

# An efficient filtered scheme for some first order Hamilton-Jacobi-Bellman equations

Smita Sahu<sup>1</sup>

joint work with Olivier Bokanowski<sup>2</sup> and Maurizio Falcone<sup>1</sup>

<sup>1</sup>SAPIENZA - Università di Roma

<sup>2</sup>Laboratoire Jacques-Louis Lions,  
Université Paris-Diderot (Paris 7)

New Prospective in Optimal Control and Games  
Rome 10-12, Nov. 2014



**SAPIENZA**  
UNIVERSITÀ DI ROMA

**ITN SADCO**

Initial Training Network

Sensitivity Analysis for Deterministic Controller Design



# Outline

- 1 Introduction
- 2 Filtered scheme
- 3 Numerical Tests
- 4 Conclusion
- 5 References

# Introduction

## Time dependent HJ equation

We are interested in computing the approximation of viscosity solution of Hamilton-Jacobi (HJ) equation:

$$\begin{cases} \partial_t v + H(x, \nabla v) = 0, & (t, x) \in (0, T) \times \mathbb{R}^d, \\ v(0, x) = v_0(x), & x \in \mathbb{R}^d. \end{cases} \quad (1)$$

(A1)  $H(x, p)$  is continuous in all its variables.

(A2)  $v_0(x)$  is Lipschitz continuous.

We aim to propose new higher order schemes, and prove their properties of consistency, stability and convergence.

Several schemes have been developed:

- Finite difference schemes (Crandall-Lions(84), Sethian(88), Osher/Shu(91), Tadmor/Lin(00)).
- Semi-Lagrangian schemes (Falcone(94,09)/ Ferretti-Carlini(03,04,13)).
- Discontinuous Galerkin approach (Hu/Shu(99), Li/Shu(05), Bokanowski/Chang/Shu(11,13,14), Cockburn(00), Capuzzo Dolcetta(83,89,90)) .
- Finite Volume schemes (Kossioris/Makridakis/Souganidis(99), Kurganov/Tadmor(00)), Abgrall(00,01).

# Monotone scheme

- **Discretization:** Let  $\Delta t > 0$  denotes the time steps and  $\Delta x > 0$  a mesh step,  $t_n = n\Delta t$ ,  $n \in [0, \dots, N]$ ,  $N \in \mathbb{N}$  and  $x_j = j\Delta x$ ,  $j \in \mathbb{Z}$ . For the given function  $u(x)$ .

Finite difference scheme (FD) Crandall-Lions (84) :

Let  $S^M$  be a **monotone FD scheme**

$$u^{n+1}(x_j) \equiv S^M(u_j^n) := u_j^n - \Delta t h^M(x_j, D^- u_j^n, D^+ u_j^n) \quad (2)$$

$$\text{with } D^\pm u_j^n := \pm \frac{u_{j\pm 1}^n - u_j^n}{\Delta x}.$$

**Assumptions on  $h^M$  :**

- (A3)  $h^M$  is Lipschitz continuous function.
- (A4) (Consistency)  $\forall x, \forall u, h^M(x, v, v) = H(x, v)$ .
- (A5) (Monotonicity) for any functions  $u, v$ ,  
 $u \leq v \implies S^M(u) \leq S^M(v)$ .

# Monotone scheme

- **Discretization:** Let  $\Delta t > 0$  denotes the time steps and  $\Delta x > 0$  a mesh step,  $t_n = n\Delta t$ ,  $n \in [0, \dots, N]$ ,  $N \in \mathbb{N}$  and  $x_j = j\Delta x$ ,  $j \in \mathbb{Z}$ . For the given function  $u(x)$ .

Finite difference scheme (FD) Crandall-Lions (84) :

Let  $S^M$  be a **monotone FD scheme**

$$u^{n+1}(x_j) \equiv S^M(u_j^n) := u_j^n - \Delta t h^M(x_j, D^- u_j^n, D^+ u_j^n) \quad (2)$$

$$\text{with } D^\pm u_j^n := \pm \frac{u_{j\pm 1}^n - u_j^n}{\Delta x}.$$

**Assumptions on  $h^M$  :**

- (A3)  $h^M$  is Lipschitz continuous function.
- (A4) (Consistency)  $\forall x, \forall u, h^M(x, v, v) = H(x, v)$ .
- (A5) (Monotonicity) for any functions  $u, v$ ,  
 $u \leq v \implies S^M(u) \leq S^M(v)$ .

## Consistency error estimate:

For any  $v \in C^2([0, T] \times \mathbb{R})$ , there exists a constant  $C_M \geq 0$  independent of  $\Delta x$  such that

$$\mathcal{E}_{SM}(v)(t, x) := \frac{v(t + \Delta t, x) - S^M(v(t, \cdot))(x)}{\Delta t} - (v_t(t, x) + H(x, v_x(t, x)))$$

$$|\mathcal{E}_{SM}(v)(t, x)| \leq C_M \left( \Delta t \|\partial_{tt} v\|_\infty + \Delta x \|\partial_{xx} v\|_\infty \right).$$

## Theorem (Crandall-Lions (84))

Let Hamiltonian  $H$  and initial data  $v_0$  be Lipschitz continuous. Let the monotone finite difference scheme (2) (with numerical hamiltonian  $h^M$  satisfies (A3)-(A5)). Let  $v_i^n := v(t_n, x_i)$ , where  $v$  is the exact solution of (1). Then there is a constant  $C$  such that for any  $n \leq T/\Delta t$  and  $i \in \mathbb{Z}$ , we have

$$|v_i^n - u_i^n| \leq C\sqrt{\Delta x}. \quad (3)$$

for  $\Delta t \rightarrow 0$ ,  $\Delta x = c\Delta t$ .

## Consistency error estimate:

For any  $v \in C^2([0, T] \times \mathbb{R})$ , there exists a constant  $C_M \geq 0$  independent of  $\Delta x$  such that

$$\mathcal{E}_{SM}(v)(t, x) := \frac{v(t + \Delta t, x) - S^M(v(t, \cdot))(x)}{\Delta t} - (v_t(t, x) + H(x, v_x(t, x)))$$

$$|\mathcal{E}_{SM}(v)(t, x)| \leq C_M \left( \Delta t \|\partial_{tt} v\|_\infty + \Delta x \|\partial_{xx} v\|_\infty \right).$$

## Theorem (Crandall-Lions (84))

Let Hamiltonian  $H$  and initial data  $v_0$  be Lipschitz continuous. Let the monotone finite difference scheme (2) (with numerical hamiltonian  $h^M$  satisfies (A3)-(A5)). Let  $v_i^n := v(t_n, x_i)$ , where  $v$  is the exact solution of (1). Then there is a constant  $C$  such that for any  $n \leq T/\Delta t$  and  $i \in \mathbb{Z}$ , we have

$$|v^n(x_i) - u^n(x_i)| \leq C\sqrt{\Delta x}. \quad (3)$$

for  $\Delta t \rightarrow 0$ ,  $\Delta x = c\Delta t$ .

# High order schemes

Let  $S^A$  denote a **high order** (possibly unstable) scheme:

$$S^A(u^n)(x) := u^n(x) - \Delta t h^A(x, D^{k,-} u, \dots, D^- u^n(x), D^+ u^n(x), \dots, D^{k,+} u^n(x)) \quad (4)$$

(for instance, the central FD scheme  $h_j^A := H\left(\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}\right)$  and TVD-RK2 time).

Assumptions on  $S^A$ :

(A6)  $h^A$  is a Lipschitz continuous function.

(A7) (*High order consistency*) There exists  $k \geq 2$ , and  $1 < \ell < k$ , for any  $v = v(t, x)$  of class  $C^{\ell+1}$ , there exists  $C_{A,\ell} \geq 0$ ,

$$\mathcal{E}_{S^A}(v)(t, x) := \left| \frac{v(t + \Delta t, x) - S^A(v(t, \cdot))(x)}{\Delta t} - (v_t(t, x) + H(x, v_x(t, x))) \right|$$

$$|\mathcal{E}_{S^A}(v)(t, x)| \leq C_{A,\ell} \left( \Delta t^\ell \|\partial_t^{\ell+1} v\|_\infty + \Delta x^\ell \|\partial_x^{\ell+1} v\|_\infty \right).$$

# Filtered scheme

It is known (Godunov's Theorem) that a monotone scheme can be at most of first order. Therefore it is needed to look for non-monotone schemes.

The difficulty is then to combine non-monotony with a convergence to viscosity solution of (1), and also obtain error estimates.

This is the core of the present work . In our approach we adapt an idea of Froese and Oberman (13) (for second order HJ equations) to treat mainly the case of evolutive first order PDEs.

# Filtered scheme

It is known (Godunov's Theorem) that a monotone scheme can be at most of first order. Therefore it is needed to look for non-monotone schemes.

The difficulty is then to combine non-monotony with a convergence to viscosity solution of (1), and also obtain error estimates.

This is the core of the present work . In our approach we adapt an idea of Froese and Oberman (13) (for second order HJ equations) to treat mainly the case of evolutive first order PDEs.

# Filtered scheme

It is known (Godunov's Theorem) that a monotone scheme can be at most of first order. Therefore it is needed to look for non-monotone schemes.

The difficulty is then to combine non-monotony with a convergence to viscosity solution of (1), and also obtain error estimates.

This is the core of the present work . In our approach we adapt an idea of Froese and Oberman (13) (for second order HJ equations) to treat mainly the case of evolutive first order PDEs.

## filtered scheme:

The scheme we propose is then

$$u_j^{n+1} \equiv S^F(u_j^n) := S^M(u_j^n) + \epsilon \Delta t F \left( \frac{S^A(u_j^n) - S^M(u_j^n)}{\epsilon \Delta t} \right) \quad (5)$$

with a proper initialization of  $u_j^0$ .

- Where  $\epsilon = \epsilon(\Delta t, \Delta x) > 0$  is the switching parameter that will satisfy

$$\lim_{(\Delta t, \Delta x) \rightarrow 0} \epsilon = 0.$$

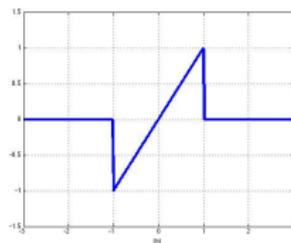
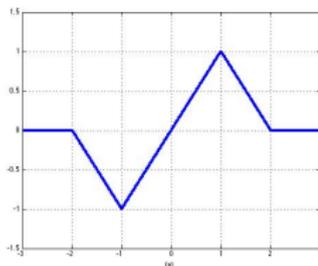
More precision on the choice of  $\epsilon$  will be given later on.

## Filtered function

Instead of the Froese and Oberman's filter function i.e.

$F(x) = \text{sign}(x) \max(1 - ||x| - 1|, 0)$ , we used **new filter function i.e.**

$F(x) := x 1_{|x| \leq 1}$ :



Froese and Oberman's filter function and new filter function.

- To keep high order when  $|h^A - h^M| \leq \epsilon$  i.e.  $\left| \frac{S^A(u_j^n) - S^M(u_j^n)}{\epsilon \Delta t} \right| \leq 1 \Rightarrow S^F \equiv S^A$
- Otherwise  $F = 0$  and  $S^F = S^M$ , i.e., the monotone scheme itself.

## Consistency error estimate

For any regular function  $v = v(t, x)$ , for all  $x \in \mathbb{R}$  and  $t \in [0, T]$ , we have

$$\begin{aligned} |\mathcal{E}_{S^F}(v)(t, x)| &= \left| \frac{v(t + \Delta t, x) - S^F(v(t, \cdot))(x)}{\Delta t} - (v_t + H(x, v_x)) \right| \\ &\leq C_M \left( \Delta t \|\partial_{tt} v\|_\infty + \Delta x \|\partial_{xx} v\|_\infty \right) + \epsilon C_0 \Delta t. \end{aligned} \quad (6)$$

### Definition ( $\epsilon$ -monotonicity)

Filtered scheme is  $\epsilon$ -monotone i.e. For any functions  $u, v$ ,

$$u \leq v \implies S(u) \leq S(v) + C\epsilon\Delta t,$$

where  $C$  is constant and  $\epsilon \rightarrow 0$  as  $\Delta t \rightarrow 0$ .

## Consistency error estimate

For any regular function  $v = v(t, x)$ , for all  $x \in \mathbb{R}$  and  $t \in [0, T]$ , we have

$$\begin{aligned} |\mathcal{E}_{S^F}(v)(t, x)| &= \left| \frac{v(t + \Delta t, x) - S^F(v(t, \cdot))(x)}{\Delta t} - (v_t + H(x, v_x)) \right| \\ &\leq C_M \left( \Delta t \|\partial_{tt} v\|_\infty + \Delta x \|\partial_{xx} v\|_\infty \right) + \epsilon C_0 \Delta t. \end{aligned} \quad (6)$$

## Definition ( $\epsilon$ -monotonicity)

Filtered scheme is  $\epsilon$ -monotone i.e. For any functions  $u, v$ ,

$$u \leq v \implies S(u) \leq S(v) + C\epsilon\Delta t,$$

where  $C$  is constant and  $\epsilon \rightarrow 0$  as  $\Delta t \rightarrow 0$ .

## Limiter correction:

The filtered scheme works fine for linear PDEs but may switch to first order for nonlinear PDEs in particular when some viscosity aspect occurs. Therefore we have added a limiter correction when some viscous behavior is detected.

1D front propagation case:  $H(x, v_x) := \max_{a \in A}(f(x, a)v_x)$ .

we say that a viscosity aspect occurs at a minimum  $x_j$  if

$$\min_a f(x_j, a) \leq 0 \quad \text{and} \quad \max_a f(x_j, a) \geq 0. \quad (7)$$

In that case, the limiter is such that the scheme iterate will not go below the local minima around the current point:

$$u_j^{n+1} \geq u_{\min,j} := \min(u_{j-1}^n, u_j^n, u_{j+1}^n), \quad (8)$$

and, in the same way, also demand that

$$u_j^{n+1} \leq u_{max,j} := \max(u_{j-1}^n, u_j^n, u_{j+1}^n). \quad (9)$$

If we consider the high-order scheme to be of the form (4) then, at points where holds, the limiting process amounts to replace  $h_j^A$  by  $\bar{h}_j^A$  such that:

$$\bar{h}_i^A := \max \left( \min \left( h^A(u^n)_j, \frac{u_j^n - u_{min,j}}{\Delta t} \right), \frac{u_j^n - u_{max,j}}{\Delta t} \right)$$

## 2D limiter

$$u_{ij}^{n+1} = \min(\max(S^A(u^n)_{ij}, u_{ij}^{min}), u_{ij}^{max}),$$

where

$$u_{ij}^{min} = \min(u_{ij}^n, u_{i\pm 1,j}^n, u_{i,j\pm 1}^n) \quad \text{and} \quad u_{ij}^{max} = \max(u_{ij}^n, u_{i\pm 1,j}^n, u_{i,j\pm 1}^n).$$

# Convergence Theorem

## Theorem

Assume (A1)-(A2), and  $v_0$  bounded. We assume also that  $S^M$  satisfies (A3)-(A5), and  $|F| \leq 1$ . Let  $u^n$  denote the filtered scheme (5). Let  $v_j^n := v(t_n, x_j)$  where  $v$  is the exact solution of (1). Assume

$$0 < \epsilon \leq c_0 \sqrt{\Delta x} \quad (10)$$

for some constant  $c_0 > 0$ .

(i) The scheme  $u^n$  satisfies the Crandall-Lions estimate

$$\|u^n - v^n\|_\infty \leq C\sqrt{\Delta x}, \quad \forall n = 0, \dots, N. \quad (11)$$

for some constant  $C$  independent of  $\Delta x$ .

## Theorem ((Cont.))

(ii) (First order convergence for classical solutions.) If furthermore the exact solution  $v$  belongs to  $C^2([0, T] \times \mathbb{R})$ , and  $\epsilon \leq c_0 \Delta x$  (instead of (10)). Then it holds

$$\|u^n - v^n\|_\infty \leq C \Delta x, \quad n = 0, \dots, N, \quad (12)$$

for some constant  $C$  independent of  $\Delta x$ .

## Theorem ((Cont.))

(iii) (Local high-order consistency.) Let  $\mathcal{N}$  be a neighborhood of a point  $(t, x) \in (0, T) \times \mathbb{R}$ . Assume that  $S^A$  is a high order scheme satisfying (A7) for some  $k \geq 2$ . Let  $1 \leq \ell \leq k$  and  $v$  be a  $C^{\ell+1}$  function on  $\mathcal{N}$ . Assume that

$$(C_{A,1} + C_M) \left( \|v_{tt}\|_{\infty} \Delta t + \|v_{xx}\|_{\infty} \Delta x \right) \leq \epsilon. \quad (13)$$

Then, for sufficiently small  $t_n - t$ ,  $x_j - x$ ,  $\Delta t$ ,  $\Delta x$ , it holds

$$S^F(v^n)_j = S^A(v^n)_j$$

and, in particular, a local high-order consistency error for the filtered scheme  $S^F$ :

$$\mathcal{E}_{S^F}(v^n)_j \equiv \mathcal{E}_{S^A}(v^n)_j = O(\Delta x^{\ell})$$

(the consistency error  $\mathcal{E}_{S^A}$  is defined as before).

## Proof.

(i) Let  $w_j^{n+1} = S^M(w^n)_j$  be defined with the monotone scheme only, with  $w_j^0 = v_0(x_j) = u_j^0$ . By definitions,

$$u_j^{n+1} - w_j^{n+1} = S^M(u^n)_j - S^M(w^n)_j + \epsilon \Delta t F(\cdot)$$

Hence, by using the monotonicity of  $S^M$ ,

$$\max_j |u_j^{n+1} - w_j^{n+1}| \leq \max_j |u_j^n - w_j^n| + \epsilon \Delta t,$$

and by recursion, for  $n \leq N$ ,

$$\max_j |u_j^n - w_j^n| \leq \epsilon n \Delta t \leq T \epsilon.$$

(Cont.)

On the other hand, by Crandall and Lions, an error estimate holds for the monotone scheme:

$$\max_j |w_j^n - v_j^n| \leq C\sqrt{\Delta x},$$

for some  $C \geq 0$ . By summing up the previous bounds, we deduce

$$\max_j |u_j^n - v_j^n| \leq C\sqrt{\Delta x} + T\epsilon,$$

and together with the assumption on  $\epsilon$ , it gives the desired result.

(Cont.)

(ii) Let  $\mathcal{E}_j^n := \frac{v_j^{n+1} - S^M(v^n)_j}{\Delta t}$ . If the solution is  $C^2$  regular with bounded second order derivatives, then the consistency error is bounded by

$$|\mathcal{E}_j^n| \leq C_M(\Delta t + \Delta x). \quad (14)$$

Hence

$$\begin{aligned} |u_j^{n+1} - v_j^{n+1}| &= |S^M(u^n)_j - S^M(v^n)_j + \Delta t \mathcal{E}_j^n + \Delta t \epsilon F(\cdot)| \\ &\leq \|u^n - v^n\|_\infty + \Delta t \|\mathcal{E}^n\|_\infty + \Delta t \epsilon. \end{aligned}$$

(Cont.)

By recursion, for  $n\Delta t \leq T$ ,

$$\|u^n - v^n\|_\infty \leq \|u^0 - v^0\|_\infty + T \left( \max_{0 \leq k \leq N-1} \|\mathcal{E}^k\|_\infty + \epsilon \right).$$

By assumption on  $\epsilon$ , the bound and the fact that  $\Delta t = O(\Delta x)$  (CFL condition) we get the desired result.

(iii) To prove that  $S^F(v^n)_j = S^A(v^n)_j$ , one has to check that

$$\frac{|S^A(v^n)_j - S^M(v^n)_j|}{\epsilon \Delta t} \leq 1$$

as  $(\Delta t, \Delta x) \rightarrow 0$ .

(Cont.)

By using the consistency error definitions,

$$\begin{aligned} \frac{|S^A(v^n)_j - S^M(v^n)_j|}{\Delta t} &= \left| \frac{v_j^{n+1} - S^M(v^n)_j}{\Delta t} - (v_t(t_n, x_j) + H(x_j, v_x(t_n, x_j))) \right. \\ &\quad \left. - \left( \frac{v_j^{n+1} - S^A(v^n)_j}{\Delta t} - (v_t(t, x) + H(x_j, v_x(t_n, x_j))) \right) \right| \\ &\leq |\mathcal{E}_{S^A}(v^n)_j| + |\mathcal{E}_{S^M}(v^n)_j| \\ &\leq (C_{A,1} + C_M)(\Delta t \|v_{tt}\|_\infty + \Delta x \|v_{xx}\|_\infty) \end{aligned}$$

Hence the desired result follows.

# Filtered Semi-Lagrangian scheme

Let us consider

$$H(x, p) := \min_{b \in B} \max_{a \in A} \{-f(x, a, b) \cdot p - \ell(x, a, b)\},$$

where  $A \subset \mathbb{R}^m$  and  $B \subset \mathbb{R}^n$  are non-empty compact sets (with  $m, n \geq 1$ ),

$f : \mathbb{R}^d \times A \times B \rightarrow \mathbb{R}^d$  and  $\ell : \mathbb{R}^d \times A \times B \rightarrow \mathbb{R}$  are Lipschitz continuous w.r.t.  $x$ :  $\exists L \geq 0, \forall (a, b) \in A \times B, \forall x, y$ :

$$\max(|f(x, a, b) - f(y, a, b)|, |\ell(x, a, b) - \ell(y, a, b)|) \leq L|x - y|.$$

Let  $[u]$  denote the  $P^1$ -interpolation of  $u$  in dimension one on the mesh  $(x_j)$ , i.e.

$$x \in [x_j, x_{j+1}] \quad \Rightarrow \quad [u](x) := \frac{x_{j+1} - x}{\Delta x} u_j + \frac{x - x_j}{\Delta x} u_{j+1}.$$

Then a monotone SL scheme can be defined as follows:

$$S^M(u^n)_j := \min_{a \in A} \max_{b \in B} \left( [u^n](x_j + \Delta t f(x_j, a, b)) + \Delta t \ell(x_j, a, b) \right). \quad (15)$$

A filtered scheme based on SL can then be defined by (5) and (15). Convergence result as well as error estimates could also be obtained in this framework. (For error estimates for the monotone SL scheme, we refer to FF 14 and Soravia 98)

## Tuning of the parameter $\epsilon$

- In order to be in the convergence setting  $\epsilon$  should be small enough i.e.  $\epsilon \leq O(\sqrt{\Delta x})$ . Thus we have, Upper bound  $\epsilon \leq C\sqrt{\Delta x}$ , with constant  $C > 0$ .
- The scheme

$$u_i^{n+1} = S^M(u_i^n) + \epsilon \Delta t F \left( \frac{S^A(u_i^n) - S^M(u_i^n)}{\epsilon \Delta t} \right)$$

has to switch to high order scheme when some regularity is detected, i.e., when

$$\left| \frac{(S^A(u_i^n) - S^M(u_i^n))}{\epsilon \Delta t} \right| = \left| \frac{(h^A(u_i^n) - h^M(u_i^n))}{\epsilon} \right| \leq 1.$$

If we assume that  $v$  is regular enough,

## Tuning of the parameter $\epsilon$

- In order to be in the convergence setting  $\epsilon$  should be small enough i.e.  $\epsilon \leq O(\sqrt{\Delta x})$ . Thus we have, Upper bound  $\epsilon \leq C\sqrt{\Delta x}$ , with constant  $C > 0$ .
- The scheme

$$u_i^{n+1} = S^M(u_i^n) + \epsilon \Delta t F \left( \frac{S^A(u_i^n) - S^M(u_i^n)}{\epsilon \Delta t} \right)$$

has to switch to high order scheme when some regularity is detected, i.e., when

$$\left| \frac{(S^A(u_i^n) - S^M(u_i^n))}{\epsilon \Delta t} \right| = \left| \frac{(h^A(u_i^n) - h^M(u_i^n))}{\epsilon} \right| \leq 1.$$

If we assume that  $v$  is regular enough,

## Tuning of the parameter $\epsilon$

- In order to be in the convergence setting  $\epsilon$  should be small enough i.e.  $\epsilon \leq O(\sqrt{\Delta x})$ . Thus we have, Upper bound  $\epsilon \leq C\sqrt{\Delta x}$ , with constant  $C > 0$ .
- The scheme

$$u_i^{n+1} = S^M(u_i^n) + \epsilon \Delta t F \left( \frac{S^A(u_i^n) - S^M(u_i^n)}{\epsilon \Delta t} \right)$$

has to switch to high order scheme when some regularity is detected, i.e., when

$$\left| \frac{(S^A(u_i^n) - S^M(u_i^n))}{\epsilon \Delta t} \right| = \left| \frac{(h^A(u_i^n) - h^M(u_i^n))}{\epsilon} \right| \leq 1.$$

If we assume that  $v$  is regular enough,

## Tuning of the parameter $\epsilon$

- In order to be in the convergence setting  $\epsilon$  should be small enough i.e.  $\epsilon \leq O(\sqrt{\Delta x})$ . Thus we have, Upper bound  $\epsilon \leq C\sqrt{\Delta x}$ , with constant  $C > 0$ .
- The scheme

$$u_i^{n+1} = S^M(u_i^n) + \epsilon \Delta t F \left( \frac{S^A(u_i^n) - S^M(u_i^n)}{\epsilon \Delta t} \right)$$

has to switch to high order scheme when some regularity is detected, i.e., when

$$\left| \frac{(S^A(u_i^n) - S^M(u_i^n))}{\epsilon \Delta t} \right| = \left| \frac{(h^A(u_i^n) - h^M(u_i^n))}{\epsilon} \right| \leq 1.$$

If we assume that  $v$  is regular enough,

by using Taylor series expansions and the high order property we have  $h^A(v)_i = H(v_x(x_i)) + O(\Delta x^2)$ , and also,

$$Dv_i^\pm = v_x(x_i) \pm \frac{1}{2} v_{xx}(x_i) \Delta x + O(\Delta x^2).$$

$$h^M(x_i, Dv_i^-, Dv_i^+) = H(v_x(x_i)) + \frac{1}{2} v_{xx}(x_i) \Delta x \left( \frac{\partial h^M(v)_i}{\partial Dv^+} - \frac{\partial h^M(v)_i}{\partial Dv^-} \right) + O(\Delta x^2).$$

Therefore,

$$\frac{1}{2} |v_{xx}(x_i)| \left| \frac{\partial h^M(v)_i}{\partial Dv^+} - \frac{\partial h^M(v)_i}{\partial Dv^-} \right| \Delta x \leq \epsilon.$$

by using Taylor series expansions and the high order property we have  $h^A(v)_i = H(v_x(x_i)) + O(\Delta x^2)$ , and also,

$$Dv_i^\pm = v_x(x_i) \pm \frac{1}{2} v_{xx}(x_i) \Delta x + O(\Delta x^2).$$

$$h^M(x_i, Dv_i^-, Dv_i^+) = H(v_x(x_i)) + \frac{1}{2} v_{xx}(x_i) \Delta x \left( \frac{\partial h^M(v)_i}{\partial Dv^+} - \frac{\partial h^M(v)_i}{\partial Dv^-} \right) + O(\Delta x^2).$$

Therefore,

$$\frac{1}{2} |v_{xx}(x_i)| \left| \frac{\partial h^M(v)_i}{\partial Dv^+} - \frac{\partial h^M(v)_i}{\partial Dv^-} \right| \Delta x \leq \epsilon.$$

# Tuning of the parameter $\epsilon$

## Bounds for $\epsilon$ :

- **Upper bound:**  $\epsilon \leq C\sqrt{\Delta x}$ , with constant  $C > 0$  (  $\Rightarrow$  to have error estimates and convergence ).
- **Lower bound:**  $\frac{1}{2} |v_{xx}(x_i)| \left| \frac{\partial h^M(v)_i}{\partial DV^+} - \frac{\partial h^M(v)_i}{\partial DV^-} \right| \Delta x \leq \epsilon$ . (  $\Rightarrow$  to have high-order behavior )

$\Rightarrow$  Typically the choice  $\epsilon := C\Delta x$  is made, for some constant  $C$  of the order of  $\|v_{xx}\|_{L^\infty}$ .

# Tuning of the parameter $\epsilon$

## Bounds for $\epsilon$ :

- **Upper bound:**  $\epsilon \leq C\sqrt{\Delta x}$ , with constant  $C > 0$  (  $\Rightarrow$  to have error estimates and convergence ).
- **Lower bound:**  $\frac{1}{2} \|v_{xx}(x_i)\| \left| \frac{\partial h^M(v)_i}{\partial Dv^+} - \frac{\partial h^M(v)_i}{\partial Dv^-} \right| \Delta x \leq \epsilon$ . (  $\Rightarrow$  to have high-order behavior )

$\Rightarrow$  Typically the choice  $\epsilon := C\Delta x$  is made, for some constant  $C$  of the order of  $\|v_{xx}\|_{L^\infty}$ .

# Tuning of the parameter $\epsilon$

## Bounds for $\epsilon$ :

- **Upper bound:**  $\epsilon \leq C\sqrt{\Delta x}$ , with constant  $C > 0$  (  $\Rightarrow$  to have error estimates and convergence ).
- **Lower bound:**  $\frac{1}{2} \|v_{xx}(x_i)\| \left| \frac{\partial h^M(v)_i}{\partial Dv^+} - \frac{\partial h^M(v)_i}{\partial Dv^-} \right| \Delta x \leq \epsilon$ . (  $\Rightarrow$  to have high-order behavior )

$\Rightarrow$  Typically the choice  $\epsilon := C\Delta x$  is made, for some constant  $C$  of the order of  $\|v_{xx}\|_{L^\infty}$ .

# Eikonal equation in 1D

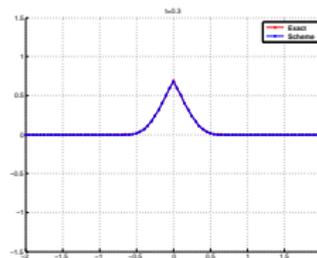
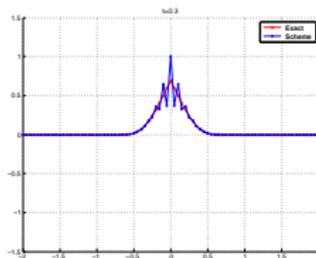
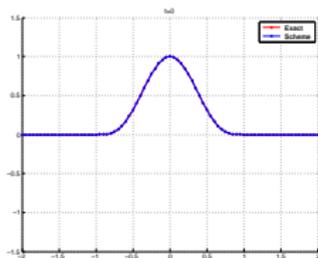
## Example 1.

$$\begin{cases} v_t + |v_x| = 0, & t > 0, x \in (-2, 2), \\ v(0, x) = v_0(x), := \max(0, 1 - x^2)^4, & x \in (-2, 2), \end{cases}$$

with periodic boundary condition on  $(-2, 2)$ , terminal time  $T = 0.3$

		Filter $\epsilon = 5\Delta x$		CFD		ENO2	
$M$	$N$	$L^2$ error	order	$L^2$ error	order	$L^2$ error	order
40	8	1.06E-02	1.91	1.18E-01	-	1.65E-01	1.72
80	15	3.42E-03	1.56	1.14E-01	0.06	4.43E-03	1.90
160	30	7.12E-04	2.26	1.13E-01	0.00	1.20E-03	1.88
320	59	1.74E-04	2.03	1.13E-01	0.00	3.25E-04	1.89
640	118	4.32E-05	2.01	1.13E-01	0.00	8.29E-05	1.97

**Table :**  $L^2$  errors for filtered scheme, Central finite difference (CFD) scheme, ENO (2nd order) scheme with RK2 in time.



Initial data (left), and plots at time  $T = 0.3$ , by Central finite difference scheme - middle - and Filtered scheme - right ( $M = 160$  mesh points).

# Eikonal equation in 1D

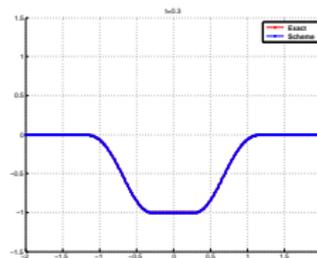
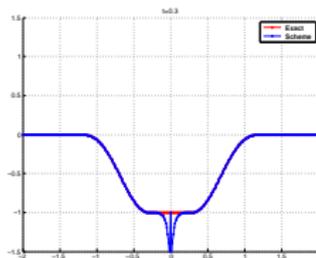
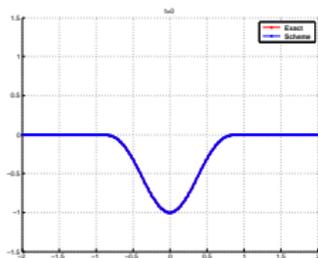
## Example 2.

$$\begin{cases} v_t + |v_x| = 0, & t > 0, x \in (-2, 2), \\ v(0, x) = v_0(x), := -\max(0, 1 - x^2)^4, & x \in (-2, 2), \end{cases}$$

with periodic boundary condition on  $(-2, 2)$ , terminal time  $T = 0.3$

Errors		Filter $\epsilon = 5\Delta x$		CFD		ENO2	
$M$	$N$	$L^2$ error	order	$L^2$ error	order	$L^2$ error	order
40	8	1.24E-02	1.93	2.02E-02	1.42	2.62E-02	1.49
80	16	3.05E-03	2.02	8.76E-03	1.20	8.08E-03	1.70
160	32	7.65E-04	2.00	1.04E-02	-0.24	2.52E-03	1.68
320	64	1.90E-04	2.01	1.23E-01	-3.57	7.88E-04	1.69
640	128	4.76E-05	2.00	1.03E+02	-9.70	2.47E-04	1.67

**Table :**  $L^2$  errors for filtered scheme, Central finite difference (CFD) scheme, ENO (2nd order) scheme with RK2 in time.



Initial data (left), and plots at time  $T = 0.3$ , by Central finite difference scheme - middle - and Filtered scheme - right ( $M = 160$  mesh points).

# 1D steady equation

Example 3. (as in Abgrall [6])  $\begin{cases} |v_x| = f(x), \text{ on } (0, 1) \\ v(0) = v(1) = 0 \end{cases} \quad f(x) := 3x^2 + a, a := \frac{1-2x_0^3}{2x_0-1},$

$$x_0 := \frac{\sqrt[3]{2}+2}{4\sqrt[3]{2}}.$$

M	Filter ( $\epsilon = 5\Delta x$ )		CFD		ENO	
	$L^\infty$ error	order	$L^\infty$ error	order	$L^\infty$ error	order
100	1.15E-03	2.25	+ inf	-	3.54E-02	0.99
200	2.68E-04	2.10	+ inf	-	1.78E-02	1.00
400	8.74E-05	1.62	+ inf	-	8.89E-03	1.00
800	4.20E-05	1.06	+ inf	-	4.45E-03	1.00

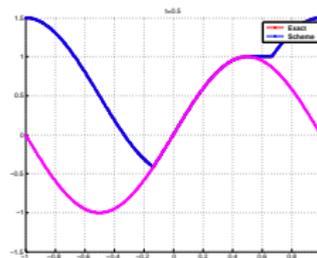
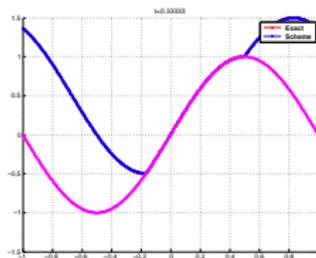
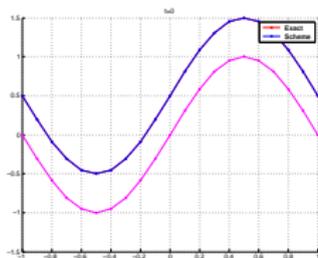
Table :  $L^\infty$  Errors for filtered scheme, CFD scheme, and RK2-2nd order ENO scheme.  $\Rightarrow$  **Filter can stabilize an otherwise unstable scheme**

# Advection with obstacle

Example 4. (as in Bokanowski et al. [9])  $\begin{cases} \min(v_t + v_x, g(x)) = 0, & t > 0, x \in [-1, 1], \\ u(0, x) = 0.5 + \sin(\pi x) & x \in [-1, 1], \end{cases}$   
 with periodic boundary condition,  $g(x) = \sin(\pi x)$ , and where the terminal time  $T = 0.5$ .

Errors		Filter $\epsilon = 5\Delta x$		CFD		ENO2	
$M$	$N$	$L^\infty$ error	order	$L^\infty$ error	order	$L^\infty$ error	order
40	20	7.93E-03	2.03	1.63E-02	1.54	2.14E-02	1.59
80	40	1.84E-03	2.10	2.98E-02	-0.87	7.75E-03	1.46
160	80	3.92E-04	2.24	4.46E-02	1.03	1.07E-03	2.86
320	160	9.67E-05	2.02	8.02E-03	0.86	2.72E-04	1.97
640	320	2.40E-05	2.01	4.10E-03	0.97	6.92E-05	1.98

Table : Local  $L^\infty$  errors for filtered scheme, Central finite difference (CFD) scheme, ENO (second order) scheme and RK2 in time.



Initial data (left), and plots at time  $T = 0.3$  - middle and  $T = 0.5$  - right by filtered scheme.

# Advection in 2D

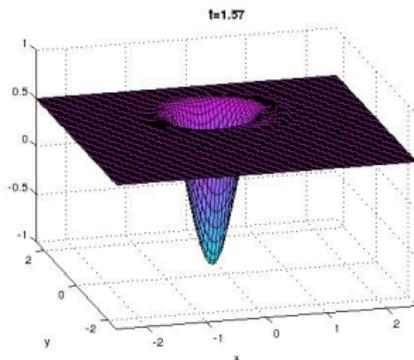
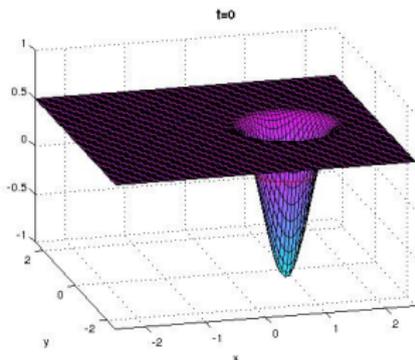
**Example 5.** We consider the domain  $\Omega = (-2.5, 2.5)^2$  and

$$\begin{cases} v_t - yv_x + xv_y = 0, & (x, y) \in \Omega, t > 0, \\ v(0, x, y) = v_0(x, y) = v_0(x, y) := 0.5 - 0.5 \max(0, \frac{1-(x-1)^2-y^2}{1-r_0^2})^4 \end{cases}$$

$r_0 = 0.5$  and with Dirichlet boundary condition  $v(t, x) = 0.5, x \in \partial\Omega, T := \frac{\pi}{2}$ .

		Filter ( $\epsilon = 20\Delta x$ )		CFD		ENO2	
$Mx = Nx$	$N$	$L^2$ error	order	$L^2$ error	order	$L^2$ error	order
20	20	5.05E-01	-	5.05E-01	-	6.99E-01	-
40	40	1.48E-01	1.77	1.48E-01	1.77	4.66E-01	0.58
80	80	3.77E-02	1.98	3.77E-02	1.98	2.04E-01	1.19
160	160	9.40E-03	2.00	9.40E-03	2.00	5.50E-02	1.89
320	320	2.34E-03	2.01	2.34E-03	2.01	1.29E-02	2.10

**Table :**  $L^2$  errors for filtered scheme, Central finite difference (CFD) scheme, ENO (second order) scheme and RK2 in time.



Initial data (left), and plots at time  $T = \pi/2$ , by Filtered scheme ( $M = 160$  mesh points).

# Eikonal in 2D

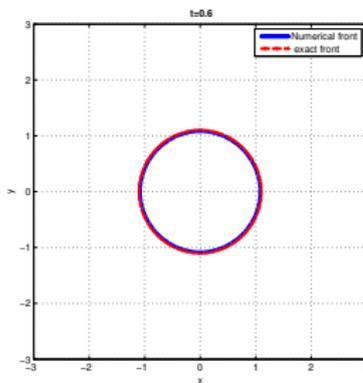
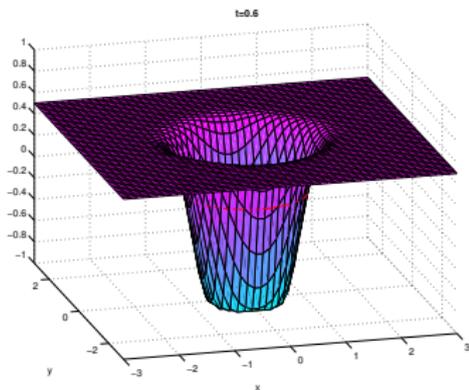
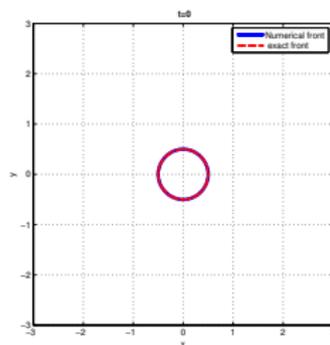
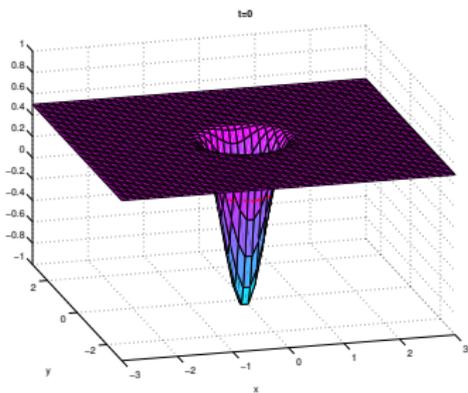
**Example 6.** In this example we consider HJB equation with smooth initial data and  $\Omega = (-3, 3)^2$

$$\begin{cases} v_t + |\nabla v| = 0, & (x, y) \in \Omega, t > 0, \\ v(0, x, y) = v_0(x, y) = v_0(x, y) := 0.5 - 0.5 \max(0, \frac{1-(x-1)^2-y^2}{1-r_0^2})^4, \end{cases}$$

where  $|\cdot|$  is the Euclidean norm and  $r_0 = 0.5$  with Dirichlet boundary conditions.

		Filter ( $\epsilon = 20\Delta x$ )		CFD		ENO2	
$Mx = Nx$	$N$	$L^2$ error	order	$L^2$ error	order	$L^2$ error	order
25	25	3.39E-01	-	4.54E-01	-	4.22E-01	-
50	50	1.14E-01	1.57	2.11E-01	1.11	1.57E-01	1.42
100	100	2.77E-02	2.04	8.89E-02	1.24	5.12E-02	1.62
200	200	6.81E-03	2.02	3.99E-02	1.16	.48E-02	1.80
400	400	1.70E-03	2.00	1.87E-02	1.10	4.34E-03	1.77

**Table :**  $L^2$  errors for filtered scheme, Central finite difference (CFD) scheme, ENO (second order) scheme and RK2 in time.



# Eikonal in 2D

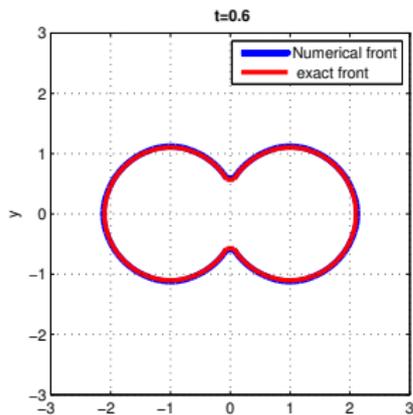
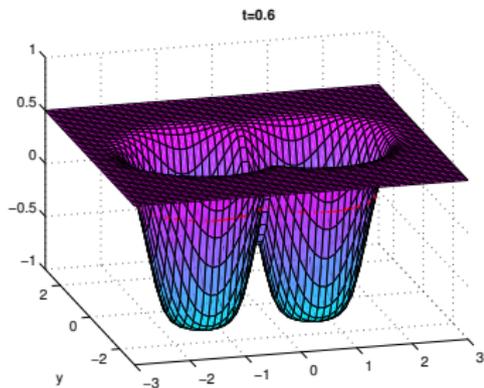
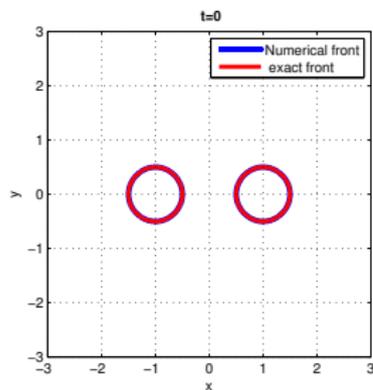
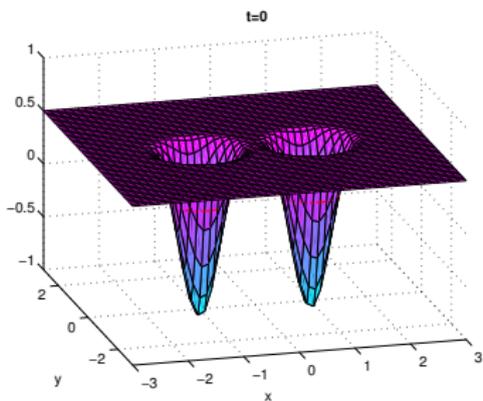
Example 7.  $\Omega = (-3, 3)^2$

$$\left\{ v(0, x, y) = v_0(x, y) = 0.5 - 0.5 \max \left( \max(0, \frac{1-(x-1)^2-y^2}{1-r_0^2})^4, \max(0, \frac{1-(x+1)^2-y^2}{1-r_0^2})^4 \right) \right\}.$$

where  $|\cdot|$  is the Euclidean norm and  $A_{\pm} := (\pm 1, 0)$  with Dirichlet boundary conditions and CFL condition  $\mu = 0.37$

		Filter ( $\epsilon = 20\Delta x$ )		CFD		ENO2	
$Mx = Nx$	$N$	$L^2$ error	order	$L^2$ error	order	$L^2$ error	order
25	25	5.39E-01	-	3.73E-01	-	4.22E-01	-
50	50	1.82E-01	1.57	1.42E-01	1.39	1.57E-01	1.42
100	100	3.72E-02	2.29	4.72E-02	1.59	5.12E-02	1.62
200	200	9.36E-03	1.99	1.66E-02	1.51	1.48E-02	1.80
400	400	2.36E-03	1.99	7.23E-03	1.20	4.34E-03	1.77

Table :  $L^2$  errors for filtered scheme, Central finite difference (CFD) scheme, ENO (second order) scheme and RK2 in time.



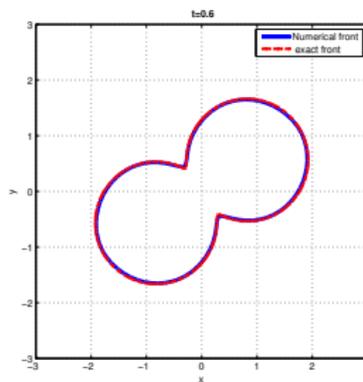
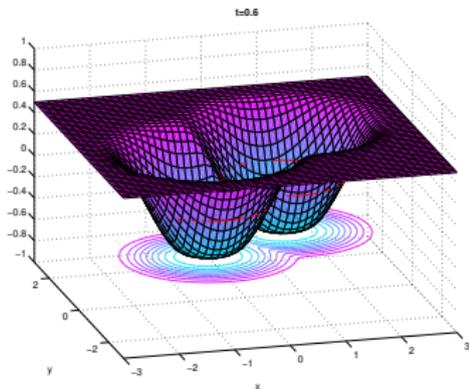
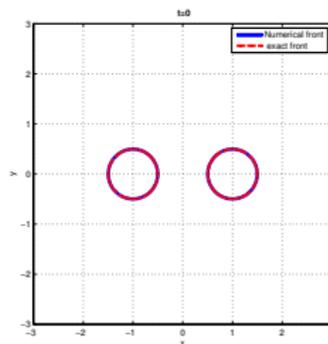
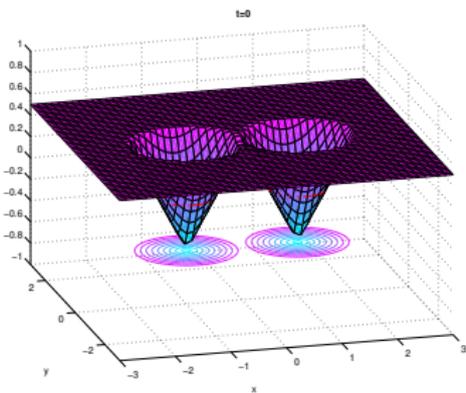
## Example 8:

$$\Omega = (-3, 3)^2$$

$$\begin{cases} v_t + -yv_x + xv_y + |\nabla v| = 0, & (x, y) \in \Omega, t > 0, \\ v(0, x, y) = v_0(x, y) = 0.5 - 0.5 \max \left( \max(0, \frac{1-(x-1)^2-y^2}{1-r_0^2})^4, \max(0, \frac{1-(x+1)^2-y^2}{1-r_0^2})^4 \right). \end{cases}$$

		Filter ( $\epsilon = 10\Delta x$ )		CFD		ENO2	
$Mx = Nx$	$N$	$L^2$ error	order	$L^2$ error	order	$L^2$ error	order
25	25	1.02E-01	-	1.11E-01	-	1.14E-01	-
50	50	2.12E-02	2.45	1.99E-02	2.48	2.12E-02	2.43
100	100	9.02E-03	1.23	2.04E-02	-0.03	3.67E-03	2.53
200	200	1.90E-03	2.25	1.27E-02	0.69	8.61E-04	2.09
400	400	3.67E-04	2.38	1.13E-02	0.17	2.12E-04	2.02

**Table :**  $L^2$  errors for filtered scheme, Central finite difference (CFD) scheme, ENO (second order) scheme and RK2 in time.



# Conclusion

- A general and simple presentation of filtered scheme and easy to implement.
  - Convergence of filtered scheme is confirmed. Error estimate is of  $O(\sqrt{\Delta x})$  and numerically observed that  $O(\Delta x^2)$  behavior in smooth regions.
  - Remark : We propose a general strategy of taking a good scheme (like ENO second order) but for which there is no convergence proof, and use the filter to assure convergence and error estimate (the theoretical  $\sqrt{\Delta x}$  as for the monotone scheme), and numerically show that we almost keep the same precision as ENO (i.e. second order) on basic linear and non linear examples.

# Conclusion

- A general and simple presentation of filtered scheme and easy to implement.
- Convergence of filtered scheme is confirmed. Error estimate is of  $O(\sqrt{\Delta x})$  and numerically observed that  $O(\Delta x^2)$  behavior in smooth regions.
- Remark : We propose a general strategy of taking a good scheme (like ENO second order) but for which there is no convergence proof, and use the filter to assure convergence and error estimate (the theoretical  $\sqrt{\Delta x}$  as for the monotone scheme), and numerically show that we almost keep the same precision as ENO (i.e. second order) on basic linear and non linear examples.

# Conclusion

- A general and simple presentation of filtered scheme and easy to implement.
- Convergence of filtered scheme is confirmed. Error estimate is of  $O(\sqrt{\Delta x})$  and numerically observed that  $O(\Delta x^2)$  behavior in smooth regions.
- Remark : We propose a general strategy of taking a good scheme (like ENO second order) but for which there is no convergence proof, and use the filter to assure convergence and error estimate (the theoretical  $\sqrt{\Delta x}$  as for the monotone scheme), and numerically show that we almost keep the same precision as ENO (i.e. second order) on basic linear and non linear examples.

## References

- [1] O. Bokanowski, M. Falcone, and S. Sahu, "*An efficient filtered scheme for some first order Hamilton-Jacobi-Bellman equations*", preprint.
- [2] M. G. Crandall and P.L. Lions, *Two approximations of solution of Hamilton-Jacobi equations. Comput. Method Appl. Mech. Engrg.*, 195:1344-1386,1984.
- [3] G. Barles and P. E. Souganidis. *Convergence of approximation schemes for fully nonlinear second order equations. Asymptot. Anal.*, 4:271-283, 1991.
- [4] B. D. Froese and A. M. Oberman, *Convergent filtered schemes for the Monge-Ampère partial differential equation. SIAM J. Numer. Anal.*, 51:423-444,2013.
- [5] M. Falcone, and R. Ferretti, *Semi-Lagrangian Approximation schemes for linear and Hamilton-Jacobi equations*, SIAM-Society for Industrial and Applied Mathematics, Philadelphia, 2014.

- [6] R. Abgrall. *Construction of simple stable and convergent high order schemes for steady first order HJ equations*. SIAM J. Comput.,31(4):2419-2446, 2009.
- [7] S. Osher, C.W. Shu, *High order essential non-oscillatory schemes for Hamilton-Jacobi equation*. SIAM J. Numer. Anal., 4:907-922, 1991.
- [8] O. Bokanowski, M. Falcone, R. Ferretti, L. Grüne, D. Kalise, H. Zidani, *Value iteration convergence of epsilon-monotone schemes for stationary HJ equations*. Preprint HAL 2014.
- [9] O. Bokanowski, Y. Cheng, and C-W Shu, *A discontinuous Galerkin scheme for front propagation with obstacle*. Numer. Math.(2014) 126:1-31.