H1)$ : $F(.,.)$ takes values closed **convex** sets and $F(.,x)$ is measurable for each $x$;

$(H2)$ : There exist $k(.), c(.) \in L^1$, and $\varepsilon > 0$ s.t.:

$$a) \;\; F(t,x) \subset F(t,x') + k(t)|x - x'|B$$

$$b) \;\; F(t,x) \subset c(t)B$$

for all $x, x' \in \bar{x}(t) + \varepsilon B$, a.e. $t \in [0,1]$.

(for $F$-trajectory $\bar{x}(.)$ of interest).

# Hamiltonian Inclusion N. C.

Take $\bar{x}(.)$ $L^\infty$-**local minimizer** for $(P)$.
There exist $\lambda \geq 0$, $p(.) \in W^{1,1}$ and a measure $\mu$ s.t.:

- $\lambda + ||p(.)||_{L^\infty} + ||\mu||_{T.V.} = 1$, $\quad$ $\text{supp}(\mu) \subset \{t : h(\bar{x}(t)) = 0\}$
- $(-\dot{p}(t), \dot{\bar{x}}(t)) \in \text{co}\, \partial_{x,p} H(t, \bar{x}(t), q(t))$,
- $-q(T) \in \lambda\, \partial g(\bar{x}(T)) + N_C(\bar{x}(T))$,

where

$$q(t) = \begin{cases} p(t) + \int_{[S,t)} \nabla h(\bar{x}(s))\mu(ds) & \text{if } t \in [S,T) \\ p(T) + \int_{[S,T]} \nabla h(\bar{x}(s))\mu(ds) & \text{if } t = T\,. \end{cases}$$

## .... plus

**1)** If $F(t,x) \equiv F(x)$ (autonomous problem):

- $H(t, \bar{x}(t), q(t)) = c$, for $t \in (S,T)$ **(open)**

**2)** If $t \mapsto F(t,x)$ Lipschitz continuous:

- $t \mapsto H(t, \bar{x}(t), q(t))$ Lipschitz on $(S,T)$ **(open)**

Not obvious because $t \mapsto q(t)$ is **not** continuous!!

# Refined necessary conditions

**Theorem: (Arutyunov-Aseev '94)**

Assume **(SH)** and $t \mapsto F(t,x)$ Lipschitz. For $\bar{x}(.)$ minimizer, then

$$t \mapsto H[t] := H(t, \bar{x}(t), q(t))$$

is Lipschitz on the **closed** interval $[S,T]$.

Again,

$$q(t) = \begin{cases} p(t) + \int_{[S,t)} \nabla h(\bar{x}(s))\mu(ds) & \text{if } t \in [S,T) \\ p(T) + \int_{[S,T]} \nabla h(\bar{x}(s))\mu(ds) & \text{if } t = T\,. \end{cases}$$

# When do $H(.,.,.)$ regularity issue arise?

1) higher order analysis of optimal control problems. (**singular optimal solutions**).

2) non degeneracy for first order necessary conditions.

3) regularity properties of minimizers.

We will subsequently concentrate on 2) and 3).

# Questions

We know:

- if $F(t,x) \equiv F(x) \Longrightarrow H(t,\bar{x}(t),q(t))$ is constant.

- if $F(.,x)$ Lipsch. $\Longrightarrow t \mapsto H(t,\bar{x}(t),q(t))$ is Lipsch.

**Question**:

If $t \mapsto F(t,x)$ contin. $\Longrightarrow t \mapsto H(t,\bar{x}(t),q(t))$ is contin.?

We answer to a **related question**...

# B.V. Multifunctions

$t \mapsto F(t,x)$ is bounded variation **(B.V.)**, uniformly along $\bar{x}(.)$ if

$$\eta(T) < \infty,$$

where

$$\eta(t) := \sup_{\tau} \{ \sum_{i=0}^{N-1} \sup\{ d_H(F(t_{i+1},x), F(t_i,x)) : x \in G \},$$

sup. over partitions $\tau = \{t_0 = S, ..., t_N = t\}$ of $[S,t]$,

$$G = \{x \ : x \in \bar{x}([S,T])\}$$

$t \mapsto \eta(t)$ is denoted as cummulative variation function.

# Properties of B.V. Multifunctions

Take $F : [S,T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ B.V. multifunction.

**1)** $\eta(.)$ cummulative variation function of $F(.,.)$. Then

$$d_H(F(t,x), F(s,x)) \leq \eta(t) - \eta(s)$$

for all $[s,t] \subset [S,T]$, $x \in G$.

**2)** Take $s \in [S,T)$ and $t \in (S,T]$; then

$$\lim_{s' \to s^+} d_H(F(s',x), F(s,x)) = 0, \qquad \lim_{t' \to t^-} d_H(F(t',x), F(t,x)) = 0$$

(there exist limits from the left and from the right!)

**3)** There exists a countable set $\mathcal{A}$ such that

$$\lim_{t' \to t} d_H(F(t',x), F(t,x)) = 0 \qquad \forall\, t \in (S,T) \setminus \mathcal{A},\ x \in G.$$

# Main Result

**Theorem: (Palladino, Vinter)**

Take $\bar{x}(.)$ minimizer for $(P)$. Assume hypotheses $(SH)$ and $t \mapsto F(t,x)$ is **B.V.**

Then the multipliers $(\lambda, p(.), \mu)$ can be chosen to satisfy the additional condition:

$$|H(t,\bar{x}(t),q(t)) - H(s,\bar{x}(s),q(s))| \leq |q(.)|_{L^\infty} \times (\eta(t) - \eta(s))$$

and $H[.]$ is *right* and *left* **continuous** at the respective **endpoints**.

# Sketch of the Proof

**1)** Discrete Approximation

Take a partition $\tau = \{t_0 = S, \ldots, t_N = T\}$

$$(P_N) \begin{cases} \text{Minimize } g(x(T))dt \\ \dot{x}(t) \in \sum_{j=0}^{N-1} F(t_j, x(t))\chi_{[t_j, t_{j+1}]}(t) \quad t \in [S,T], \\ h(x(t)) \leq 0\,, \quad \text{for all } t \in [S,T] \\ x(S) = x_0, \qquad x(T) \in C\,, \end{cases}$$

**Convexity** implies existence of a minimizer $x_N(.)$.

By **Filippov Theorem** and **Compactness Arguments**,

$$x_N(.) \to \bar{x}(.) \qquad \text{uniformly.}$$

# Sketch of the Proof (Continued)

**2)** Use of the Multistage Necessary Conditions.

$H[.]$ is **piecewise constant**.

Jumps:

$$\Delta_j = H(t_j^+, x_N(t_j), q(t_j)) - H(t_j^-, x_N(t_j), q(t_j)) \quad j = 1, \ldots, N-1$$

and

$$|\Delta_j| \leq K(\eta(t_j^+) - \eta(t_j^-)) \quad j = 1, \ldots, N-1$$

**Jumps** are controlled by the cummulative variation function!

# Application 1: Regularity

$$(CV) \begin{cases} \text{Minimize } \int_S^T L(t, x(t), \dot{x}(t))dt \\ \text{over } x(.) \in W^{1,1}([S,T]) \text{ s.t.} \\ x(S) = x_0 \text{ and, } x(T) = x_1 \ . \end{cases}$$

$(CV)$ admits a minimizer $\bar{x}(.)$ if:

**(HE)** (i): $L(.,.,.)$ is $\mathcal{L} \times \mathcal{B}^{n \times n}$ **measurable**, and $L(t,.,.)$ is **lower semicontinuous** for each $t \in [S,T]$.

(ii): $L(t,x,.)$ is **convex** for each $(t,x) \in \mathbb{R}^n$.

(iii): There exist $\theta(.) : \mathbb{R}^+ \to \mathbb{R}^+$ convex s.t. $\lim_{r \uparrow \infty} \theta(r)/r = \infty$, and $\alpha > 0$ s.t.

$$L(t,x,v) \geq \theta(|v|) - \alpha |x| \quad \forall \ (t,x,v) \in [S,T] \times \mathbb{R}^n \times \mathbb{R}^n$$

# Ball-Mizel Example

$$(Q) \begin{cases} \text{Minimize } \int_0^1 \{r\dot{x}^2(t) + (x^3(t) - t^2)^2 \dot{x}^{14}(t)\} dt \\ \text{over } x(.) \in W^{1,1}([0,1]) \quad \text{ s.t.} \\ x(0) = 0 \text{ and } x(1) = k \ . \end{cases} ,$$

where $r > 0$ and $k > 0$ are constants such that

$$r = (2k/3)^{12}(1 - k^3)(13k^3 - 7).$$

There exists $\varepsilon > 0$ s.t., for all $k \in (1 - \varepsilon, 1)$,

$$\bar{x}(t) = kt^{2/3}$$

is the unique **(non-Lipschitz)** minimizer for $(Q)$.

# Application 1 (Continued)

**Corollary:** Take $\bar{x}(.)$ minimizer. Assume **(HE)** and

- $(x, v) \mapsto L(t, x, v)$ is loc. Lipschitz a.e. $t \in [S, T]$.

- $t \mapsto L(t, x, v)$ is B.V. uniformly along $(\bar{x}(.), \dot{\bar{x}}(.))$.

Then $\bar{x}(.)$ is a **Lipschitz minimizer.**

(Replace **Lipschitz continuity** of $t \mapsto L(t, x, v)$ by **B.V.** assumption.)

# Application 2: Non degeneracy

Consider the optimal control problem

$$
(ND) \quad \begin{cases} \text{Minimize } g(x(T)) \\ \text{over } x \in W^{1,1}([S,T];\mathbb{R}^n) \\ \dot{x}(t) \in U(t) \text{ a.e.} \\ x(S) = x_0, \\ h(x(t)) \leq 0, \qquad t \in [S,T] \end{cases}
$$

Take *any* feasible trajectory $\bar{x}(.)$ s.t. $h(\bar{x}(S)) = 0$. Then

$$
p(t) \equiv -\nabla h(\bar{x}(S)),\ \mu(t) = \delta_{\{S\}}(t) \text{ and } \lambda = 0\,,
$$

satisfies necessary conditions.

(**Trivial Multiplier!**)

# Application 2: (Continued)

If the data is **B.V.**, then

$$|H(t,\bar{x}(t),q(t)) - H(s,\bar{x}(s),q(s))| \leq K \times (\eta(t) - \eta(s))$$

on every $[s,t] \subset [S,T]$ and $H[.]$ is *right* and *left continuous* at the **endpoints.**

Such strenghtened conditions give existence of non trivial multipliers.

**Extend (Arutyunov-Aseev) result:** replace $t \mapsto F(t,x)$ Lipschitz by $t \mapsto F(t,x)$ B.V.

# Application 2: (Continued)

**Corollary:** Take $\bar{x}(.)$ minimizer. Assume **(SH)**, $F(., x)$ **B.V.** and

$(\mathbf{I})$: There exists $v \in \liminf_{s \downarrow S} F(s, x_0)$

such that

$$\nabla h(x_0) \cdot v < 0 \, .$$

Then the strenghtened non triviality condition holds true:

$$\lambda + \int_{(S,T]} \mu(ds)) \neq 0$$

(**Rule out** trivial multipliers!)

# Concluding Remarks

This talk has showed that $H[.]$ inherits the same regularity of $F(.,x)$. We know:

- if $F(t,x) \equiv F(x) \Longrightarrow t \mapsto H(\bar{x}(t),q(t))$ is constant.
- if $F(.,x)$ Lipsch. $\Longrightarrow t \mapsto H(t,\bar{x}(t),q(t))$ is Lipsch.
- if $F(.,x)$ B.V. $\Longrightarrow t \mapsto H(t,\bar{x}(t),q(t))$ is B.V. (**New!**).

If $t \mapsto F(t,x)$ contin. $\Longrightarrow t \mapsto H(t,\bar{x}(t),q(t))$ is contin.?
**Open Question!**