Zubov method for controlled diffusions under state-constraints

Athena Picarelli

joint work with: Lars Grüne, Universität Bayreuth

INRIA Saclay-Ile de France & ENSTA ParisTech

New Perspectives in Optimal Control and Games
Roma, 10-12 November 2014

Outline of the talk

1. Setting
2. The domain $\mathcal{D}^{\mathcal{T},\mathcal{K}}$
3. Level set approach
4. Characterization of $\mathcal{D}^{\mathcal{T},\mathcal{K}}$ via generalized Zubov equation

1 Setting

2 The domain $\mathcal{D}^{\mathcal{T},\mathcal{K}}$

3 Level set approach

4 Characterization of $\mathcal{D}^{\mathcal{T},\mathcal{K}}$ via generalized Zubov equation

Setting

Let $(\Omega, \mathbb{F}, \{\mathbb{F}\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space. Let us consider:

$$\begin{cases} dX(t) = b(X(t), u(t))dt + \sigma(X(t), u(t))dW(t), \quad t > 0 \\ X(0) = x \in \mathbb{R}^d \end{cases} \tag{1}$$

where

- Controls:
  $\mathcal{U} := \{\text{Progr. meas. processes} : u(s) \in U, \forall s \geq 0 \text{ a.s.}\}$;
  $U \subset \mathbb{R}^m$, compact set;
- $W(\cdot)$ p-dimensional Brownian motion;
- Drift b, volatility σ.

$\rightsquigarrow$ $X_x^u(\cdot)$: unique solution of (1) associated to $u \in \mathcal{U}$.

Setting

- Target: $\mathcal{T} \subset \mathbb{R}^d$ nonempty and compact set;
- State-constraints: $\mathcal{K} \subseteq \mathbb{R}^d$ open set.

AIM: Characterize and compute, under suitable stabilizability assumptions, the domain of asymptotic controllability (with positive probability) of $\mathcal{T}$

$$\mathcal{D}^{\mathcal{T},\mathcal{K}} := \Big\{ x \in \mathbb{R}^d : \exists u \in \mathcal{U} \text{ s.t.} \\ \mathbb{P}\Big[\lim_{t \to +\infty} \mathrm{dist}_{\mathcal{T}}(X_x^u(t)) = 0 \text{ and } X_x^u(t) \in \mathcal{K}, \forall t \geq 0 \Big] > 0 \Big\}$$

1 Setting

2 The domain $\mathcal{D}^{\mathcal{T},\mathcal{K}}$

3 Level set approach

4 Characterization of $\mathcal{D}^{\mathcal{T},\mathcal{K}}$ via generalized Zubov equation

Assumptions:

(H_v) $\mathcal{T}$ is viable;

(H_s) $\mathcal{T}$ is locally exponentially stabilizable in probability:
$\exists r, \lambda > 0$ s.t. $\forall \varepsilon > 0, \exists C$ s.t.
$\forall x \in \mathcal{T}_r := \{x \in \mathbb{R}^d : \text{dist}_{\mathcal{T}}(x) \leq r\}$, $\exists u_x \in \mathcal{U}$ s.t. one has

$$\mathbb{P}\left[\sup_{t\geq 0} \text{dist}_{\mathcal{T}}(X_x^{u_x}(t))e^{\lambda t} \leq C\,\text{dist}_{\mathcal{T}}(x) \text{ and } X_x^{u_x}(t) \in \mathcal{K}, \forall t \geq 0\right] \geq 1-\varepsilon.$$

Let us define, for any $u \in \mathcal{U}$, the hitting time

$$\tau(x, u) := \inf\left\{t \geq 0 : X_x^u(t) \in \mathcal{T}_r\right\}.$$

Proposition (hitting time characterization of $\mathcal{D}^{\mathcal{T},\mathcal{K}}$)

Let assumptions (H_v)-(H_s) be satisfied. Then

$$\mathcal{D}^{\mathcal{T},\mathcal{K}} = \Big\{ x \in \mathbb{R}^d : \sup_{u \in \mathcal{U}} \mathbb{P}\Big[\tau(x,u) < +\infty \text{ and } X_x^u(t) \in \mathcal{K}, \forall t \in [0, \tau(x,u)] \Big] > 0 \Big\}$$

Sketch of the proof.

"($\subseteq$)": ok (because of the continuity of the paths).

"($\supseteq$)": Let $\bar{u} \in \mathcal{U}$ be such that $\exists \Omega_1 \subseteq \Omega$ with $\mathbb{P}[\Omega_1] > 0$ such that $\forall \omega \in \Omega_1$

$$\tau(x,\bar{u})(\omega) < +\infty \text{ and } X_x^{\bar{u}}(t)(\omega) \in \mathcal{K}, \forall t \in [0, \tau(x,\bar{u})(\omega)].$$

In particular this implies $X_x^{\bar{u}}(\tau(x,\bar{u}))(\omega) \in \mathcal{T}_r$, $\forall \omega \in \Omega_1$.
We can thus apply the local stability assumption (H_s) for this point.
The result is then proved considering the control

$$\nu(t) := \bar{u}\mathbb{1}_{\{t \le \tau(x,\bar{u})\}} + \left(\bar{u}\mathbb{1}_{\{\tau(x,\bar{u}) = +\infty\}} + u_{X_x^{\bar{u}}(\tau)} \mathbb{1}_{\{\tau(x,\bar{u}) < \infty\}} \right) \mathbb{1}_{\{t > \tau(x,\bar{u})\}}$$

under the assumption of stability under bifurcation of the set $\mathcal{U}$, that ensures $\nu \in \mathcal{U}$. □

Level set approach

1. Setting
2. The domain $\mathcal{D}^{\mathcal{T},\mathcal{K}}$
3. Level set approach
4. Characterization of $\mathcal{D}^{\mathcal{T},\mathcal{K}}$ via generalized Zubov equation

APPROACH: Characterize $\mathcal{D}^{\mathcal{T},\mathcal{K}}$ as a suitable sublevel set of a continuous function, that solves a (Zubov-type) PDE .

SOME REFERENCES:

- Camilli-Cesaroni-Grüne-Wirth '06 (unconstrained);
- Camilli-Grüne '03, Camilli-Loreti '06 (uncontrolled, unconstrained);
- Camilli-Grüne-Wirth '08 (deterministic, unconstrained), Grüne-Zidani '14 (deterministic, constrained);
- Barles-Daher-Romano '94, Bokanowski-AP-Zidani '14 (optimal control problems with maximum cost).

Level set approach

Let us consider $g : \mathbb{R}^d \times U \to [0, +\infty)$ and $h : \mathbb{R}^d \to [0, +\infty]$ such that

(H_g) $|g(x,u) - g(x',u)| \leq L_g |x - x'|$;
$g(x,u) \leq M_g$;
$g(x,u) = 0 \Leftrightarrow x \in \mathcal{T}$.
$\inf_{u \in U} g(x,u) \geq g_0 > 0, \quad \forall x \in \mathbb{R}^d \setminus \mathcal{T}_r$.

(H_h) h is locally Lipschitz in $\mathcal{K}$;
$h(x) = +\infty \Leftrightarrow x \notin \mathcal{K}$;
$h(x_n) \to +\infty, \quad \forall x_n \to x \notin \mathcal{K}$;
$h(x) = 0, \quad \forall x \in \mathcal{T}$;
$\left| e^{-h(x)} - e^{-h(x')} \right| \leq L_h |x - x'|, \quad \exists L_h \geq 0, \forall x, x' \in \mathbb{R}^d$.

The level set function we consider is:

$$v(x) := \inf_{u \in \mathcal{U}} \left\{ 1 + \mathbb{E}\left[\sup_{t \geq 0} \; - e^{-\int_0^t g(X_x^u(s), u(s))ds - h(X_x^u(t))} \right] \right\}.$$

Theorem

Let assumptions (H_s), (H_v),(H_g),(H_h) be satisfied. Then

$$\mathcal{D}^{\mathcal{T},\mathcal{K}} = \left\{ x \in \mathbb{R}^d : v(x) < 1 \right\}.$$

Level set approach

Sketch of the proof: "$(x \notin \mathcal{D}^{\mathcal{T},\mathcal{K}} \Rightarrow v(x) = 1)$"

If $x \notin \mathcal{D}^{\mathcal{T},\mathcal{K}}$ one has $\forall u \in \mathcal{U}, \omega \in \Omega$:

$$\tau(x,u) = +\infty \Rightarrow g(X_x^u(t),u) > g_0, \forall t \Rightarrow \sup_{t\geq 0}(-e^{-\int_0^t g(X_x^u(s),u(s))ds}) \geq \sup_{t\geq 0} -e^{-g_0 t} = 0$$

OR

$$\exists \bar{t} \in [0,\tau(x,u)] : X_x^u(\bar{t}) \notin \mathcal{K} \Rightarrow h(X_x^u(\bar{t})) = +\infty \Rightarrow \sup_{t\geq 0} \; -e^{-h(X_x^u(t))} = 0.$$

In any case we get $\forall u \in \mathcal{U}$

$$\sup_{t\geq 0}\left(-e^{-\int_0^t g(X_x^u(s),u(s))ds - h(X_x^u(t))}\right) = 0 \qquad \text{a.s.}$$

and thus

$$v(x) := \inf_{u\in\mathcal{U}} \left\{ 1 + \mathbb{E}\left[\sup_{t\geq 0} \; -e^{-\int_0^t g(X_x^u(s),u(s))ds - h(X_x^u(t))} \right] \right\} = 1$$

Sketch of the proof: "$(x \in D^{\mathcal{T},\mathcal{K}} \Rightarrow v(x) < 1)$"

If $x \in D^{\mathcal{T},\mathcal{K}}$ then $\exists \bar{u} \in \mathcal{U}$ s.t.

$$\mathbb{P}\Big[\tau(x,\bar{u}) < +\infty \text{ and } X_x^{\bar{u}}(t) \in \mathcal{K}, \forall t \in [0, \tau(x,\bar{u})]\Big] > 0.$$

Moreover $\exists T, M \geq 0$ s.t.

$$\mathbb{P}\Big[A_x^{\bar{u}}\Big] := \mathbb{P}\Big[\tau(x,\bar{u}) < T \text{ and } \max_{t \in [0,\tau(x,\bar{u})]} h(X_x^{\bar{u}}(t)) \leq M\Big] > 0.$$

We have

$$\sup_{u \in \mathcal{U}} \mathbb{E}\Big[\inf_{t \geq 0} e^{-\int_0^t g(X_x^u(s),u(s))ds - h(X_x^u(t))}\Big]$$

$$\geq \sup_{u \in \mathcal{U}} \int_{A_x^u} e^{-\int_0^{+\infty} g(X_x^u(\xi),u(\xi))d\xi - \max_{\xi \in [0,+\infty)} h(X_x^u(\xi))} d\mathbb{P}$$

$$\geq e^{-g_0 T - M} \sup_{u \in \mathcal{U}} \int_{A_x^u} e^{-\int_{\tau(x,u)}^{+\infty} g(X_x^u(\xi),u(\xi))d\xi - \max_{\xi \in [\tau(x,u),+\infty)} h(X_x^u(\xi))} d\mathbb{P}$$

Level set approach

Let $y \in \mathcal{T}_r$.
The exponential stability assumption ensures that there exists $\lambda > 0$ such that $\forall \varepsilon > 0$, $\exists C$ such that there exists $u_y \in \mathcal{U}$ s.t.

$$\mathbb{P}[\Omega_y] := \mathbb{P}\Big[\sup_{t\geq 0} \text{dist}_{\mathcal{T}}(X_y^{u_y}(t))e^{\lambda t} \leq C \text{ and } X_y^{u_y}(t) \in \mathcal{K}, \forall t\Big] \geq 1 - \varepsilon.$$

On Ω_y, thanks to the Lipschitz continuity of g and h, we get:

$$g(X_y^{u_y}(t), u_y(t)) \leq L_g \, \text{dist}_{\mathcal{T}}(X_y^{u_y}(t)) \leq CL_g e^{-\lambda t}$$
$$h(X_y^{u_y}(t)) \leq L \, \text{dist}_{\mathcal{T}}(X_y^{u_y}(t)) \leq CLe^{-\lambda t}$$

for any $t \geq 0$.

Level set approach

Putting the two estimates together we obtain:

$$\sup_{u\in\mathcal{U}} \mathbb{E}\left[\inf_{t\geq 0} e^{-\int_0^t g(X_x^u(s),u(s))ds - h(X_x^u(t))}\right]$$

$$\geq e^{-g_0 T - M} \sup_{u\in\mathcal{U}} \int_{A_x^u} e^{-\int_{\tau(x,u)}^{+\infty} g(X_x^u(\xi),u(\xi))d\xi - \max_{\xi\in[\tau(x,u),+\infty)} h(X_x^u(\xi))} d\mathbb{P}$$

$$\geq e^{-g_0 T - M} e^{-\frac{CL_g}{\lambda}} e^{-LC} \underbrace{\sup_{u\in\mathcal{U}} \mathbb{P}\left[A_x^u \cap \Omega_{X_x^u(\tau)}\right]}_{>0} > 0.$$

Therefore

$$v(x) = \inf_{u\in\mathcal{U}} \left\{1 + \mathbb{E}\left[\sup_{t\geq 0} - e^{-\int_0^t g(X_x^u(s),u(s))ds - h(X_x^u(t))}\right]\right\} < 1$$

□

1 Setting

2 The domain $\mathcal{D}^{\mathcal{T},\mathcal{K}}$

3 Level set approach

4 Characterization of $\mathcal{D}^{\mathcal{T},\mathcal{K}}$ via generalized Zubov equation

PROBLEM: v does not satisfy a Dynamic Programming Principle

$$(\;!!\; \mathbb{E}[\sup_{t\geq 0} \ldots] \;!!\;)$$

⬇

AUXILIARY OPTIMAL CONTROL PROBLEM: $y \in [-1, 0]$

$$w(x,y) := \inf_{u\in\mathcal{U}} \left\{ 1 + \mathbb{E}\left[\sup_{t\geq 0} \left(-e^{-\int_0^t g(X_x^u(s),u(s))ds - h(X_x^u(t))} \right) \vee y \right] \right\}.$$

One has $0 \leq w \leq 1$ and for any $x \in \mathbb{R}^d$:

$$w(x,-1) = v(x).$$

Then

$$\mathcal{D}^{\mathcal{T},\mathcal{K}} = \left\{ x \in \mathbb{R}^d : w(x,-1) < 1 \right\}.$$

Let us denote for any $x \in \mathbb{R}^d$, $y \in [-1, 0]$ and $u \in \mathcal{U}$

$$G(\cdot, x, u) = \int_0^{\cdot} g(X_x^u(s), u(s))ds$$

and

$$Y_{x,y}^u(\cdot) := e^{G(\cdot,x,u)}\left(y \vee \sup_{t\in[0,\cdot\,]} (-e^{-G(t,x,u)-h(X_x^u(t))})\right)$$

Proposition (Dynamic Programming Principle, DPP)

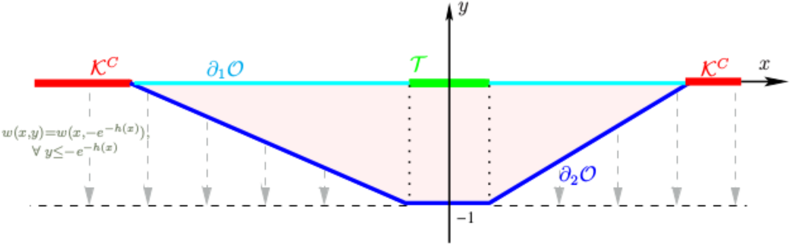
The function w is continuous and for any finite $\mathbb{F}$-stopping time $\theta \geq 0$ it satisfies

$$w(x, y) = \inf_{u\in\mathcal{U}} \mathbb{E}\Big[e^{-G(\theta,x,u)} w(X_x^u(\theta), Y_{x,y}^u(\theta)))$$
$$+ \int_0^{\theta} g(X_x^u(s), u(s))e^{-G(s,x,u)} ds\Big].$$

Let us define:

$$\mathcal{O} := \left\{(x, y) \in \mathbb{R}^{d+1} : -e^{-h(x)} < y < 0\right\},$$

$$\partial_1 \mathcal{O} := \left\{(x, y) \in \overline{\mathcal{O}} : y = 0\right\} \quad \partial_2 \mathcal{O} := \left\{(x, y) \in \overline{\mathcal{O}} : y = -e^{-h(x)}, y < 0\right\}$$



The domain $\mathcal{O}$ $(d = 1)$

Let

$$H(x,w,Dw,D^2w) := \sup_{u\in U}\Big\{ g(x,u)(w-1) - b(x,u)D_x w - \frac{1}{2}Tr[\sigma\sigma^T(x,u)D^2_{xx}w] - g(x,u)y\partial_y w \Big\}.$$

By using the DPP we can prove the following result:

Theorem

The function w is a bounded viscosity solution of the following generalized Zubov equation:

$$\begin{cases} H(x,w,Dw,D^2w) = 0 & \text{if } (x,y)\in\mathcal{O} \\ w(x,y) = 1 & \text{if } (x,y)\in\partial_1\mathcal{O} \\ -\partial_y w(x,y) = 0 & \text{if } (x,y)\in\partial_2\mathcal{O} \end{cases} \qquad (2)$$

UNIQUENESS: Is w the unique bounded viscosity solution of (2)?
(!! $g(x,u)$ degenerates near $\mathcal{T}$!!)

Uniqueness can be proved by using sub- and super- optimality principles.

Theorem

w is the unique continuous and bounded viscosity solution of equation (2) such that

$$w(x,y) = 1 + y \quad \text{on } \{(x,y) \in \overline{\mathcal{O}} : x \in \mathcal{T}\}$$

and

$$w(x,y) = 1 \quad \text{on } \{(x,y) \in \overline{\mathcal{O}} : y = 0\}.$$

Sketch of the proof.

We prove a comparison principle between any sub-solution ($\underline{w}$) and any super-solution ($\overline{w}$) satisfying

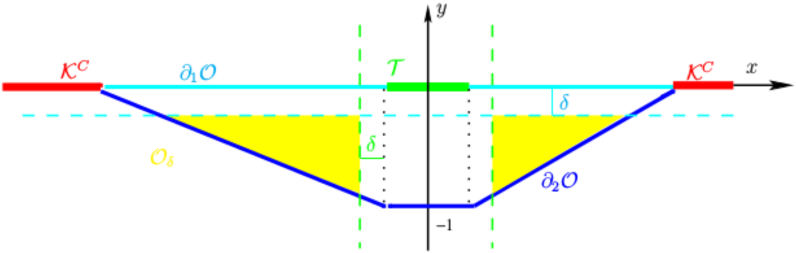
$$\underline{w}(x,y) \leq 1 + y \leq \overline{w}(x,y), \qquad \text{if } x \in \mathcal{T}$$

and

$$\underline{w}(x,0) = \overline{w}(x,0) = 1.$$

Characterization of $\mathcal{D}^{\mathcal{T},\mathcal{K}}$ via generalized Zubov equation

Let $\mathcal{O}_\delta := \{(x,y) \in \overline{\mathcal{O}} : \text{dist}_\mathcal{T}(x) > \delta, y < \delta\}$:



- Any bounded viscosity sub-solution $\underline{w}$ satisfies a sub-optimality principle in O_δ.
  Any bounded viscosity super-solution $\overline{w}$ satisfies a super-optimality principle in O_δ.

Characterization of $\mathcal{D}^{\mathcal{T},\mathcal{K}}$ via generalized Zubov equation

- We get for any $(x, y) \in \mathcal{O}_\delta,\ T \geq 0$:

$$\underline{w}(x,y) - \overline{w}(x,y) \leq \sup_{u \in \mathcal{U}} \mathbb{E}\Big[e^{-G(\theta_\delta \wedge T, x, u)} \Big(\underline{w}\big(X_x^u(\theta_\delta \wedge T), Y_{x,y}^u(\theta_\delta \wedge T)\big) - \overline{w}\big(X_x^u(\theta_\delta \wedge T), Y_{x,y}^u(\theta_\delta \wedge T)\big) \Big) \Big]$$

  where $\theta_\delta = \inf\{t \geq 0 : (X_x^u(t), Y_{x,y}^u(t)) \notin \mathcal{O}_\delta\}$.
- The proof is concluded using the properties of g and the values assumed by $\overline{w}$ and $\underline{w}$ for $x \in \mathcal{T}$ and $y = 0$, at the limit for $\delta \to 0$.

Conclusion and perspectives

Conclusions:

- Characterization of the domain of non-null asymptotic controllability as the 1-sublevel set of a continuous function;
- Link between the domain of non null asymptotic controllability and the unique viscosity of a generalized Zubov equation with mixed Dirichlet-Neumann boundary conditions.

Perspectives:

- Numerical tests and examples;
- Characterization of the domain of asymptotic controllability with a given probability: given $p \in [0,1]$

$$\mathcal{D}_p^{\mathcal{T},\mathcal{K}} := \Big\{ x \in \mathbb{R}^d : \exists u \in \mathcal{U} \text{ s.t.} \\ \mathbb{P}\Big[\lim_{t \to +\infty} \text{dist}_{\mathcal{T}}(X_x^u(t)) = 0 \text{ and } X_x^u(t) \in \mathcal{K}, \forall t \geq 0 \Big] = p \Big\}$$

(see Camilli-Cesaroni-Grüne-Wirth ('06), Camilli-Grüne ('03))

Thank you for your attention!!