Robustness of Performance & Stability for Multistep & Updated Multistep MPC

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- 3 algorithms: standard, m-step and updated m-step MPC
- Nominal stability and performance of the m-step MPC
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- The m-step and the updated m-step MPC in the perturbed setting

Preliminary Setup

Consider the nonlinear discrete time control system

$$x(k+1) = f(x(k), u(k))$$

- state $x \in \mathbb{X} \subseteq X$ and control value $u \in \mathbb{U} \subseteq U$

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- $x_u(\cdot, x_0)$ – trajectory when initial state x_0 is driven by control sequence $u(\cdot)$
- a time-dependent feedback law $\mu : \mathbb{X} \times \mathbb{N}_0 \to \mathbb{U}$ yields the **feedback controlled system**

$$x(k+1) = f(x(k), \mu(x(\tilde{k}), k))$$

where $\tilde{k} = \tilde{k}(k) \leq k$

Preliminary Setup

- Problem: find μ that 'solves' the **infinite horizon OCP**

$$\begin{aligned} \text{min} \quad & J_\infty(x_0, u(\cdot)) := \sum_{k=0}^{\infty} \ell\left(x_u(k, x_0), u(k)\right) \\ \text{w.r.t.} \quad & u(\cdot) \in \mathbb{U}^\infty(x_0) \end{aligned}$$

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- **optimal value function:**

$$V_\infty(x_0) := \inf_{u(\cdot) \in \mathbb{U}^\infty(x_0)} J_\infty\left(x_0, u(\cdot)\right)$$

- **closed-loop performance of μ:**

$$J_\infty^{\mathsf{cl}}\left(x_0, \mu\right) := \sum_{k=0}^{\infty} \ell\left(x_\mu(k, x_0), \mu(x_\mu(\tilde{k}, x_0), k)\right)$$

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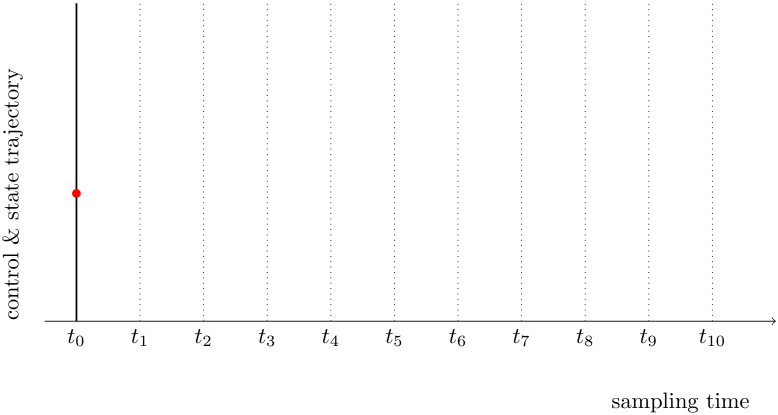
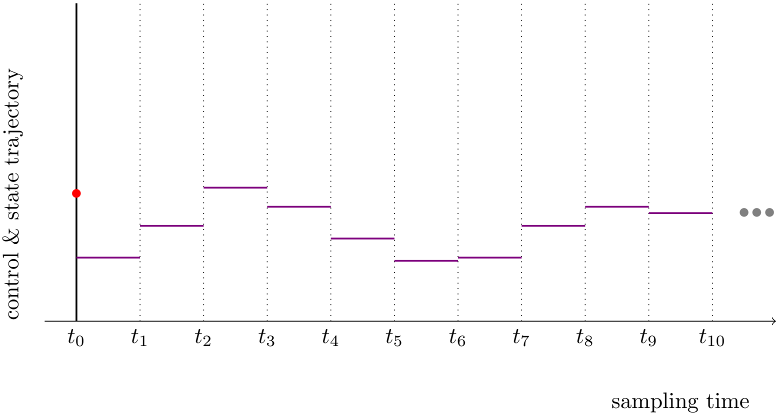
- find μ such that

$$V_\infty(x_0) = J_\infty^{\mathsf{cl}}(x_0, \mu)$$

Infinite Horizon Problem



Infinite Horizon Problem



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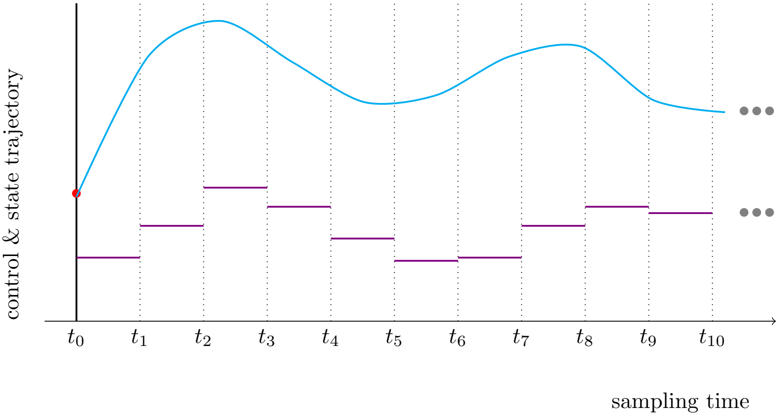

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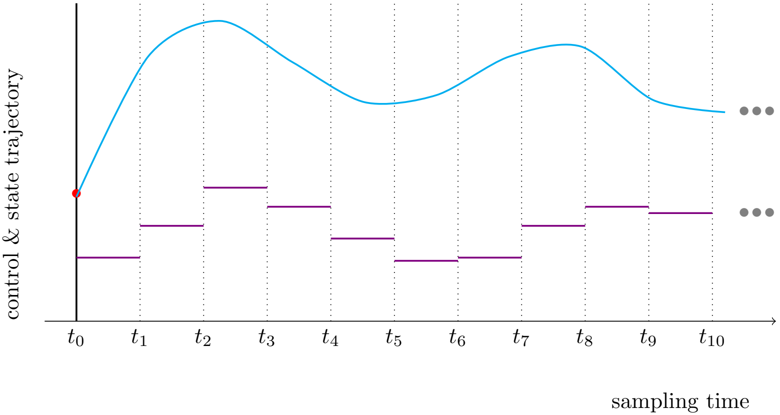

Issue: solving infinite horizon OCP is difficult.

Finite Horizon Problem

- **MPC:** receding horizon strategy to obtain the feedback law

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- the **finite horizon OCP** counterpart:

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- **optimal control sequence:** $u^*(\cdot)$

Dynamic Programming Principle

Theorem

Let x_0 be an initial state. Let $u^*(0), u^*(1), \ldots, u^*(N-1)$ be an optimal control for $\mathcal{P}_N(x_0)$ and $x_{u^*}(0), x_{u^*}(1), \ldots, x_{u^*}(N)$ be the corresponding optimal state trajectory.

Then for any i, $i = 0, 1, \ldots, N-1$, the control sequence $u^*(i), u^*(i+1), \ldots, u^*(N-1)$ is optimal control for $\mathcal{P}_{N-i}(x_{u^*}(i))$.

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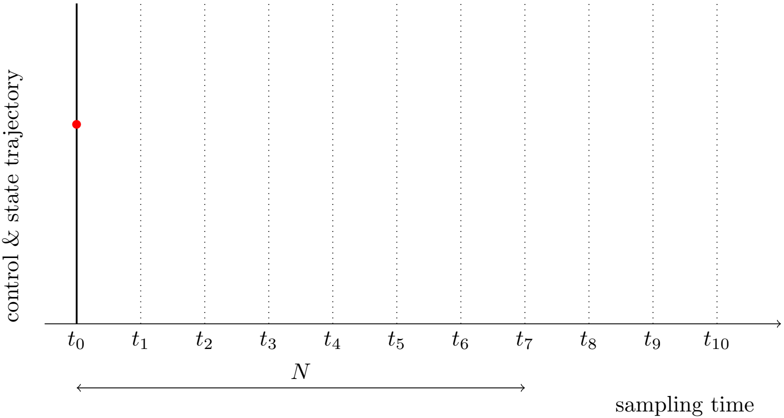
Standard MPC Algorithm



Given prediction horizon N and initial value x_0.

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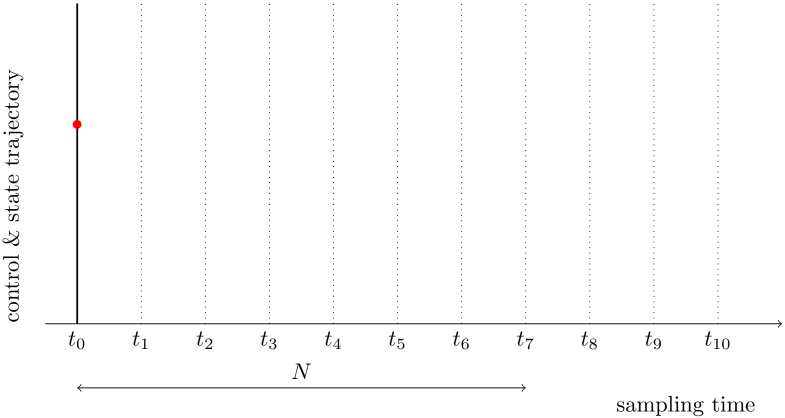

Solve the OCP $\mathcal{P}_N(x_0)$

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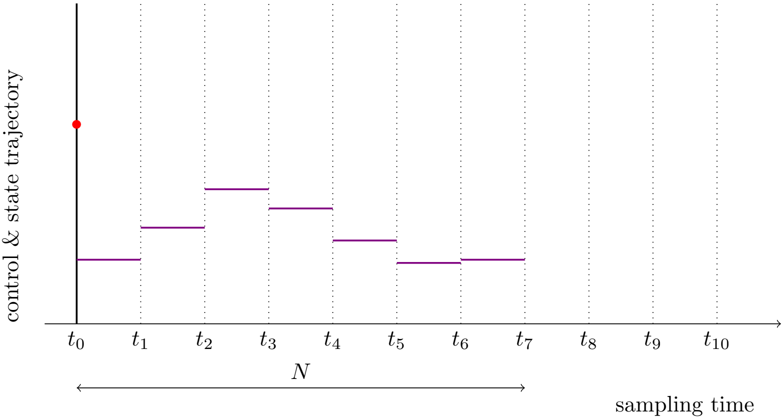

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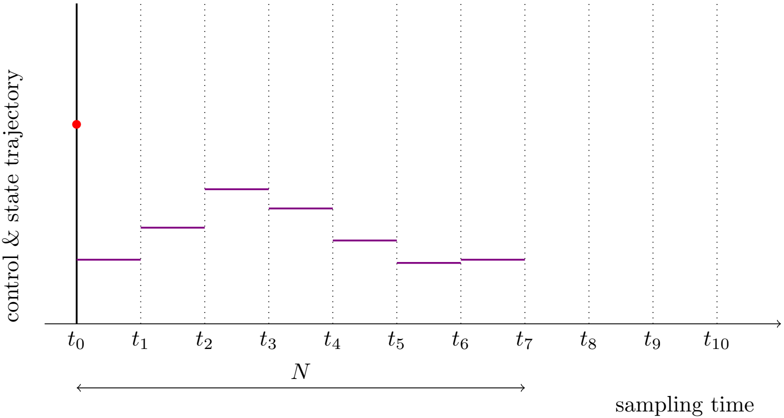

Denote optimal control sequence by $u^*(\cdot)$.

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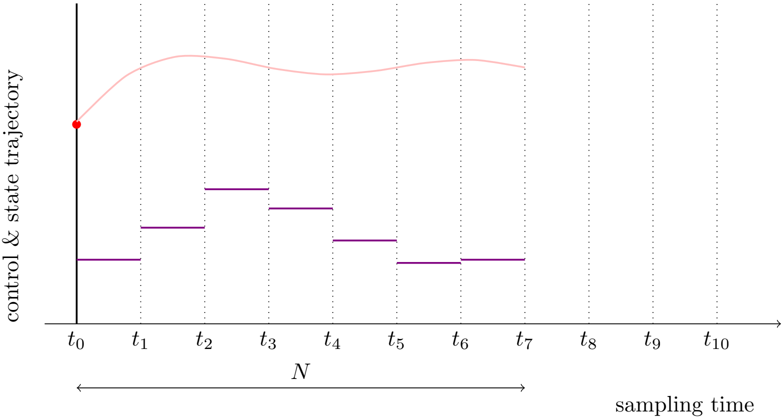

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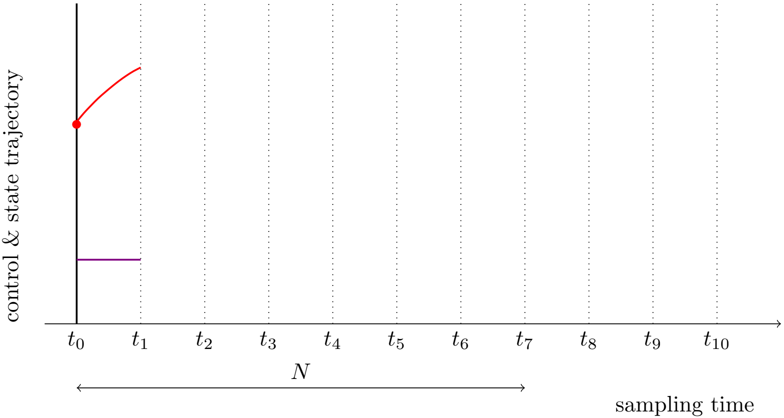
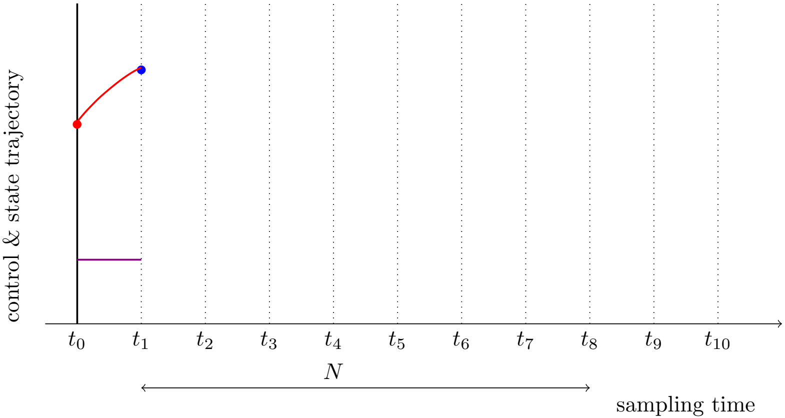


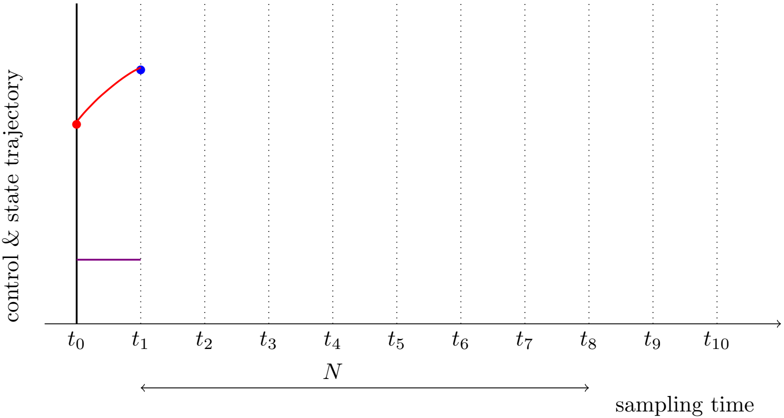
Define NMPC-feedback $\mu_N(x_0, t_0) := u^*(0)$.

Standard MPC Algorithm



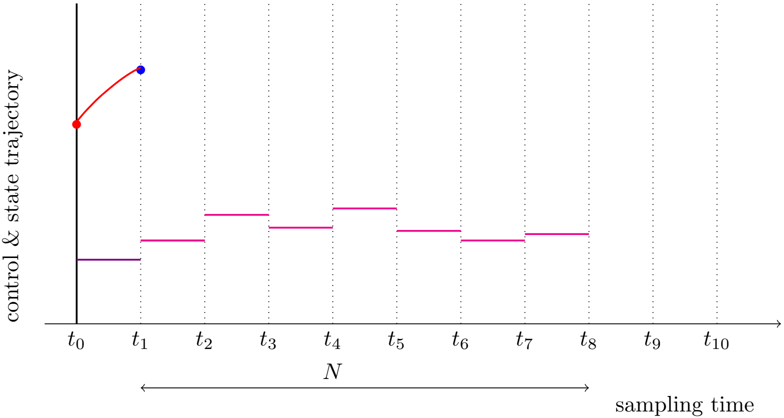
Use the feedback to generate the next state.

Standard MPC Algorithm



Repeat the process.

Standard MPC Algorithm



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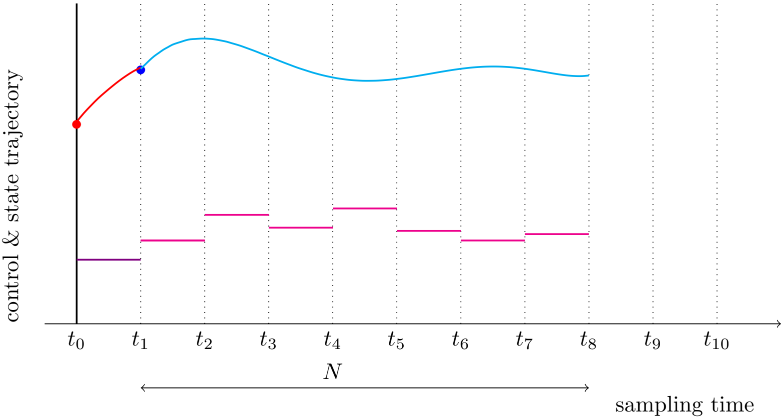

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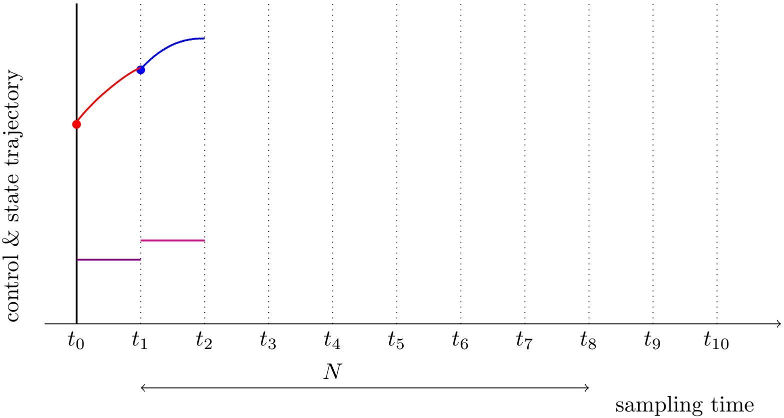
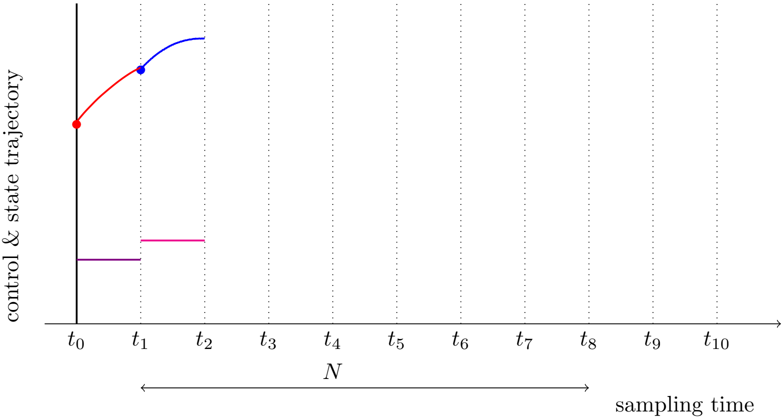


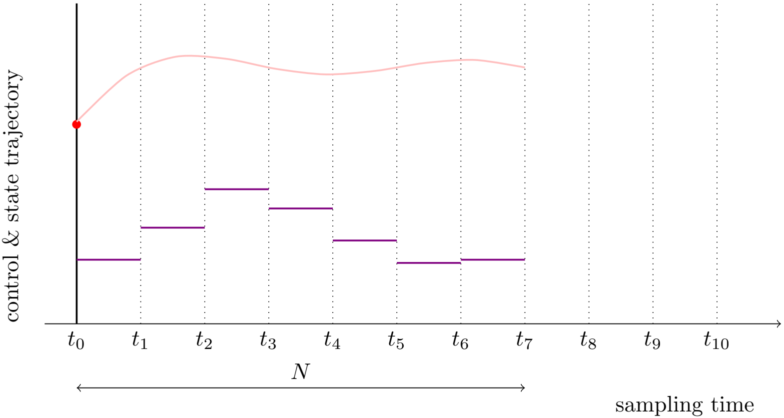
feedback law: μ_N trajectory/solution: $x_{\mu_N}(k, x_0)$

Standard MPC Algorithm



Challenge/Issue: Solving OCP every step is expensive.

Multistep or m-step MPC Algorithm



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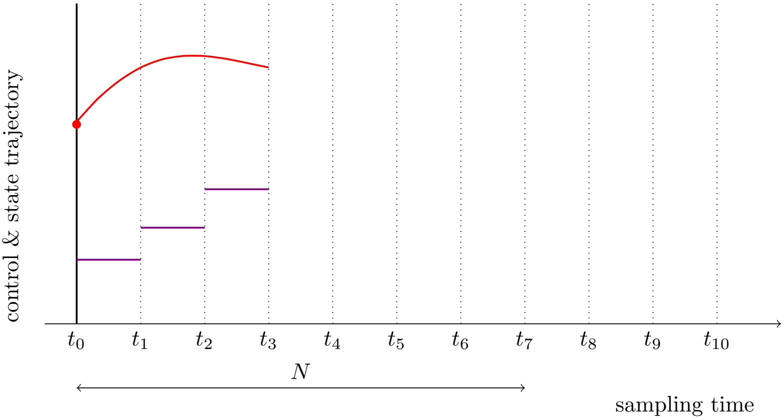
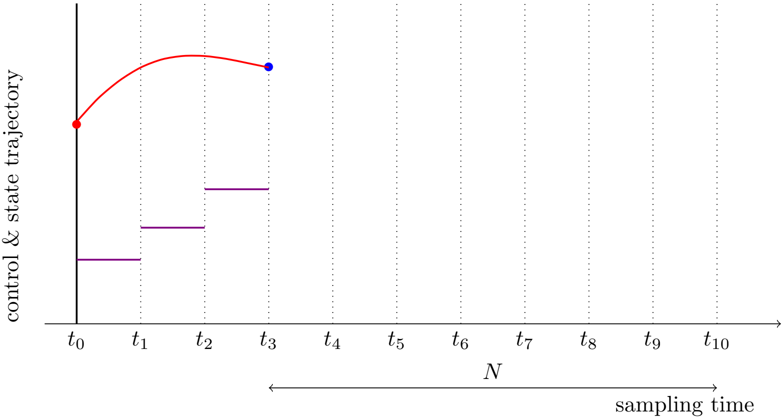


Multistep feedback - implement the first m elements.

Multistep or m-step MPC Algorithm



$u^\star(0), u^\star(1), \ldots, u^\star(m-1).$

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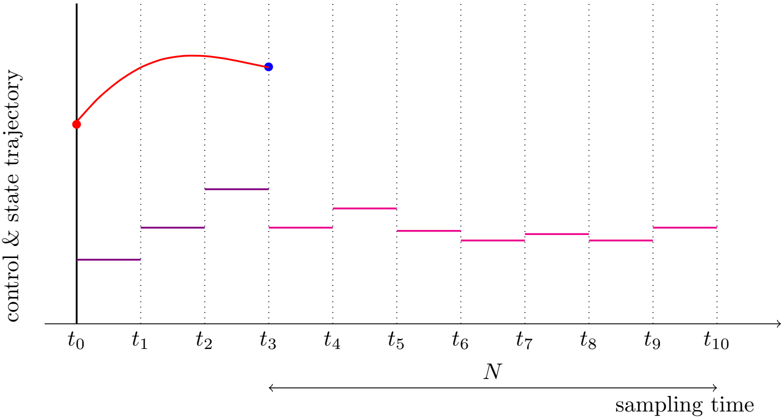
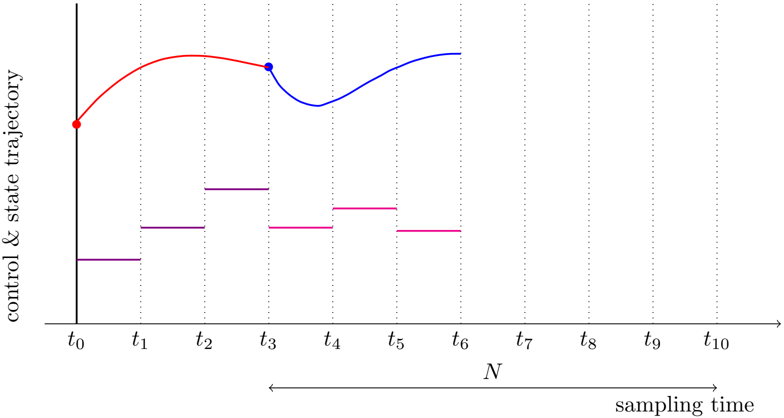


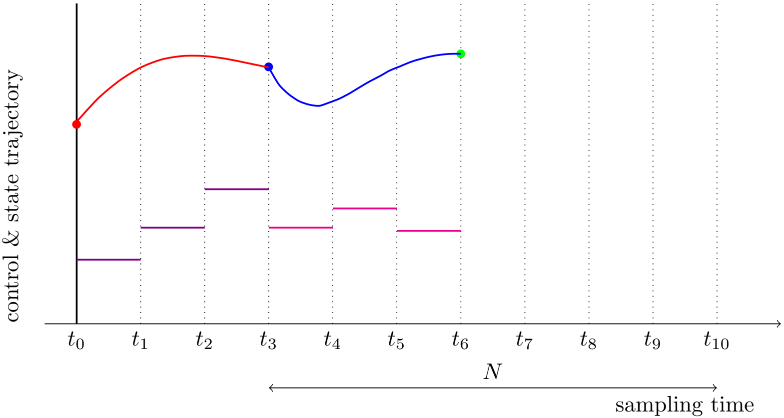
Then proceed with the next optimization.

Multistep or m-step MPC Algorithm



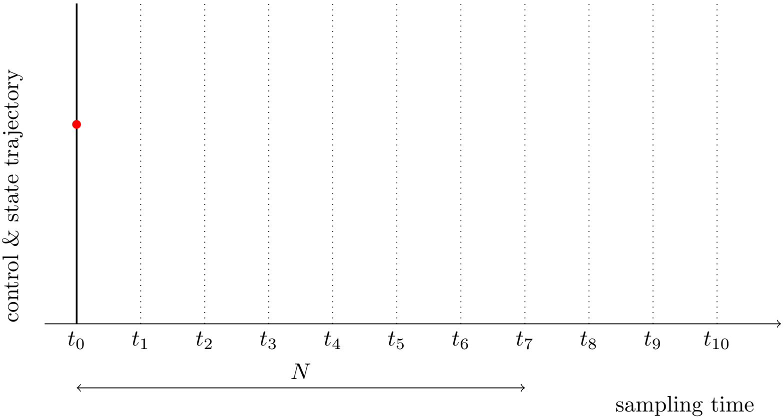
m is called **control horizon**.

Multistep or m-step MPC Algorithm



feedback law: $\mu_{N,m}$ trajectory/solution: $x_{\mu_{N,m}}(k, x_0)$

Updated m-step MPC Algorithm



Given prediction horizon N and initial value x_0.

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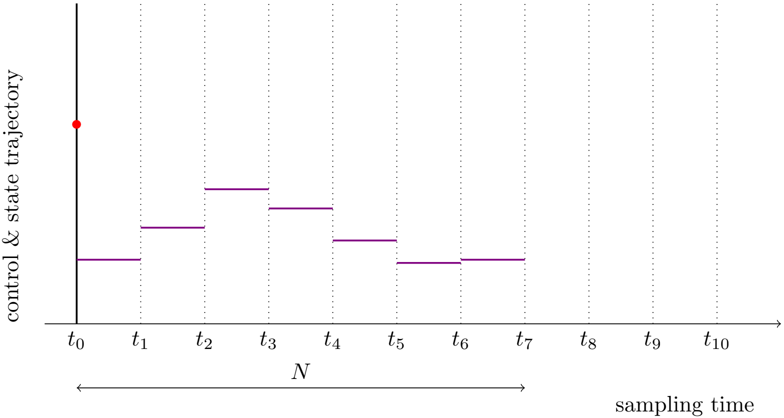

Solve $\mathcal{P}_N(x_0)$

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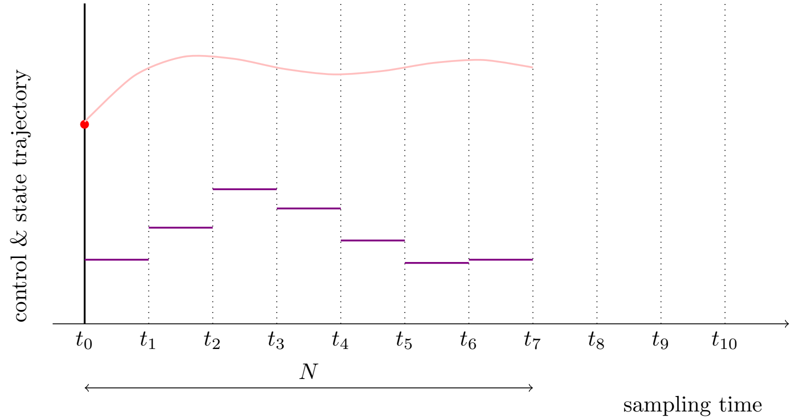

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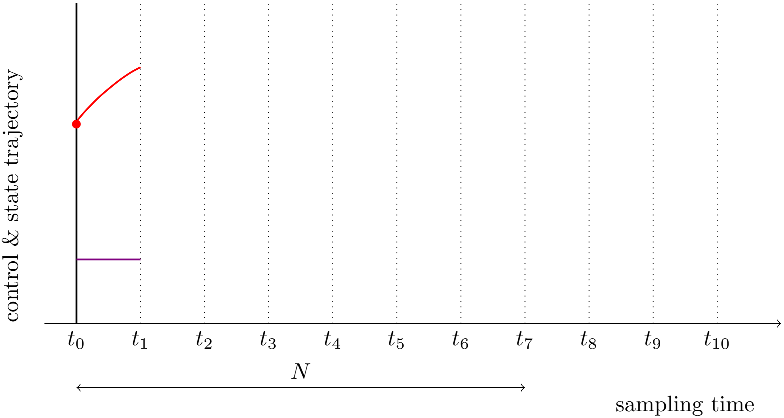
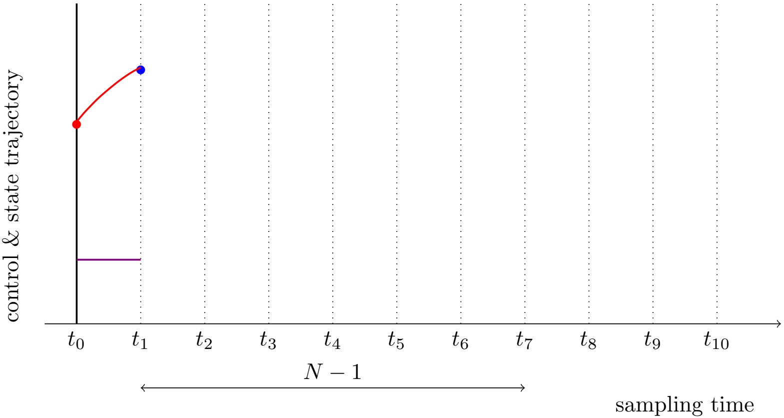


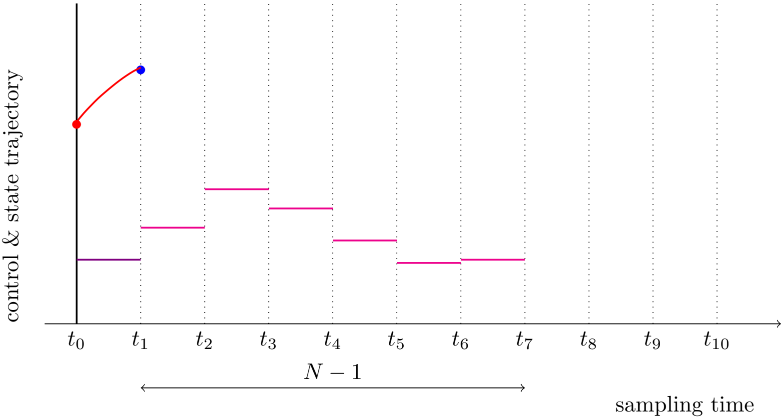
Implement the first element of the optimal control.

Updated m-step MPC Algorithm



Re-optimize over shrunken horizon, i.e., solve $\mathcal{P}_{N-1}(x(t_1))$.

Updated m-step MPC Algorithm



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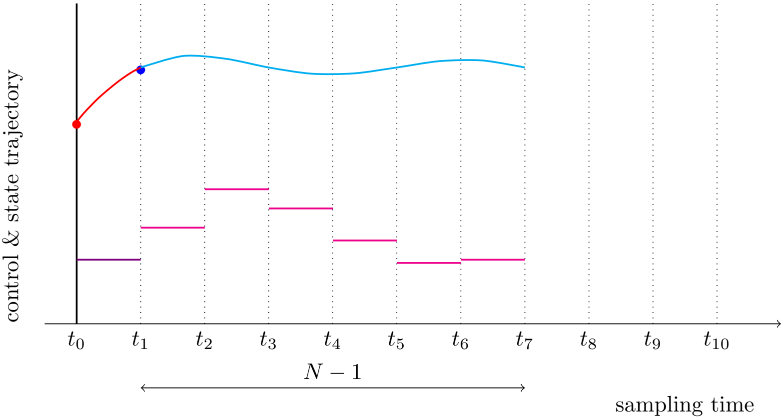

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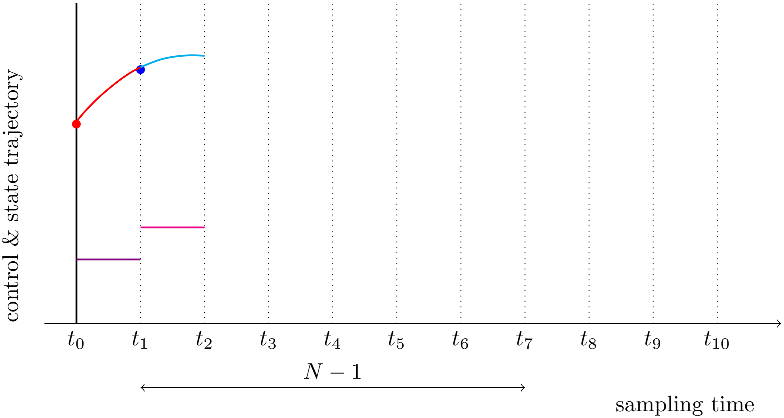

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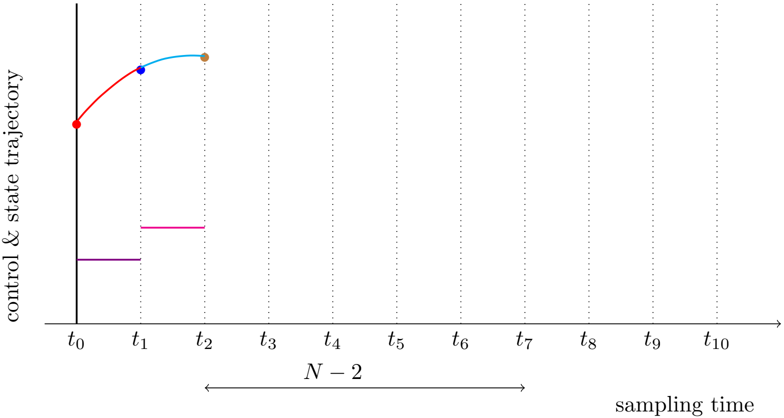
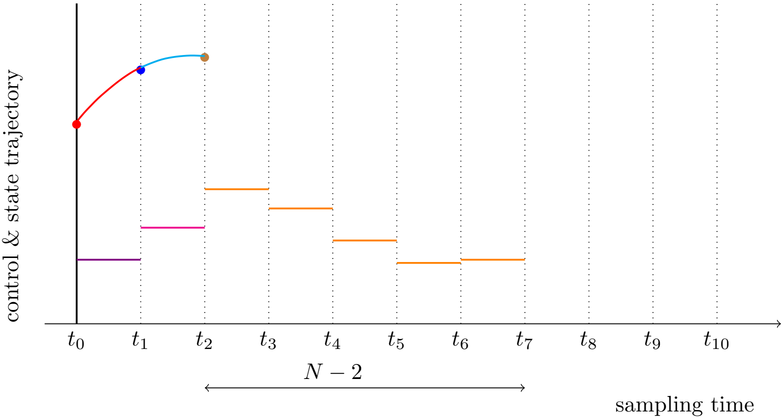


Solve $\mathcal{P}_{N-2}(x(t_2))$.

Updated m-step MPC Algorithm



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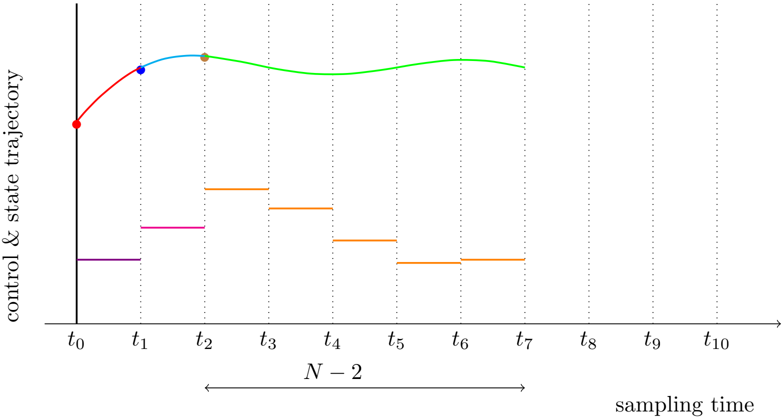

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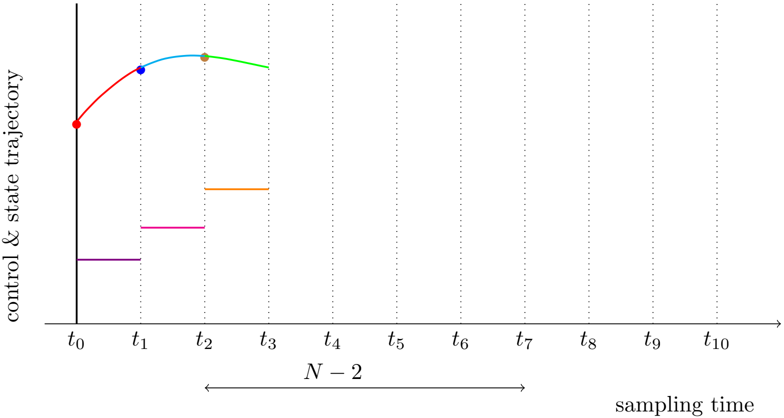
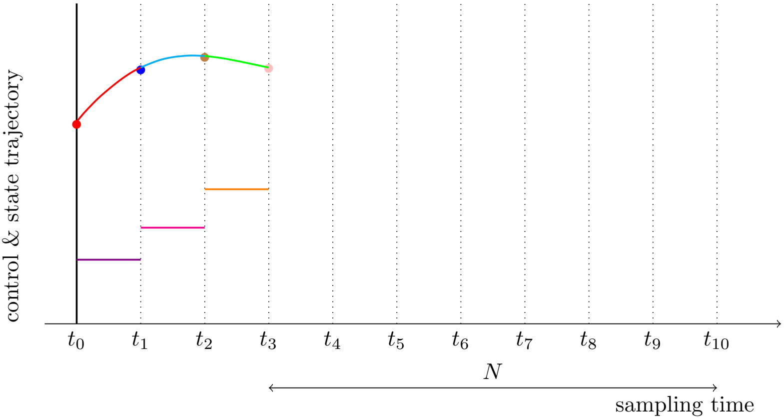


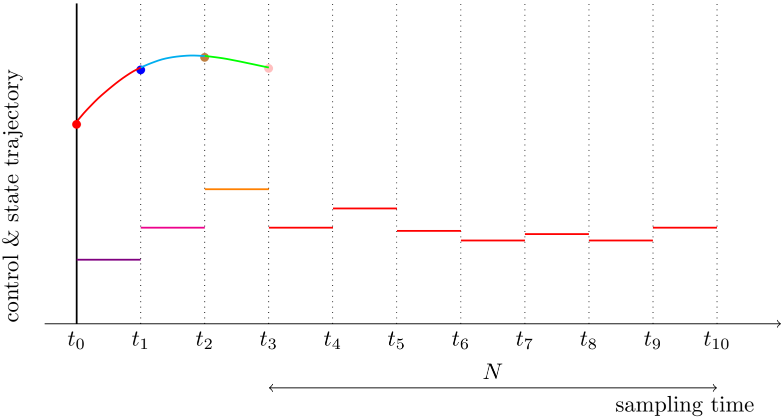
Implement the first element of the optimal control.

Updated m-step MPC Algorithm



Solve $\mathcal{P}_N(x(t_3))$.

Updated m-step MPC Algorithm



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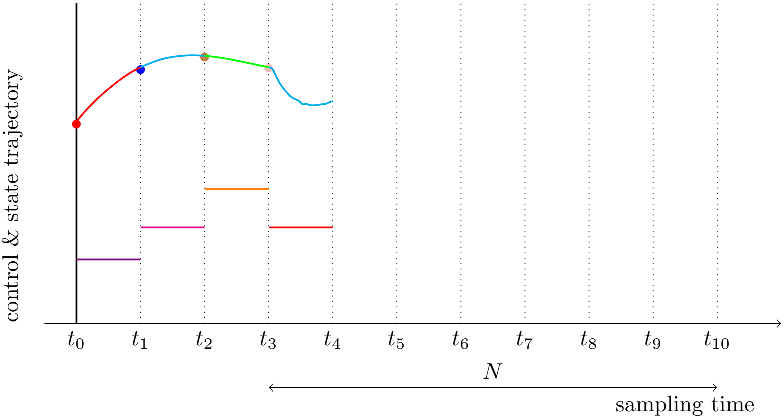
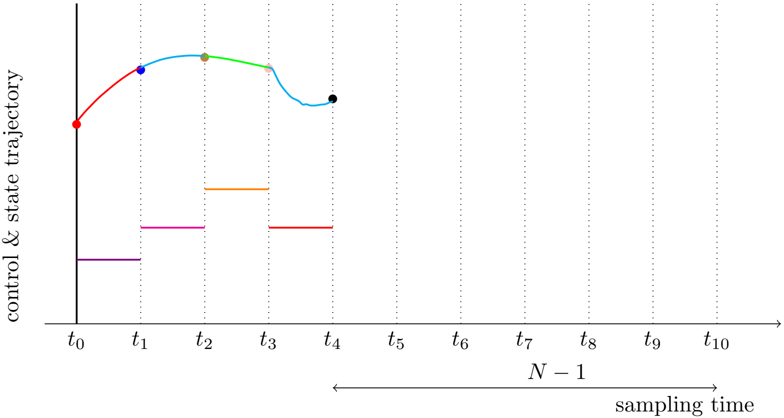


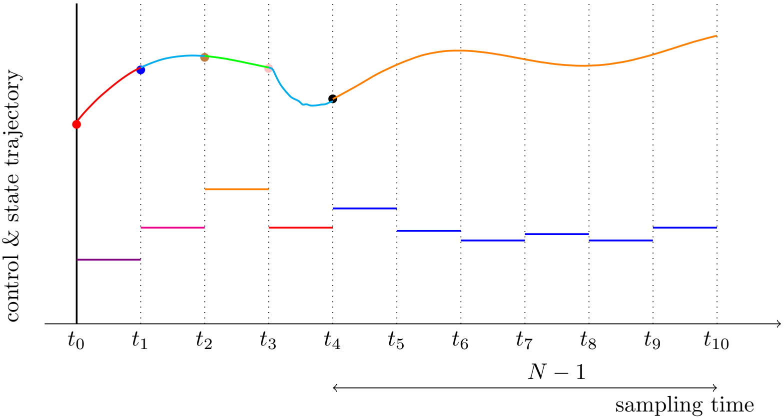
Implement the first element of the optimal control.

Updated m-step MPC Algorithm



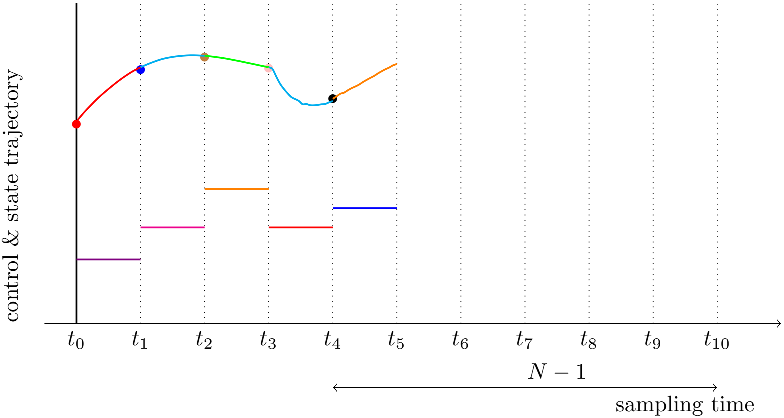
Solve $\mathcal{P}_{N-1}(x(t_4))$.

Updated m-step MPC Algorithm



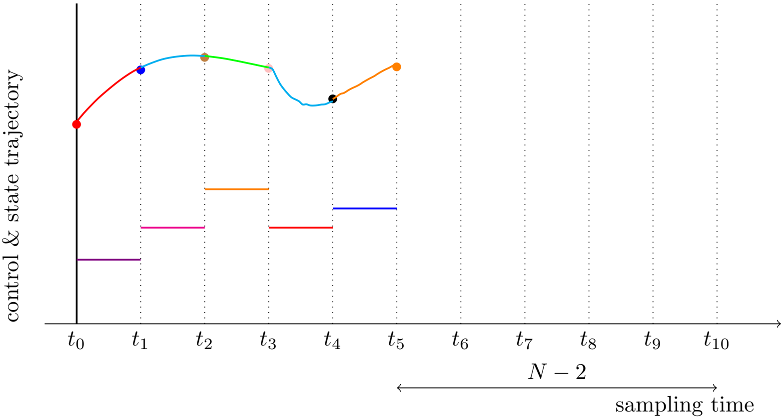
Solve $\mathcal{P}_{N-1}(x(t_4))$.

Updated m-step MPC Algorithm



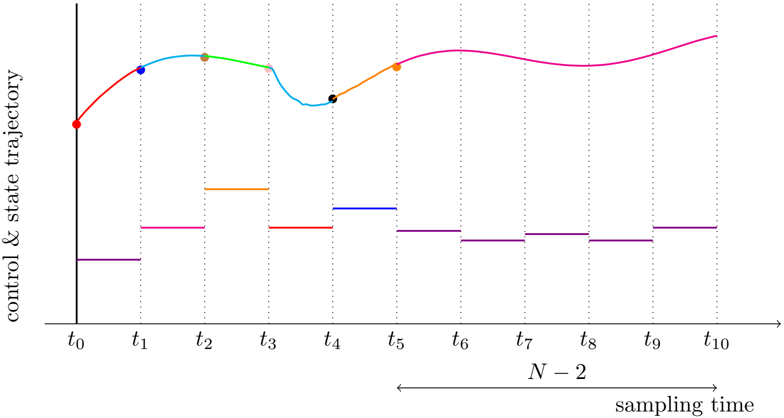
Implement the first element of the optimal control.

Updated m-step MPC Algorithm



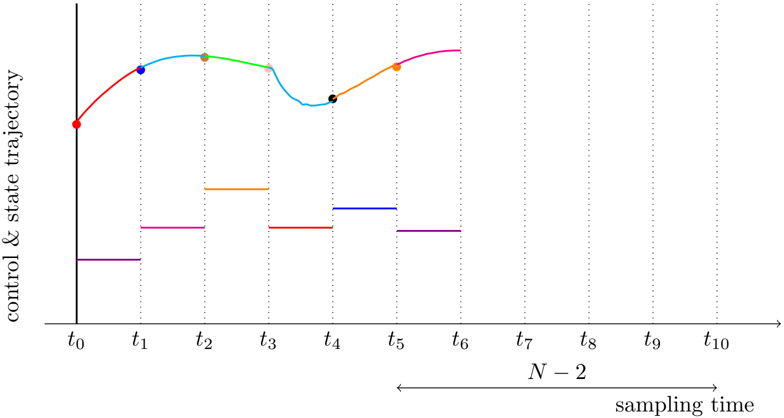
Solve $\mathcal{P}_{N-2}(x(t_5))$.

Updated m-step MPC Algorithm



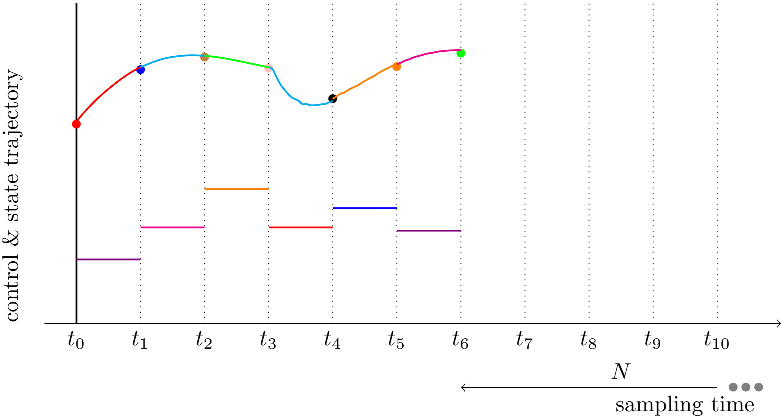
Solve $\mathcal{P}_{N-2}(x(t_5))$.

Updated m-step MPC Algorithm



Implement the first element of the optimal control.

Updated m-step MPC Algorithm



feedback law: $\hat{\mu}_{N,m}$ trajectory/solution: $x_{\hat{\mu}_{N,m}}(k, x_0)$

Nominal Setting

Inverted Pendulum

$$
\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= -\frac{g}{\ell}\sin(x_1(t)) - \frac{k_L}{l}\arctan(1000x_2(t))x_2^2(t) - \frac{u(t)}{l}\cos(x_1(t)) \\
&\quad -k_R\left(\frac{4ax_2(t)}{1+4(ax_2(t))^2} + \frac{2\arctan(bx_2(t))}{\pi}\right) \\
\dot{x}_3(t) &= x_4(t) \\
\dot{x}_4(t) &= u(t)
\end{aligned}
$$

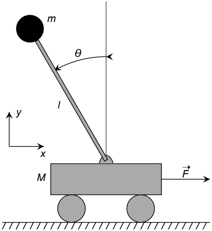


Image from http://en.wikipedia.org/wiki/Inverted_pendulum

Nominal Setting

Inverted Pendulum

$$
\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= -\frac{g}{\ell}\sin(x_1(t)) - \frac{k_L}{l}\arctan(1000x_2(t))x_2^2(t) - \frac{u(t)}{l}\cos(x_1(t)) \\
&\quad -k_R\left(\frac{4ax_2(t)}{1+4(ax_2(t))^2} + \frac{2\arctan(bx_2(t))}{\pi}\right) \\
\dot{x}_3(t) &= x_4(t) \\
\dot{x}_4(t) &= u(t)
\end{aligned}
$$

- stabilize the upright position $x_* = ((2k+1)\pi, 0, 0, 0),\ k \in \mathbb{N}$,
- sampling period $T = 0.2$, prediction horizon $N = 15$, initial value $x_0 = (-\pi - 0.1, 0, -0.1, 0)$

Nominal Setting

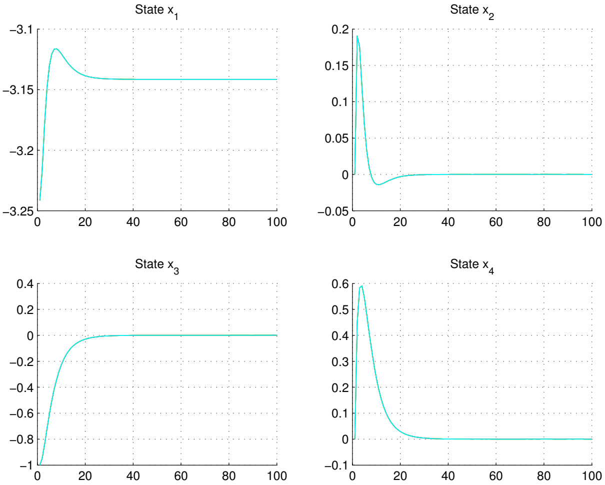


Figure: the standard MPC scheme (cyan), 1-step (red) and updated 1-step (green) MPC schemes

Nominal Setting

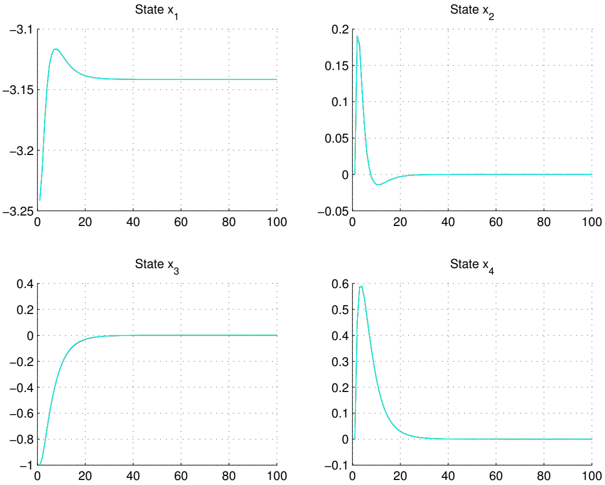


Figure: the standard MPC scheme (cyan), 2-step (red) and updated 2-step (green) MPC schemes

Nominal Setting

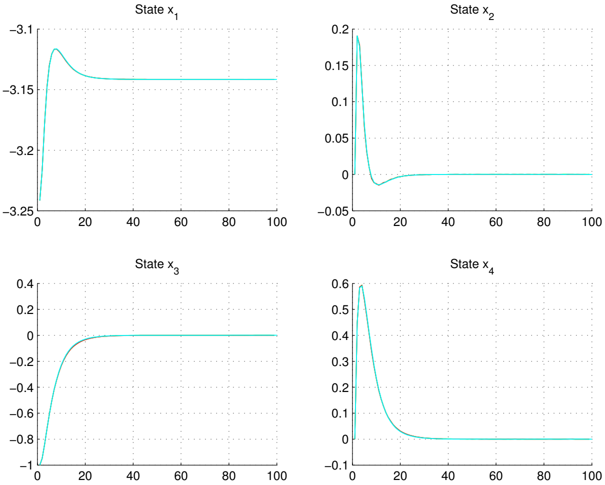


Figure: the standard MPC scheme (cyan), 3-step (red) and updated 3-step (green) MPC schemes

Nominal Setting

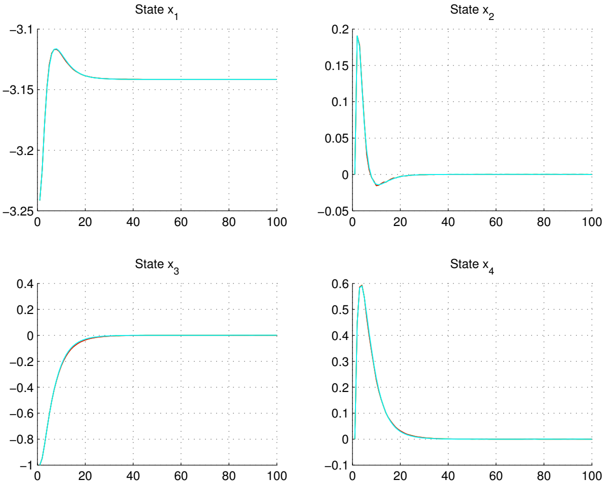


Figure: the standard MPC scheme (cyan), 4-step (red) and updated 4-step (green) MPC schemes

Nominal Setting

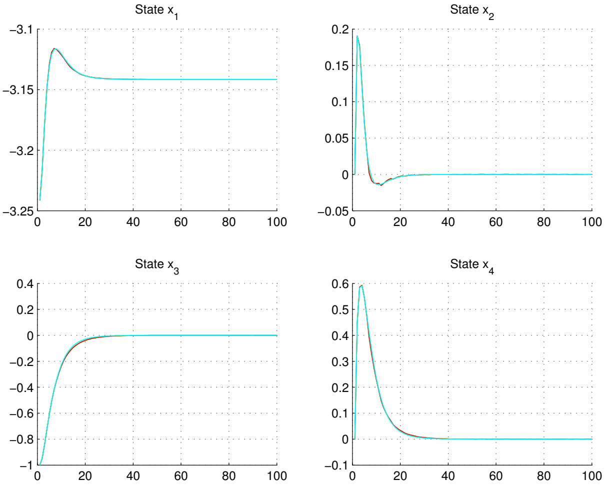


Figure: the standard MPC scheme (cyan), 5-step (red) and updated 5-step (green) MPC schemes

Nominal Setting

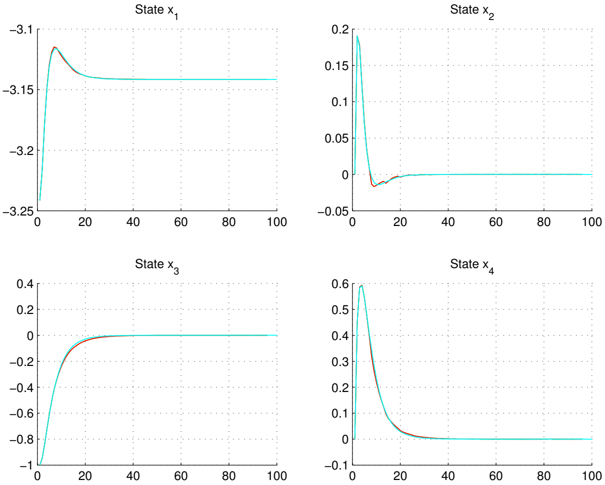
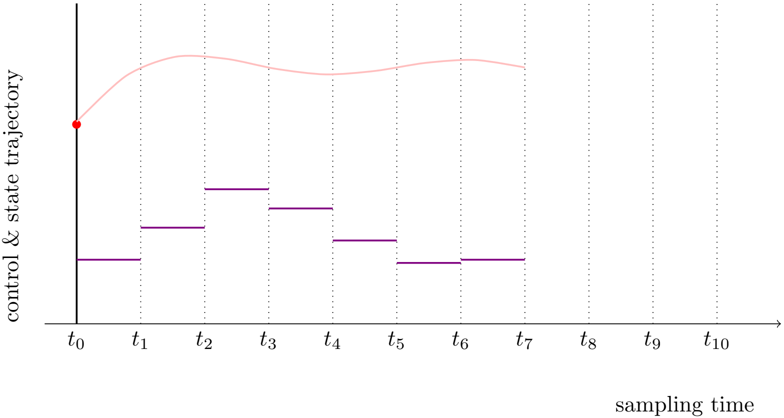
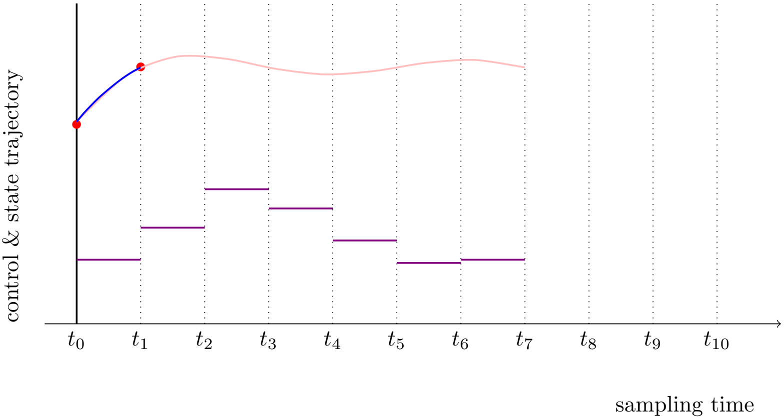


Figure: the standard MPC scheme (cyan), 6-step (red) and updated 6-step (green) MPC schemes

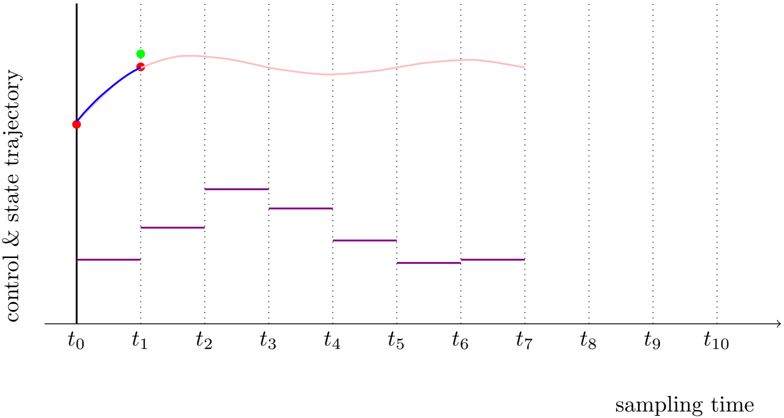
Perturbed Systems



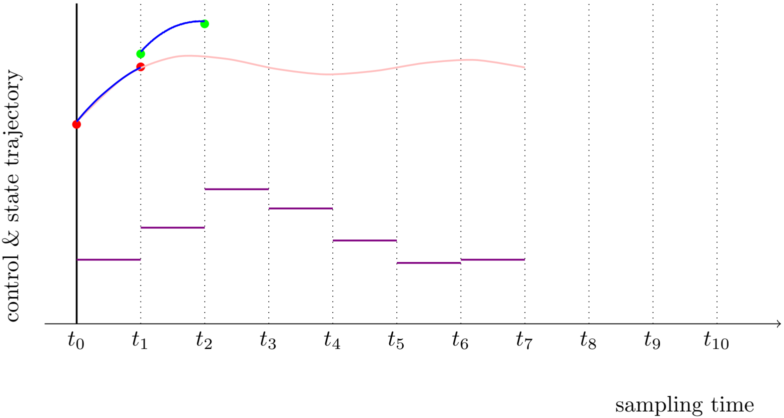
Perturbed Systems



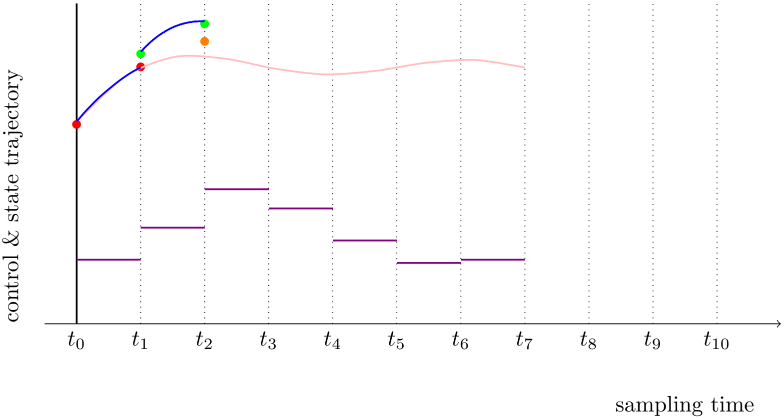
Perturbed Systems



Perturbed Systems



Perturbed Systems



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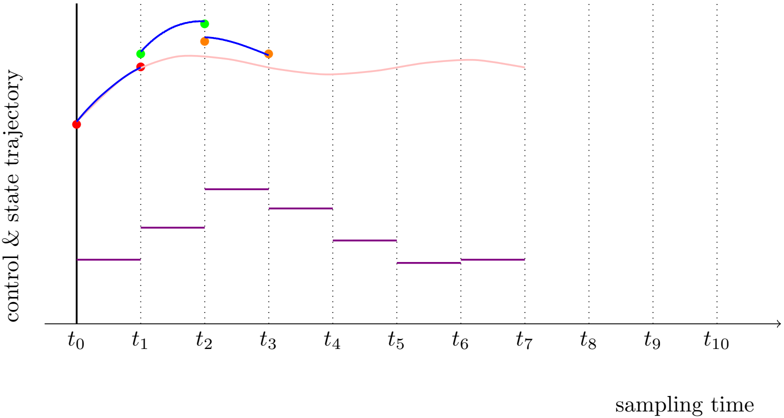
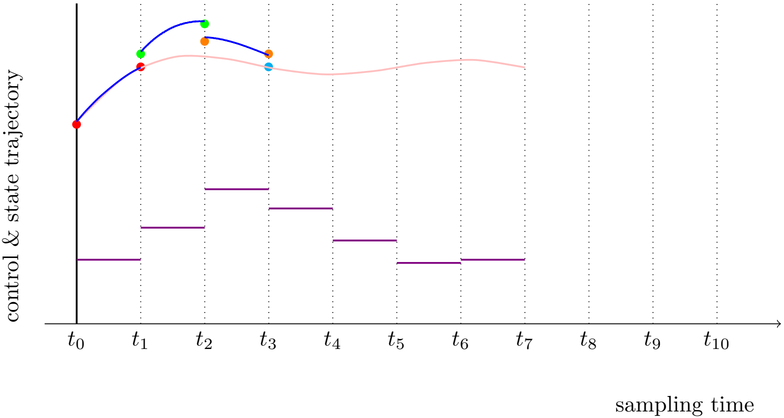
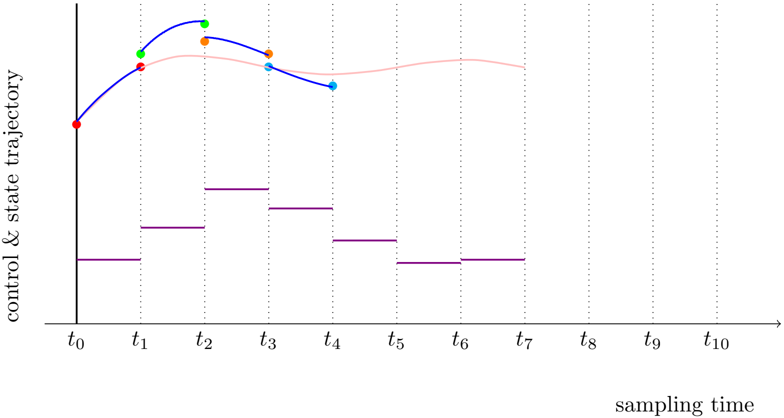


Perturbed Systems



Perturbed Systems



Perturbed Systems

- control moves of the m-step MPC are based on old info

Perturbed Systems

- control moves of the m-step MPC are based on old info
- **nominal** closed-loop system

$$x(k+1) = f(x(k), \mu(x(\tilde{k}), k))$$

- **perturbed** closed-loop system

$$\tilde{x}(k+1) = f(\tilde{x}(k), \mu(\tilde{x}(\tilde{k}), k))) + d(k)$$

Perturbed Systems

Inverted Pendulum

$$
\begin{aligned}
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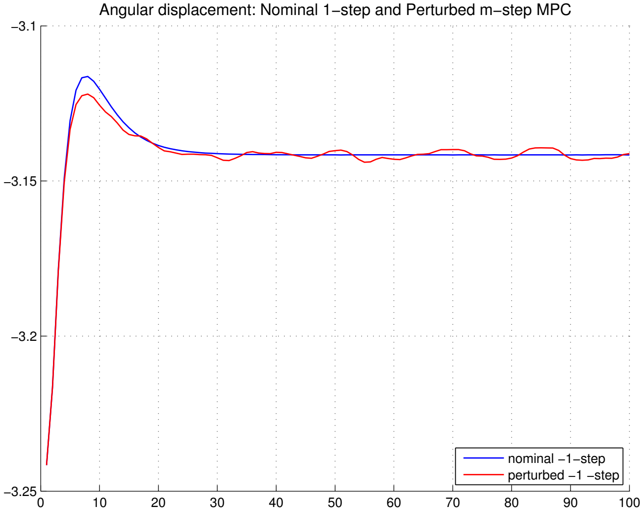
$$

- stabilize the upright position $x_* = ((2k+1)\pi, 0, 0, 0),\ k \in \mathbb{N}$,
- sampling period $T = 0.2$, prediction horizon $N = 15$, initial value $x_0 = (-\pi - 0.1, 0, -0.1, 0)$
- randomly generated perturbation sequence $d(k) = [0, 0, d_3(k), 0]^\top, \quad d_3(k) \in [-0.05, 0]$

m-step MPC in Perturbed Setting



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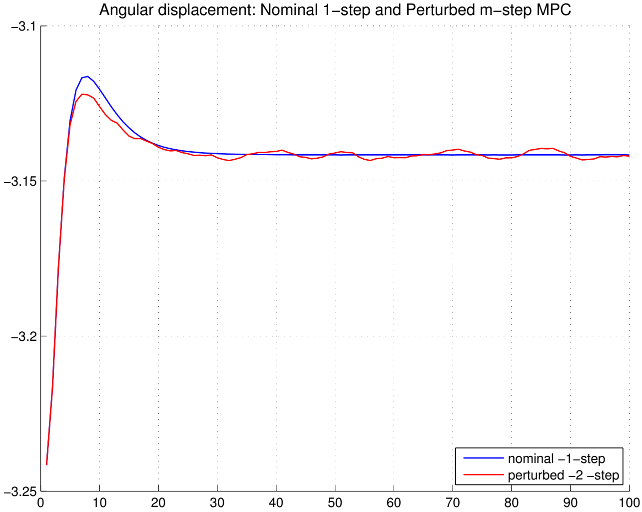
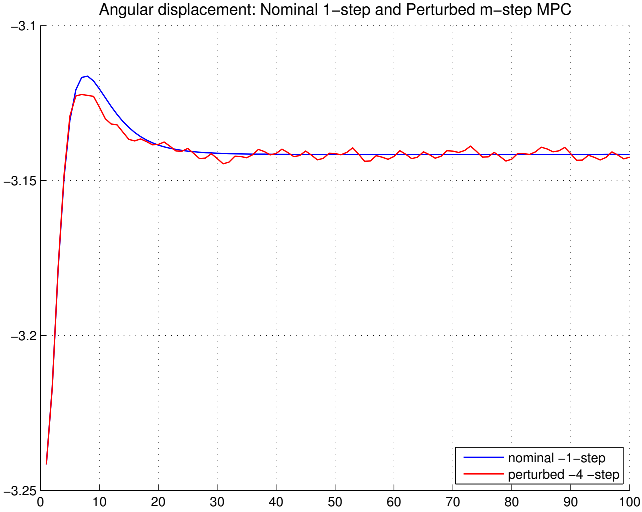
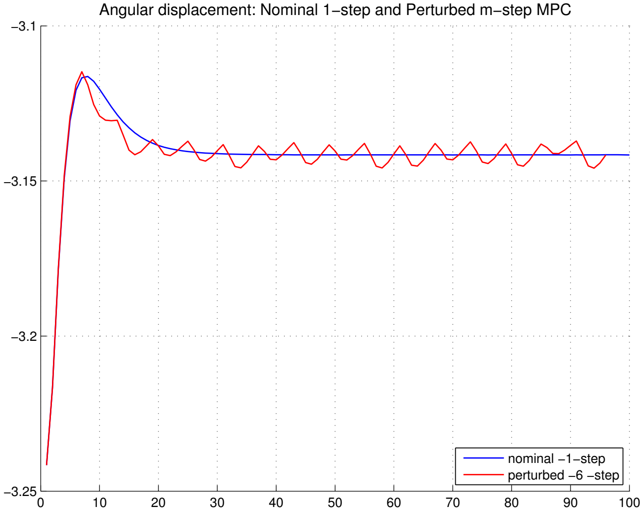


m-step MPC in Perturbed Setting



m-step MPC in Perturbed Setting



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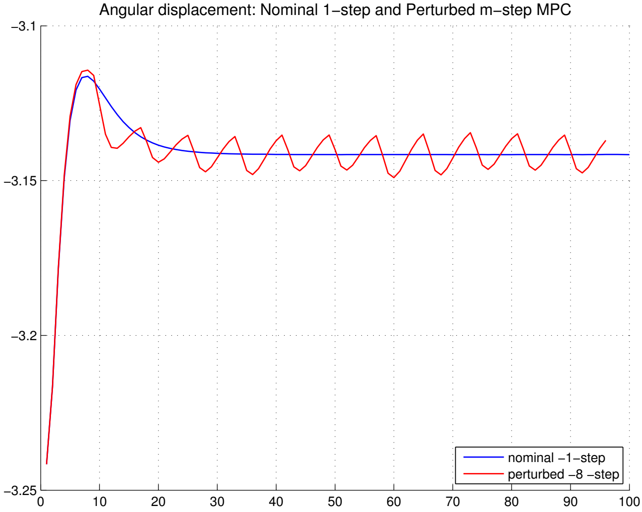

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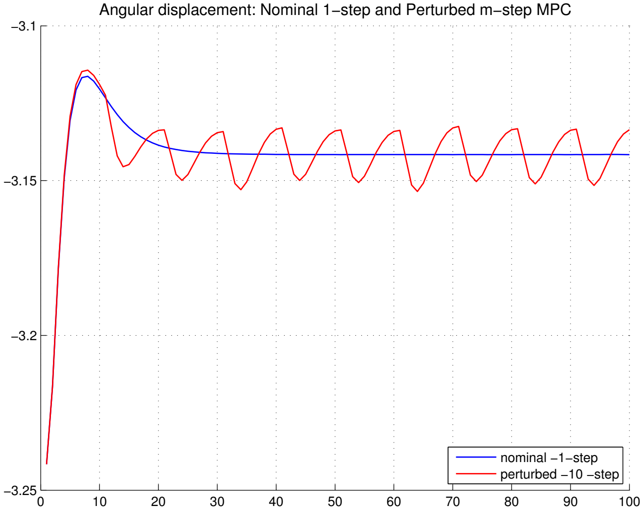
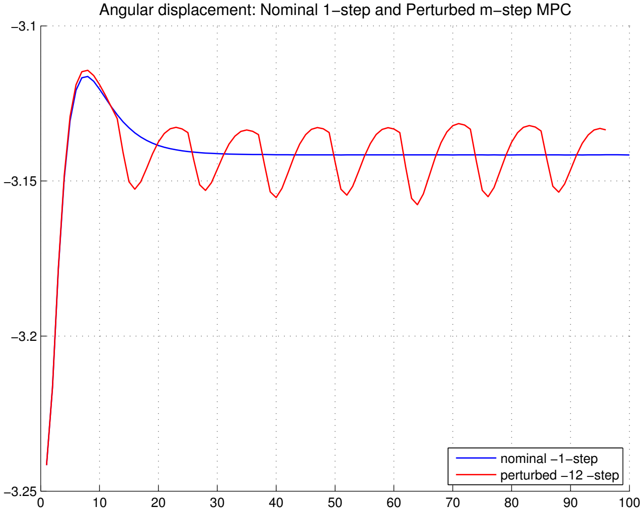
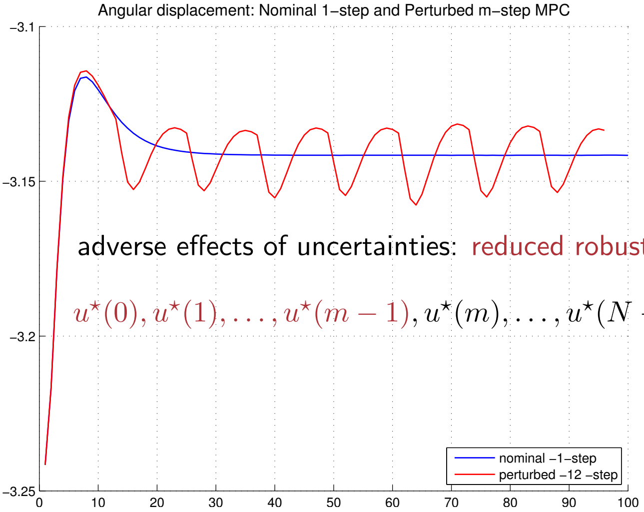


m-step MPC in Perturbed Setting



m-step MPC in Perturbed Setting



adverse effects of uncertainties: reduced robustness

$$u^\star(0), u^\star(1), \ldots, u^\star(m-1), u^\star(m), \ldots, u^\star(N-1)$$

Updated m-step MPC in Perturbed Setting

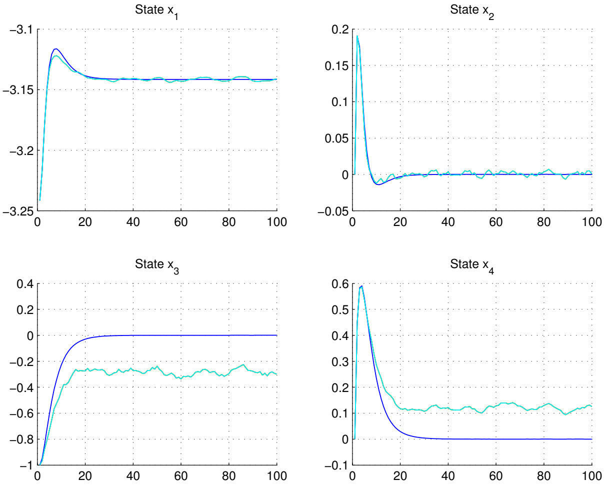


Figure: the nominal 1-step MPC scheme (blue), the standard MPC scheme (cyan), 1-step (red) and updated 1-step (green) MPC schemes for the perturbed system

Updated m-step MPC in Perturbed Setting

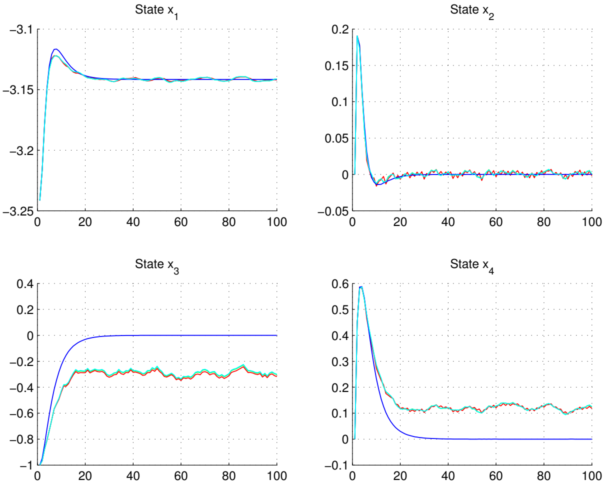


Figure: the nominal 2-step MPC scheme (blue), the standard MPC scheme (cyan), 2-step (red) and updated 2-step (green) MPC schemes for the perturbed system

Updated m-step MPC in Perturbed Setting

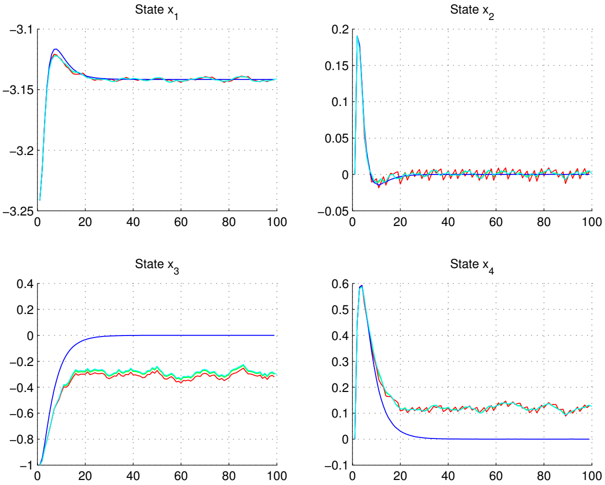


Figure: the nominal 3-step MPC scheme (blue), the standard MPC scheme (cyan), 3-step (red) and updated 3-step (green) MPC schemes for the perturbed system

Updated m-step MPC in Perturbed Setting

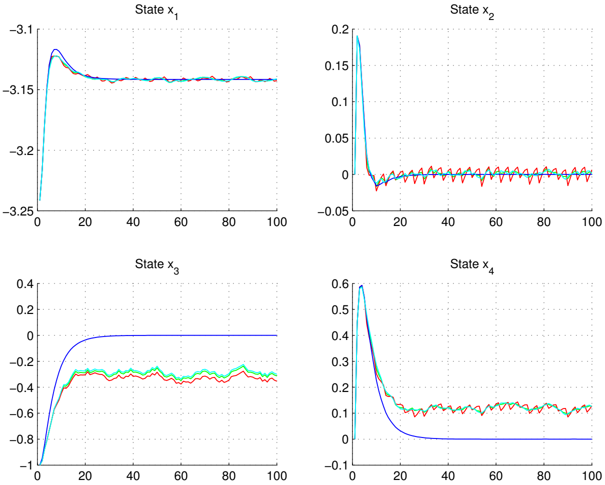


Figure: the nominal 4-step MPC scheme (blue), the standard MPC scheme (cyan), 4-step (red) and updated 4-step (green) MPC schemes for the perturbed system

Updated m-step MPC in Perturbed Setting

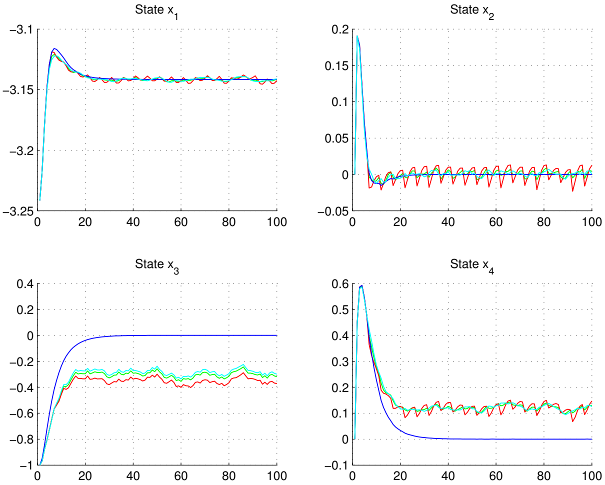


Figure: the nominal 5-step MPC scheme (blue), the standard MPC scheme (cyan), 5-step (red) and updated 5-step (green) MPC schemes for the perturbed system

Updated m-step MPC in Perturbed Setting

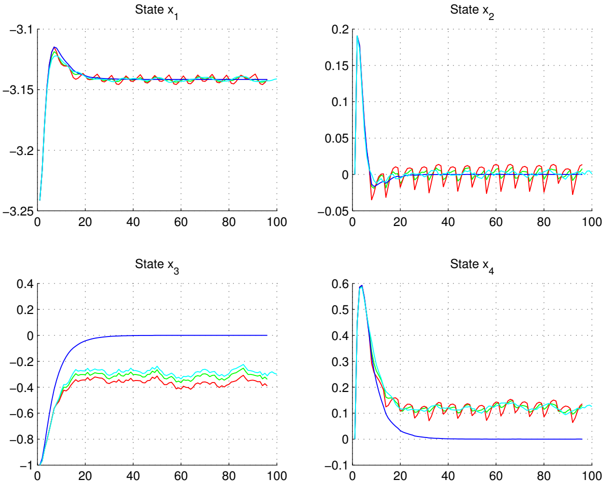


Figure: the nominal 6-step MPC scheme (blue), the standard MPC scheme (cyan), 6-step (red) and updated 6-step (green) MPC schemes for the perturbed system

Updated m-step MPC in Perturbed Setting

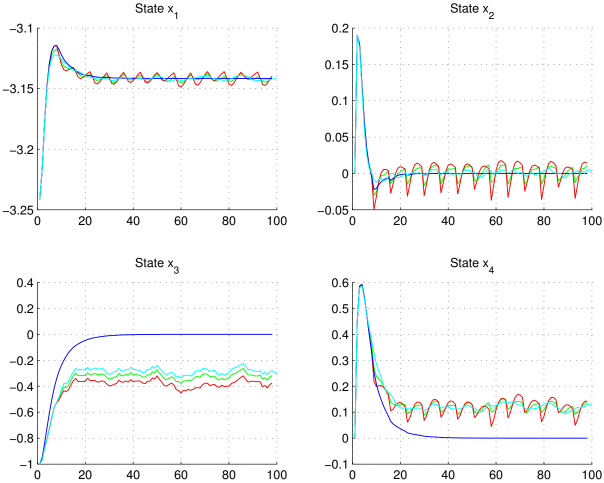


Figure: the nominal 7-step MPC scheme (blue), the standard MPC scheme (cyan), 7-step (red) and updated 7-step (green) MPC schemes for the perturbed system

Updated m-step MPC in Perturbed Setting

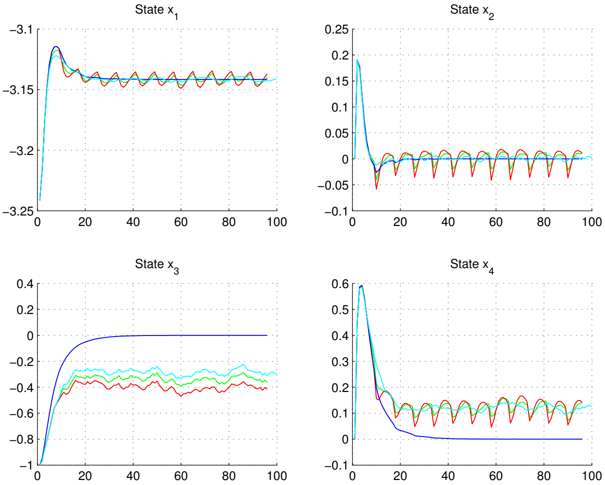


Figure: the nominal 8-step MPC scheme (blue), the standard MPC scheme (cyan), 8-step (red) and updated 8-step (green) MPC schemes for the perturbed system

Updated m-step MPC in Perturbed Setting

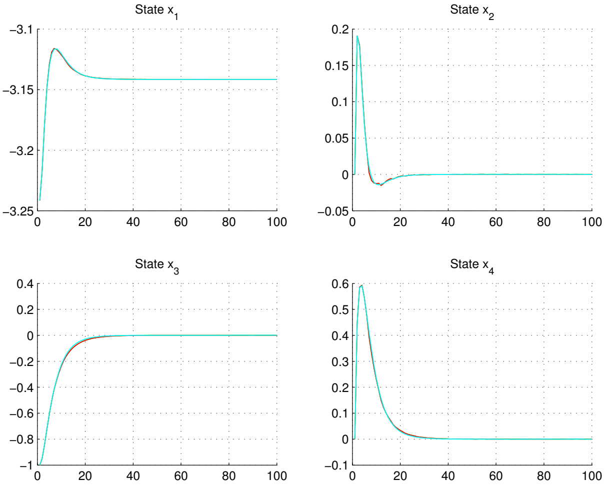


Figure: $\|d_3(k)\| = 0$, the nominal 5-step MPC scheme (blue), the standard MPC scheme (cyan), 5-step (red) and updated 5-step (green) MPC schemes for the perturbed system

Updated m-step MPC in Perturbed Setting

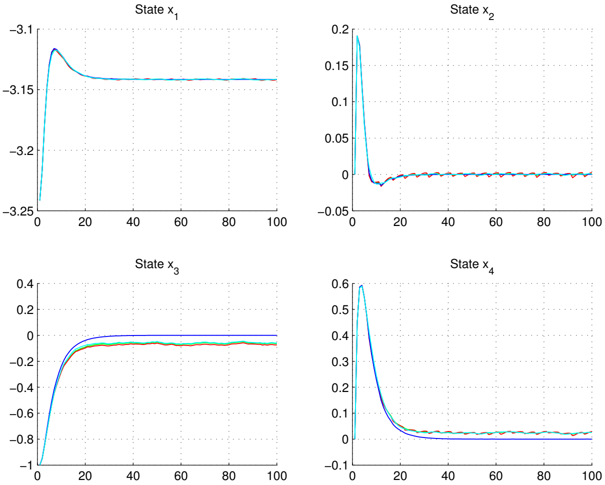


Figure: $\|d_3(k)\| = 10^{-2}$, the nominal 5-step MPC scheme (blue), the standard MPC scheme (cyan), 5-step (red) and updated 5-step (green) MPC schemes for the perturbed system

Updated m-step MPC in Perturbed Setting

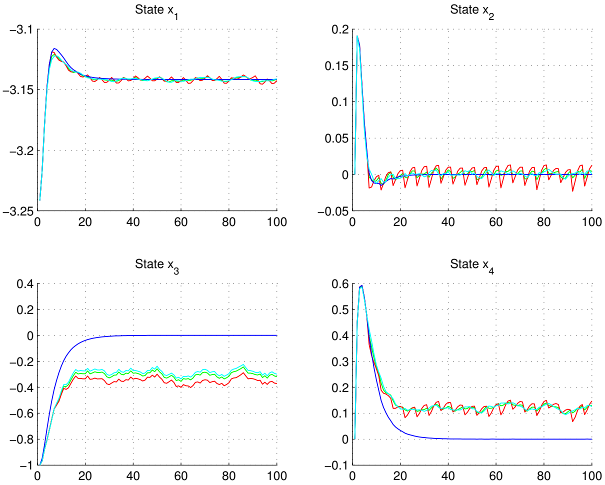


Figure: $\|d_3(k)\| = 10^{-1}/2$, the nominal 5-step MPC scheme (blue), the standard MPC scheme (cyan), 5-step (red) and updated 5-step (green) MPC schemes for the perturbed system

Updated m-step MPC in Perturbed Setting

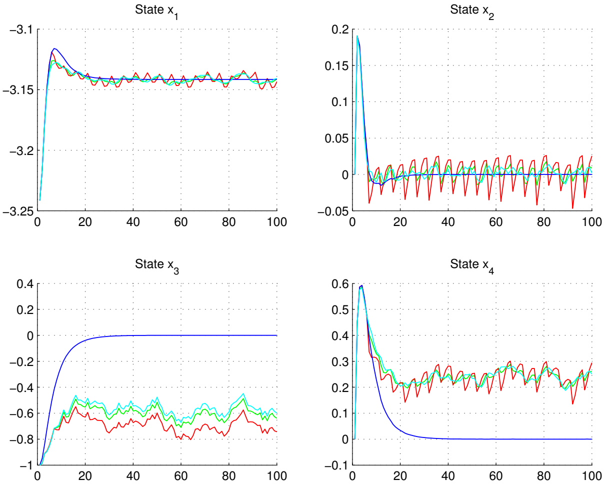


Figure: $\|d_3(k)\| = 10^{-1}$, the nominal 5-step MPC scheme (blue), the standard MPC scheme (cyan), 5-step (red) and updated 5-step (green) MPC schemes for the perturbed system

Updated m-step MPC in Perturbed Setting

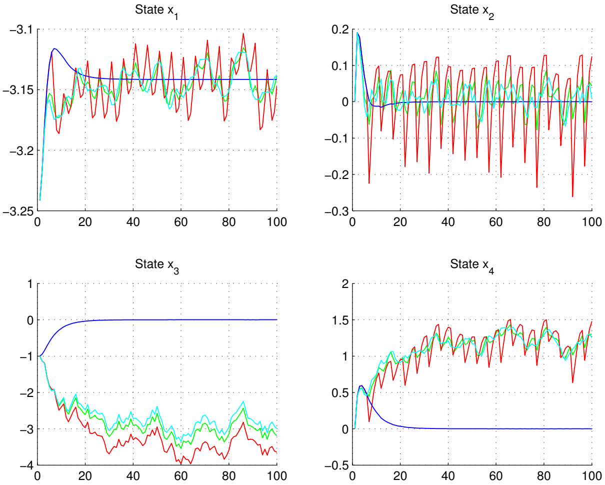


Figure: $\|d_3(k)\| = 10^{-0}/2$, the nominal 5-step MPC scheme (blue), the standard MPC scheme (cyan), 5-step (red) and updated 5-step (green) MPC schemes for the perturbed system

Updated m-step MPC in Perturbed Setting

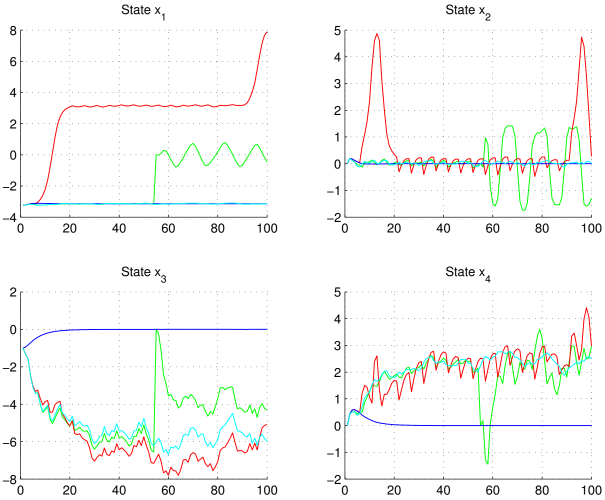


Figure: $\|d_3(k)\| = 10^{-0}$, the nominal 5-step MPC scheme (blue), the standard MPC scheme (cyan), 5-step (red) and updated 5-step (green) MPC schemes for the perturbed system

Theoretical Results

- Main Question:

  How can we quantify the benefit of the updates (i.e., re-optimization) of the updated MPC scheme?

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  How can we quantify the benefit of the updates
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- Analyze the feedback laws $\mu_{N,m}$ and $\hat{\mu}_{N,m}$

Comparison Functions

$$\mathcal{K} = \left\{ \rho : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \;\middle|\; \begin{array}{c} \rho \text{ is continuous, } \rho(0) = 0 \\ \text{and is strictly increasing} \end{array} \right\}$$

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$$\mathcal{K}_\infty = \left\{ \rho : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \;\middle|\; \rho \in \mathcal{K} \text{ and is unbounded} \right\}$$

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$$\mathcal{KL} = \left\{ \beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+ \;\middle|\; \begin{array}{c} \beta \text{ is continuous,} \\ \forall r \lim_{t\to\infty} \beta(r,t) = 0 \text{ and} \\ \forall t \geq 0,\ \beta(\cdot, t) \in \mathcal{K}_\infty \end{array} \right\}$$

Comparison Functions

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$$\mathcal{KL}_0 = \left\{ \beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+ \;\middle|\; \begin{array}{c} \beta \text{ is continuous,} \\ \forall r \lim_{t\to\infty} \beta(r,t) = 0 \text{ and} \\ \forall t \geq 0, \beta(\cdot, t) \in \mathcal{K}_\infty \\ \text{or } \beta(\cdot, t) \equiv 0 \end{array} \right\}$$

Stabilizing MPC without terminal constraints

- Aim: to yield a controller stabilizing a given equilibrium

Stabilizing MPC without terminal constraints

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x^* is **asymptotically stable** if there exists $\beta \in \mathcal{KL}$ s.t.

$$\|x_\mu(k, x_0)\|_{x^*} \leq \beta(\|x_0\|_{x^*}, k)$$

for all $x_0 \in \mathbb{X}$ and all $k \in \mathbb{N}$.

Assumption 1 (A1)

There exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ s.t.

$$\alpha_1(\|x\|_{x_*}) \leq \ell^*(x) \leq \alpha_2(\|x\|_{x_*})$$

for all $x \in \mathbb{X}$, where $\ell^*(x) \;:=\; \inf_{u \in \mathbb{U}} \ell(x, u)$.

Stabilizing MPC without terminal constraints

Proposition 1 (P1)

Consider time-dependent $\mu : \mathbb{X} \times \mathbb{N}_0 \to U$ and $V : X \to \mathbb{R}_0^+$ satisfying the relaxed dynamic programming inequality

$$V(x_0) \geq V(x_\mu(m)) + \alpha \sum_{k=0}^{m-1} \ell(x_\mu(k), \mu(x_\mu(\tilde{k}), k))$$

for some $\alpha \in (0, 1]$, some $m \geq 1$ and all $x_0 = x_\mu(0) \in \mathbb{X}$.

Stabilizing MPC without terminal constraints

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$$V_\infty(x) \leq J_\infty^{\mathsf{cl}}(x, \mu) \leq V(x)/\alpha$$

Stabilizing MPC without terminal constraints

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Then for all $x \in \mathbb{X}$, the ff. estimate holds:

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If (A1) holds and $\exists\, \alpha_3, \alpha_4 \in \mathcal{K}_\infty$ with

$$\alpha_3(\|x\|_{x_*}) \leq V(x) \leq \alpha_4(\|x\|_{x_*})$$

then x_* is asymptotically stable for the closed-loop system.

Controllability Assumption

- Show: $\mu_{N,m}$ and V_N satisfy the RDPI.

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There exists $B_k \in \mathcal{K}_\infty$ such that the optimal value functions of $\mathcal{P}_k(x_0)$ satisfy

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$$V_k(x) \leq B_k(\ell^*(x)) \text{ for all } x \in \mathbb{X} \text{ and all } k = 2, \dots, N$$

- the system is exponentially controllable w.r.t. ℓ

$\Longrightarrow$ existence of such a $B_k \in \mathcal{K}_\infty$

Controllability Assumption

- Show: $\mu_{N,m}$ and V_N satisfy the RDPI.
- How do we compute α?

Assumption 2 (A2)

There exists $B_k \in \mathcal{K}_\infty$ such that the optimal value functions of $\mathcal{P}_k(x_0)$ satisfy

$$V_k(x) \leq B_k(\ell^*(x)) \quad \text{for all} \quad x \in \mathbb{X} \text{ and all } \quad k = 2, \ldots, N$$

Consider the optimization problem

$$\begin{aligned} \alpha := \quad & \inf_{\lambda_0, \ldots, \lambda_{N-1}, \nu} \frac{\sum_{n=0}^{N-1} \lambda_n - \nu}{\sum_{n=0}^{m-1} \lambda_n} \\ \text{s.t.} \quad & \sum_{n=k}^{N-1} \lambda_n \leq B_{N-k}(\lambda_k), \quad k = 0, \ldots, N-2 \\ & \nu \leq \sum_{n=0}^{j-1} \lambda_{n+m} + B_{N-j}(\lambda_{j+m}), \quad j = 0, .., N-m-1 \\ & \sum_{n=0}^{m-1} \lambda_n > 0, \; \lambda_0, \ldots, \lambda_{N-1}, \nu \geq 0 \end{aligned} \qquad \mathcal{P}_\alpha$$

Nominal Stability and Performance [Grüne, 2009]

Assume (A2) and that the optimization problem

$$
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\alpha := \quad & \inf_{\lambda_0,\ldots,\lambda_{N-1},\nu} \frac{\sum_{n=0}^{N-1} \lambda_n - \nu}{\sum_{n=0}^{m-1} \lambda_n} \\
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& \nu \leq \textstyle\sum_{n=0}^{j-1} \lambda_{n+m} + B_{N-j}(\lambda_{j+m}), \quad j = 0, .., N-m-1 \\
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\end{aligned}
$$

has an optimal value $\alpha \in (0, 1]$.

Then the m-step feedback $\mu_{N,m}$ & optimal value fcn V_N satisfy the relaxed dynamic programming inequality

$$V_N(x_0) \geq V_N(x_{\mu_{N,m}}(m)) + \alpha \sum_{k=0}^{m-1} \ell(x_{\mu_{N,m}}(k), \mu_{N,m}(x_{\mu_{N,m}}(\tilde{k}), k))$$

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& \sum_{n=0}^{m-1} \lambda_n > 0, \; \lambda_0, \ldots, \lambda_{N-1}, \nu \geq 0
\end{aligned}
$$

has an optimal value $\alpha \in (0, 1]$.

Then the ff. suboptimality estimate holds:

$$V_\infty(x) \leq J_\infty^{\text{cl}}(x, \mu_{N,m}) \leq V_N(x)/\alpha \leq V_\infty(x)/\alpha \qquad \forall x \in \mathbb{X}$$

If (A1) also holds, then the closed loop is asymptotically stable.

α is the index of suboptimality

Nominal Stability and Performance [Grüne et al., 2010]

Theorem

Let B_k, $k = 2, \ldots, N$, be linear functions.
Define $\gamma_k := B_k(r)/r$.
Then $\alpha = 1$ if and only if $\gamma_{m+1} \leq 1$. Otherwise,

$$\alpha = 1 - \frac{(\gamma_{m+1} - 1) \prod_{i=m+2}^{N} (\gamma_i - 1) \prod_{i=N-m+1}^{N} (\gamma_i - 1)}{\left(\prod_{i=m+1}^{N} \gamma_i - (\gamma_{m+1} - 1) \prod_{i=m+2}^{N} (\gamma_i - 1) \right) \left(\prod_{i=N-m+1}^{N} \gamma_i - \prod_{i=N-m+1}^{N} (\gamma_i - 1) \right)}$$

Nominal Stability and Performance

Course of reasoning:

- (A2) ($V_k(x) \leq B_k(\ell^*(x))$) allows for the formulation of $\mathcal{P}_\alpha$
- If $\mathcal{P}_\alpha$ has a solution $\alpha \in (0, 1]$

  $\Longrightarrow$ Relaxed Dynamic Programming Inequality (RDPI)

  $\Longrightarrow$ assumptions of (P1) are fulfilled

  $\Longrightarrow$ asymptotic stability and performance estimates

Robust Stability

Consider $A \subset \mathbb{X}$ where μ is feasible.

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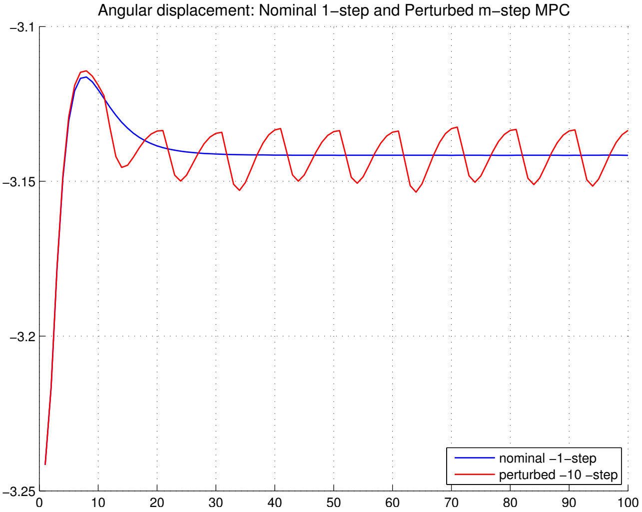
$$\tilde{x}_{\mu_N}(k, x_0) \in A \quad \text{and}$$

$$\|\tilde{x}_{\mu_N}(k, x_0)\|_{x^*} \leq \max\left\{\beta(\|x_0\|_{x^*}, k), \delta\right\} \quad \forall k \in \mathbb{N}_0.$$

Robust Stability



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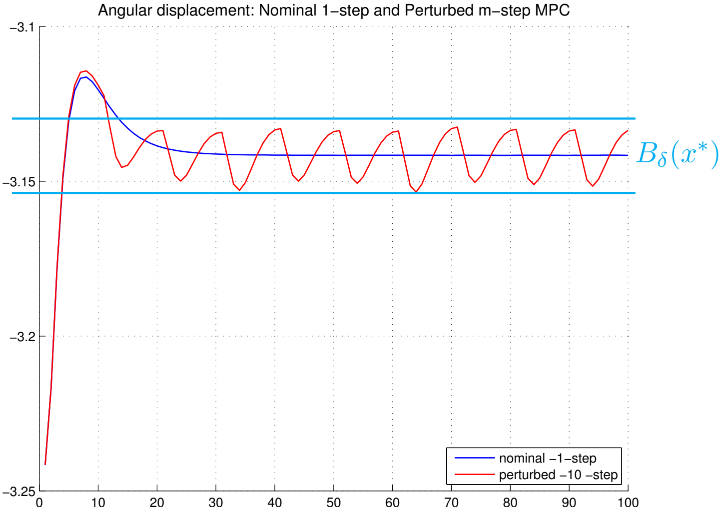

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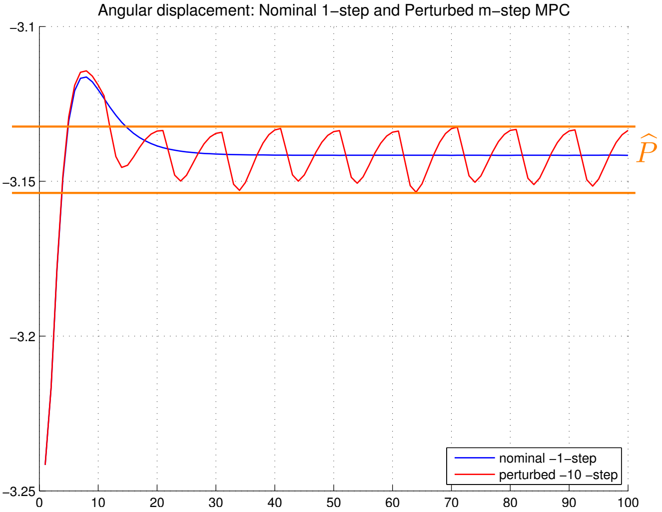

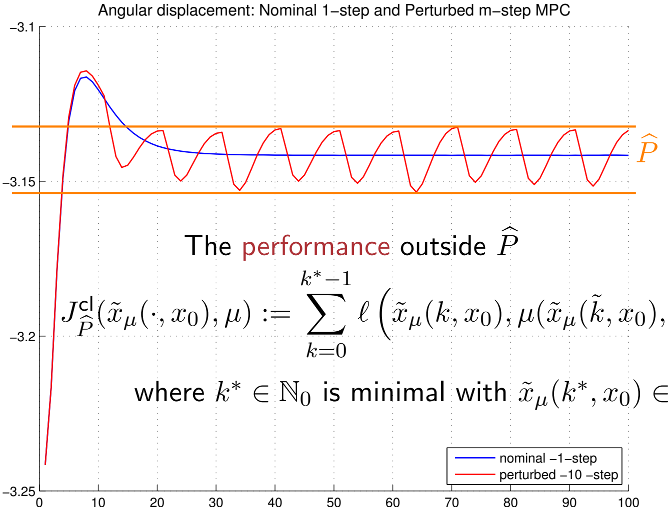

The performance outside $\widehat{P}$

$$J^{\mathsf{cl}}_{\widehat{P}}(\tilde{x}_\mu(\cdot, x_0), \mu) := \sum_{k=0}^{k^*-1} \ell\left(\tilde{x}_\mu(k, x_0), \mu(\tilde{x}_\mu(\tilde{k}, x_0), k)\right)$$

where $k^* \in \mathbb{N}_0$ is minimal with $\tilde{x}_\mu(k^*, x_0) \in \widehat{P}$

Counterpart of (P1)

Consider time-dependent $\mu : \mathbb{X} \times \mathbb{N} \to U$, function $V : X \to \mathbb{R}_0^+$ and sets $Y, P, \widehat{P}$ with appropriate invariance properties.

Assume $\exists\ \alpha \in (0,1]$ s.t. the RDPI

$$V(x_0) \geq V(\tilde{x}_\mu(m, x_0)) + \alpha \sum_{k=0}^{m-1} \ell(\tilde{x}_\mu(k, x_0), \mu(\tilde{x}_\mu(\tilde{k}, x_0), k))$$

holds $\forall x_0 \in Y \setminus P$ and $\forall$ perturbed soln $\tilde{x}_\mu(\cdot, x_0)$ with $\|d(k)\|$.

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Then $\forall x_0 \in Y \setminus \widehat{P}$ and $\forall \tilde{x}_\mu(k, x_0)$,

$$J^{\mathsf{cl}}_{\widehat{P}}(\tilde{x}_\mu(k, x_0), \mu) \leq V(x_0)/\alpha$$

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Then $\forall x_0 \in Y \setminus \widehat{P}$ and $\forall \tilde{x}_\mu(k, x_0)$,

$$J^{\mathsf{cl}}_{\widehat{P}}(\tilde{x}_\mu(k, x_0), \mu) \leq V(x_0)/\alpha$$

If (A1) holds and $\exists\ \alpha_3, \alpha_4 \in \mathcal{K}_\infty$ s.t.

$$\alpha_3(\|x\|_{x_*}) \leq V(x) \leq \alpha_4(\|x\|_{x_*}),$$

then μ renders the closed-loop system robustly stable.

Counterpart of (P1)

- Aim: using the counterpart of (P1) for perturbed system, show that $\mu_{N,m}$ and $\hat{\mu}_{N,m}$ renders the perturbed system robustly stable
- We need the corresponding α's

Uniform Continuity

Consider vector spaces Z, Y, set $A \subset Z$ and arbitrary set W

- a function $\phi : Z \to Y$ is uniformly continuous on A if $\exists\ \omega \in \mathcal{K}$ s.t. $\forall\ z_1, z_2 \in A$

$$\|\phi(z_1) - \phi(z_2)\| \leq \omega\left(\|z_1 - z_2\|\right).$$

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- a function $\phi : Z \times W \to Y$ is uniformly continuous on A uniformly in $v \in W$ if $\exists\ \omega \in \mathcal{K}$ s.t. $\forall\ z_1, z_2 \in A$ and $\forall\ v \in W$

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$$\|\phi(z_1, v) - \phi(z_2, v)\| \leq \omega\left(\|z_1 - z_2\|\right).$$

The function ω is called the modulus of continuity.

Uniform Continuity of $V_N(\cdot)$

$$\|J_N(x_1, u(\cdot)) - J_N(x_2, u(\cdot))\| \leq \omega_{J_N}(\|x_1 - x_2\|)$$

uniform continuity of J_N uniformly on $u(\cdot)$

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uniform continuity of V_N

- $\omega_{V_N} \leq \omega_{J_N}$
- $\omega_{V_N} \ll \omega_{J_N}$ for open-loop unstable and controllable

Obtaining analogous statements

Recall $\mathcal{P}_\alpha$

$$\alpha := \inf_{\lambda_0,\ldots,\lambda_{N-1},\nu} \frac{\sum_{n=0}^{N-1}\lambda_n - \nu}{\sum_{n=0}^{m-1}\lambda_n}$$

s.t. $$\sum_{n=k}^{N-1}\lambda_n \le B_{N-k}(\lambda_k) \qquad k = 0,\ldots,N-2$$

$$\nu \le \sum_{n=0}^{j-1}\lambda_{n+m} + B_{N-j}(\lambda_{j+m}) \qquad j = 0,..,N-m-1$$

$$\sum_{n=0}^{m-1}\lambda_n > 0,\ \lambda_0,\ldots,\lambda_{N-1},\nu \ge 0$$

Obtaining analogous statements

Define $\mathcal{P}_\alpha^{\mathsf{pmult}}$

$$\alpha^{\mathsf{pmult}} := \inf_{\lambda_0,\ldots,\lambda_{N-1},\nu} \frac{\sum_{n=0}^{N-1}\lambda_n - \nu}{\sum_{n=0}^{m-1}\lambda_n}$$

$$\text{s.t.} \quad \sum_{n=k}^{N-1}\lambda_n \leq B_{N-k}(\lambda_k) + \xi^{\mathsf{pmult}}, \qquad k = 0,\ldots,N-2$$

$$\nu \leq \sum_{n=0}^{j-1}\lambda_{n+m} + B_{N-j}(\lambda_{j+m}) + \xi^{\mathsf{pmult}}, \quad j = 0,..,N-m-1$$

$$\sum_{n=0}^{m-1}\lambda_n > \zeta,\ \lambda_0,\ldots,\lambda_{N-1},\nu \geq 0$$

where $\xi_k^{\mathsf{pmult}} = \sum_{j=1}^{N-k-1}\omega_{J_{N-k-j}}(\|d(k+j)\|) + \omega_{B_{N-k}}(\lambda_{k,k,0} - \lambda_{k,0,0}) + \omega_{J_{N-k}}(x_{k,k,0} - x_{k,0,0})$

Obtaining analogous statements

Define $\mathcal{P}_\alpha^{\mathsf{upd}}$

$$\alpha^{\mathsf{upd}} := \inf_{\lambda_0,\ldots,\lambda_{N-1},\nu} \frac{\sum_{n=0}^{N-1}\lambda_n - \nu}{\sum_{n=0}^{m-1}\lambda_n}$$

s.t.

$$\sum_{n=k}^{N-1}\lambda_n \leq B_{N-k}(\lambda_k) + \xi^{\mathsf{upd}}, \qquad k = 0,\ldots,N-2$$

$$\nu \leq \sum_{n=0}^{j-1}\lambda_{n+m} + B_{N-j}(\lambda_{j+m}) + \xi^{\mathsf{upd}}, \quad j = 0,..,N-m-1$$

$$\sum_{n=0}^{m-1}\lambda_n > \zeta,\ \lambda_0,\ldots,\lambda_{N-1},\nu \geq 0$$

where $\xi_k^{\mathsf{upd}} = \sum_{j=1}^{N-k-1}\omega_{V_{N-k-j}}(\|d(k+j)\|)$

Comparing $\alpha, \alpha^{\text{pmult}}, \alpha^{\text{upd}}$

- $\mathcal{P}_\alpha^{\text{pmult}}$ and $\mathcal{P}_\alpha^{\text{upd}}$ are 'perturbed' versions of $\mathcal{P}_\alpha$

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- $\mathcal{P}_\alpha^{\text{pmult}}$ and $\mathcal{P}_\alpha^{\text{upd}}$ are 'perturbed' versions of $\mathcal{P}_\alpha$

Theorem

Consider problems $\mathcal{P}_\alpha$, $\mathcal{P}_\alpha^{pmult}$ and $\mathcal{P}_\alpha^{upd}$.
Assume that the B_k, $k \in \mathbb{N}$ from (A2) are linear functions.
Then

$$\alpha^{pmult} \geq \alpha - \frac{B_{m+1}(\xi^{pmult}) + \xi^{pmult}}{\zeta}$$

$$\alpha^{upd} \geq \alpha - \frac{B_{m+1}(\xi^{upd}) + \xi^{upd}}{\zeta}$$

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$$\alpha^{upd} \geq \alpha - \frac{B_{m+1}(\xi^{upd}) + \xi^{upd}}{\zeta}$$

- lower bounds for α^{pmult} and α^{upd} in terms of α

Robust Stability rendered by $\mu_{N,m}$ and $\hat{\mu}_{N,m}$

Assume some technical assumptions.
Let α^{pmult} be the solution of $\mathcal{P}_\alpha^{\mathsf{pmult}}$ for $d(\cdot)$ and some $\zeta > 0$.

Then the RDPI

$$V_N(x_0) \geq V_N(x_{\mu_{N,m}}(m, x_0)) + \tilde{\alpha}^{\mathsf{pmult}} \sum_{k=0}^{m-1} \ell(\tilde{x}_{\mu_{N,m}}(k, x_0), \mu_{N,m}(x_0, k))$$

holds for

$$\tilde{\alpha}^{\mathsf{pmult}} = \alpha^{\mathsf{pmult}} - \frac{\sigma}{\zeta} \quad \text{where } \sigma = \sum_{j=1}^{m-1} \omega_{J_{N-j}}(\|d(j)\|)$$

Robust Stability rendered by $\mu_{N,m}$ and $\hat{\mu}_{N,m}$

Assume some technical assumptions.
Let α^{upd} be the solution of $\mathcal{P}_\alpha^{\mathsf{upd}}$ for $d(\cdot)$ and some $\zeta > 0$.

Then the RDPI

$$V_N(x_0) \geq V_N(x_{\hat{\mu}_{N,m}}(m, x_0)) + \tilde{\alpha}^{\mathsf{upd}} \sum_{k=0}^{m-1} \ell(\tilde{x}_{\hat{\mu}_{N,m}}(k, x_0), \hat{\mu}_{N,m}(x_0, k))$$

holds for

$$\tilde{\alpha}^{\mathsf{upd}} = \alpha^{\mathsf{upd}} - \frac{\sigma}{\zeta} \quad \text{where } \sigma = \sum_{j=1}^{m-1} \omega_{V_{N-j}}(\|d(j)\|)$$

Robust Stability rendered by $\mu_{N,m}$ and $\hat{\mu}_{N,m}$

- the decisive difference bet. the 2 cases: error terms
- for $\mu_{N,m}$, dependence on ω_{J_k}
  for $\hat{\mu}_{N,m}$, dependence on ω_{V_k}
- these error terms determine the bound for $\overline{d}$
  and the suboptimality index α

Numerical Example

Table: Comparison of time requirements in CPU time

m	multistep	updated
1	11.0447	11.0967
2	5.6484	10.4687
3	3.6762	10.3646
4	2.5522	10.1046
5	2.1921	9.3766
6	1.8241	8.6125
7	1.5801	7.7765
8	1.2321	7.7845
9	1.0881	7.2405
10	1.0641	6.5404
11	0.9521	6.1124
12	0.8601	5.7884
13	0.8681	5.2243

Numerical Example

Table: Suboptimality index α of the schemes for various m and iterations

m	nominal multistep	perturbed multistep	updated multistep
1	0.9908	0.8667	0.8667
2	0.9911	0.8678	0.8681
3	0.9915	0.7936	0.7955
4	0.9917	0.7672	0.7729
5	0.9916	0.7632	0.7734
6	0.9913	0.7724	0.7868
7	0.9908	0.7404	0.7629
8	0.9902	0.7103	0.7414
9	0.9895	0.7066	0.7423
10	0.9888	0.6988	0.7379
11	0.9883	0.6477	0.6953
12	0.9880	0.6183	0.6688
13	0.9879	0.6133	0.6609

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Where do we use the results?

- to be applied to a sensitivity-based m-step MPC where the sensitivity-based updates can be viewed as a less costly approximation of the updates of the updated m-step MPC

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Thank you for your attention! :)