Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz Association

Optimal control of phase field systems involving dynamic boundary conditions and singular nonlinearities

Jürgen Sprekels

WIAS and Institut für Mathematik, Humboldt-Universität zu Berlin

(joint work with P. Colli, G. Gilardi (Pavia), M. H. Farshbaf-Shaker (WIAS))

Consider the IVP with dynamic boundary condition

$$y_t - \Delta y + f'(y) = u \quad \text{a.e. in } Q, \tag{1}$$

$$\partial_t y_\Gamma - \Delta_\Gamma y_\Gamma + \partial_{\mathbf{n}} y + g'(y_\Gamma) = u_\Gamma, \quad y_{|\Gamma} = y_\Gamma, \quad \text{a.e. on } \Sigma, \tag{2}$$

$$y(0) = y_0 \quad \text{a.e. in } \Omega, \quad y_\Gamma(0) = y_{0_\Gamma} \quad \text{a.e. on } \Gamma. \tag{3}$$

Here, we have

- Δ_Γ : Laplace–Beltrami operator, n : outward unit normal derivative;
- $f\,,\,g$: given nonlinearities;
- $u\,,\,u_\Gamma$: control functions;
- $y_0 \in H^1(\Omega)$: initial datum s.t. $y_{0|_\Gamma} = y_{0_\Gamma}$.

The system (1)–(3) constitutes a model for an isothermal phase transition between two different phases that takes place in the container $\Omega \subset \mathbb{R}^3$ and is accompanied by surface diffusion on the boundary Γ. Here, y plays the role of a non-conserved "order parameter" (Allen–Cahn type) of the phase transition, which is typically the fraction of one of the phases and therefore should attain values in $[0,1]$.

The corresponding "free energy" is of the form

$$\mathcal{F}(y) = \int_\Omega \left(f(y) + \frac{1}{2}|\nabla y|^2 \right) dx + \int_\Gamma \left(g(y_\Gamma) + \frac{1}{2}|\nabla_\Gamma y_\Gamma|^2 \right) d\Gamma,$$

where ∇_Γ denotes the surface gradient.

We remark that the corresponding system for a conserved order parameter (Cahn–Hilliard type) can also be treated, but is more difficult.

We introduce the Banach spaces

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad H_\Gamma := L^2(\Gamma), \quad V_\Gamma := H^1(\Gamma),$$

$$\mathcal{H} := L^2(Q) \times L^2(\Sigma), \quad \mathcal{X} := L^\infty(Q) \times L^\infty(\Sigma),$$

$$\mathcal{Y} := \Big\{ (y, y_\Gamma) : y \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;H^2(\Omega)),$$
$$y_\Gamma \in H^1(0,T;H_\Gamma) \cap C^0([0,T];V_\Gamma) \cap L^2(0,T;H^2(\Gamma)), \quad y_\Gamma = y_{|\Gamma} \Big\},$$

endowed with their respective natural norms. We also assume:

(A1) There are given functions

$$z_Q \in L^2(Q), \quad z_\Sigma \in L^2(\Sigma), \quad z_T \in V, \quad z_{\Gamma,T} \in V_\Gamma,$$

$$\widetilde{u}_1, \widetilde{u}_2 \in L^\infty(Q) \text{ with } \widetilde{u}_1 \le \widetilde{u}_2 \text{ a.e. in } Q,$$

$$\widetilde{u}_{1_\Gamma}, \widetilde{u}_{2_\Gamma} \in L^\infty(\Sigma) \text{ with } \widetilde{u}_{1_\Gamma} \le \widetilde{u}_{2_\Gamma} \text{ a.e. on } \Sigma.$$

(CP) Minimize the (tracking-type) cost functional

$$
\begin{aligned}
J((y,y_\Gamma),(u,u_\Gamma)) \;:=\; & \frac{\beta_1}{2}\int_0^T\!\!\int_\Omega |y-z_Q|^2\,dx\,dt + \frac{\beta_2}{2}\int_0^T\!\!\int_\Gamma |y_\Gamma - z_\Sigma|^2\,d\Gamma\,dt \\
& + \frac{\beta_3}{2}\int_\Omega |y(\cdot,T)-z_T|^2\,dx + \frac{\beta_4}{2}\int_\Gamma |y_\Gamma(\cdot,T)-z_{\Gamma,T}|^2\,d\Gamma \\
& + \frac{\beta_5}{2}\int_0^T\!\!\int_\Omega |u|^2\,dx\,dt + \frac{\beta_6}{2}\int_0^T\!\!\int_\Gamma |u_\Gamma|^2\,d\Gamma\,dt \qquad (4)
\end{aligned}
$$

subject to the state system (1)–(3) and to the control constraint

$$
\begin{aligned}
(u,u_\Gamma) \in \mathcal{U}_{ad} := \{ (w,w_\Gamma) \in \mathcal{H} : \widetilde{u}_1 \le w \le \widetilde{u}_2 \ \text{ a.e. in } Q, \\
\widetilde{u}_{1_\Gamma} \le w_\Gamma \le \widetilde{u}_{2_\Gamma} \ \text{ a.e. on } \Sigma \}. \qquad (5)
\end{aligned}
$$

(A2) $f = f_1 + f_2$, $g = g_1 + g_2$, where $f_2, g_2 \in C^3[0,1]$, and where $f_1, g_1 \in C^3(0,1)$ are convex and satisfy:

$$\lim_{r \searrow 0} f_1'(r) = \lim_{r \searrow 0} g_1'(r) = -\infty, \quad \lim_{r \nearrow 1} f_1'(r) = \lim_{r \nearrow 1} g_1'(r) = +\infty \tag{6}$$

$$\exists\, M_1 \geq 0,\, M_2 > 0 : \quad |f_1'(r)| \leq M_1 + M_2\,|g_1'(r)| \quad \forall\, r \in (0,1). \tag{7}$$

(A2) $f = f_1 + f_2\,,\ g = g_1 + g_2\,,$ where $f_2, g_2 \in C^3[0,1]$, and where $f_1, g_1 \in C^3(0,1)$ are convex and satisfy:

$$\lim_{r \searrow 0} f_1'(r) = \lim_{r \searrow 0} g_1'(r) = -\infty, \quad \lim_{r \nearrow 1} f_1'(r) = \lim_{r \nearrow 1} g_1'(r) = +\infty \tag{6}$$

$$\exists\ M_1 \geq 0,\ M_2 > 0 : \quad |f_1'(r)| \leq M_1 + M_2\,|g_1'(r)| \quad \forall\, r \in (0,1). \tag{7}$$

(A3) $y_0 \in V\,,\ \ y_{0_\Gamma} = y_{0_{|\Gamma}}\,,\ \ f_1(y_0) \in L^1(\Omega)\,,\ \ g_1(y_{0_\Gamma}) \in L^1(\Gamma)$, and

$$0 < y_0 < 1 \ \text{ a.e. in } \Omega, \qquad 0 < y_{0_\Gamma} < 1 \ \text{ a.e. on } \Gamma\,. \tag{8}$$

(A2) $f = f_1 + f_2\,,\ g = g_1 + g_2\,,$ where $f_2, g_2 \in C^3[0,1]$, and where $f_1, g_1 \in C^3(0,1)$ are convex and satisfy:

$$\lim_{r \searrow 0} f_1'(r) = \lim_{r \searrow 0} g_1'(r) = -\infty, \quad \lim_{r \nearrow 1} f_1'(r) = \lim_{r \nearrow 1} g_1'(r) = +\infty \tag{6}$$

$$\exists\ M_1 \geq 0,\ M_2 > 0: \quad |f_1'(r)| \leq M_1 + M_2\,|g_1'(r)| \quad \forall\, r \in (0,1). \tag{7}$$

(A3) $y_0 \in V\,,\ \ y_{0_\Gamma} = y_{0_{|\Gamma}}\,,\ \ f_1(y_0) \in L^1(\Omega)\,,\ \ g_1(y_{0_\Gamma}) \in L^1(\Gamma)$, and

$$0 < y_0 < 1 \ \text{ a.e. in } \Omega, \qquad 0 < y_{0_\Gamma} < 1 \ \text{ a.e. on } \Gamma\,. \tag{8}$$

(A4) $\mathcal{U} \subset \mathcal{X}$ is open such that $\mathcal{U}_{\mathrm{ad}} \in \mathcal{U}$, and there is some $R > 0$ with

$$\|u\|_{L^\infty(Q)} + \|u_\Gamma\|_{L^\infty(\Sigma)} \leq R \quad \forall\,(u, u_\Gamma) \in \mathcal{U}. \tag{9}$$

Remarks:

1. (A2) implies that the singularity on the boundary grows at least with the same order as the one in the bulk.

Remarks:

1. (A2) implies that the singularity on the boundary grows at least with the same order as the one in the bulk.

2. Typical nonlinearities satisfying (5) and (6) are

$$f_1(r) = c_1(r\log(r) + (1-r)\log(1-r)),$$

$$g_1(r) = c_2(r\log(r) + (1-r)\log(1-r)),$$

where $c_1 > 0,\ c_2 > 0$.

Remarks:

1. (A2) implies that the singularity on the boundary grows at least with the same order as the one in the bulk.

2. Typical nonlinearities satisfying (5) and (6) are

$$f_1(r) = c_1(r\log(r) + (1-r)\log(1-r)),$$

$$g_1(r) = c_2(r\log(r) + (1-r)\log(1-r)),$$

   where $c_1 > 0, c_2 > 0$.

3. We assume here a differentiable situation. The results are submitted to SIAM J. Control Optimization. A non-differentiable case was studied in Colli–Farshbaf-Shaker–Sprekels (to appear 2014 in Appl. Math. Optim.): there, we assume that $f_1 = g_1 = I_{[0,1]}$, so that we have to replace f_1', g_1' in (1) and (2) by the subdifferential $\partial I_{[0,1]}$.

The following result is a special case of results proved in Calatroni–Colli (Nonlinear Anal. 2013):

Theorem 1: *Suppose that* **(A2)**, **(A3)** *are satisfied. Then we have:*

(i) *The state system* (1)–(3) *has for any* $(u, u_\Gamma) \in \mathcal{H}$ *a unique solution* $(y, y_\Gamma) \in \mathcal{Y}$ *such that*

$$0 < y < 1 \quad \text{a.e. in } Q, \qquad 0 < y_\Gamma < 1 \quad \text{a.e. on } \Sigma. \tag{10}$$

(ii) If also **(A4)** holds, $\exists\, K_1^* > 0$: for any $(u, u_\Gamma) \in \mathcal{U}$ the associated solution $(y, y_\Gamma) \in \mathcal{Y}$ satisfies

$$\|(y, y_\Gamma)\|_{\mathcal{Y}} \le K_1^*, \quad \|f'(y)\|_{L^2(Q)} + \|g'(y_\Gamma)\|_{L^2(\Sigma)} \le K_1^*. \tag{11}$$

Moreover, $\exists\, K_2^* > 0$: whenever $(u_1, u_{1_\Gamma}), (u_2, u_{2_\Gamma}) \in \mathcal{U}$ are given, then we have

$$\begin{aligned} &\|y_1 - y_2\|_{C^0([0,T];H)} + \|\nabla(y_1 - y_2)\|_{L^2(Q)} + \|y_{1_\Gamma} - y_{2_\Gamma}\|_{C^0([0,T];H_\Gamma)} \\ &+ \|\nabla_\Gamma(y_{1_\Gamma} - y_{2_\Gamma})\|_{L^2(\Sigma)} \\ &\le K_2^* \left(\|u_1 - u_2\|_{L^2(Q)} + \|u_{1_\Gamma} - u_{2_\Gamma}\|_{L^2(\Sigma)} \right). \end{aligned} \tag{12}$$

Remark:

4. Owing to **Theorem 1**, the *control-to-state mapping*

$$\mathcal{S} : (u, u_\Gamma) \mapsto \mathcal{S}(u, u_\Gamma) := (y, y_\Gamma)$$

is defined as a mapping from $\mathcal{X}$ into $\mathcal{Y}$. Moreover, $\mathcal{S}$ is Lipschitz continuous when viewed as a mapping from the subset $\mathcal{U}$ of $\mathcal{H}$ into the space $\big(C^0([0,T];H) \cap L^2(0,T;V)\big) \times \big(C^0([0,T];H_\Gamma) \cap L^2(0,T;V_\Gamma)\big)$.

We now come to a linearized version of **Theorem 1**, which will play a central role in the derivation of first-order necessary and second-order sufficient conditions for **(CP)**.

Theorem 2: *Let* $(u, u_\Gamma) \in \mathcal{H}$, $c_1 \in L^\infty(Q)$, $c_2 \in L^\infty(\Sigma)$ *, as well as* $(w_0, w_{0_\Gamma}) \in V \times V_\Gamma$ *with* $w_{0_{|\Gamma}} = w_{0_\Gamma}$ *be given. Then we have:*

(i) *The linear IBVP*

$$w_t - \Delta w + c_1(x,t)\, w = u \quad \text{a.e. in } Q, \tag{13}$$

$$\partial_t w_\Gamma - \Delta_\Gamma w_\Gamma + \partial_{\mathbf{n}} w + c_2(x,t)\, w_\Gamma = u_\Gamma\,, \quad w_{|\Gamma} = w_\Gamma\,, \quad \text{a.e. on } \Sigma, \tag{14}$$

$$w(\cdot,0) = w_0 \quad \text{a.e. in } \Omega, \qquad w_\Gamma(\cdot,0) = w_{0_\Gamma} \quad \text{a.e. on } \Gamma, \tag{15}$$

has a unique solution $(w, w_\Gamma) \in \mathcal{Y}$.

(ii) *There is some* $\widehat{C} > 0$ *such that: whenever* $w_0 = 0$ *and* $w_{0_\Gamma} = 0$ *then*

$$\|(w, w_\Gamma)\|_{\mathcal{Y}} \le \widehat{C}\, \|(u, u_\Gamma)\|_{\mathcal{H}}\,. \tag{16}$$

Idea of Proof: **(i)** is more or less a consequence of **Theorem 1**. Now let $w_0 = 0$, $w_{0_\Gamma} = 0$. Testing (13) by w_t and applying Young's and Gronwall's inequalities, we easily find

$$\|w\|_{H^1(0,T;H)\cap C^0([0,T];V)} + \|w_\Gamma\|_{H^1(0,T;H_\Gamma)\cap C^0([0,T];V_\Gamma)} \leq C_1 \,\|(u,u_\Gamma)\|_{\mathcal{H}}.$$

Comparison in (13) yields

$$\|\Delta w\|_{L^2(Q)} \leq C_2 \,\|(u,u_\Gamma)\|_{\mathcal{H}}.$$

Then, applying a standard embedding result,

$$\|w\|_{L^2(0,T;H^{3/2}(\Omega))} \leq C_3 \,\|(u,u_\Gamma)\|_{\mathcal{H}},$$

whence, by the trace theorem,

$$\|\partial_{\mathbf{n}} w\|_{L^2(0,T;H_\Gamma)} \leq C_4 \,\|(u,u_\Gamma)\|_{\mathcal{H}}.$$

But then, by comparison in (14),

$$\|\Delta_\Gamma w_\Gamma\|_{L^2(\Sigma)} \le C_5 \,\|(u, u_\Gamma)\|_{\mathcal{H}}\,,$$

whence

$$\|w_\Gamma\|_{L^2(0,T;H^2(\Gamma))} \le C_6 \,\|(u, u_\Gamma)\|_{\mathcal{H}}\,.$$

Standard elliptic estimates then yield

$$\|w\|_{L^2(0,T;H^2(\Omega))} \le C_7 \,\|(u, u_\Gamma)\|_{\mathcal{H}}\,.$$

$\blacksquare$

Remark:

5. It cannot be expected that $(w, w_\Gamma) \in L^\infty(Q) \times L^\infty(\Sigma)$, in general.

An L^∞ bound for (y, y_Γ)

(A5) It holds $y_0 \in L^\infty(\Omega)$, $y_{0_\Gamma} \in L^\infty(\Gamma)$, as well as

$$0 < \operatorname*{ess\,inf}_{x\in\Omega} y_0(x), \quad \operatorname*{ess\,sup}_{x\in\Omega} y_0(x) < 1,$$

$$0 < \operatorname*{ess\,inf}_{x\in\Gamma} y_{0_\Gamma}(x), \quad \operatorname*{ess\,sup}_{x\in\Gamma} y_{0_\Gamma}(x) < 1.$$

(A5) It holds $y_0 \in L^\infty(\Omega)$, $y_{0_\Gamma} \in L^\infty(\Gamma)$, as well as

$$0 < \operatorname*{ess\,inf}_{x\in\Omega} y_0(x), \quad \operatorname*{ess\,sup}_{x\in\Omega} y_0(x) < 1,$$

$$0 < \operatorname*{ess\,inf}_{x\in\Gamma} y_{0_\Gamma}(x), \quad \operatorname*{ess\,sup}_{x\in\Gamma} y_{0_\Gamma}(x) < 1.$$

Lemma 3: *Let* **(A2)–(A5)** *hold. Then* $\exists\; 0 < r_* < r^* < 1$ *such that:*

whenever $(y, y_\Gamma) = \mathcal{S}(u, u_\Gamma)$ *for some* $(u, u_\Gamma) \in \mathcal{U}$, *then it holds*

$$r_* \le y \le r^* \text{ a.e. in } Q, \quad r_* \le y_\Gamma \le r^* \text{ a.e. on } \Sigma. \tag{17}$$

(A5) It holds $y_0 \in L^\infty(\Omega)$, $y_{0_\Gamma} \in L^\infty(\Gamma)$, as well as

$$0 < \operatorname*{ess\,inf}_{x\in\Omega} y_0(x), \quad \operatorname*{ess\,sup}_{x\in\Omega} y_0(x) < 1,$$

$$0 < \operatorname*{ess\,inf}_{x\in\Gamma} y_{0_\Gamma}(x), \quad \operatorname*{ess\,sup}_{x\in\Gamma} y_{0_\Gamma}(x) < 1.$$

Lemma 3: *Let* **(A2)–(A5)** *hold. Then* $\exists\, 0 < r_* < r^* < 1$ *such that:*

whenever $(y, y_\Gamma) = \mathcal{S}(u, u_\Gamma)$ *for some* $(u, u_\Gamma) \in \mathcal{U}$, *then it holds*

$$r_* \le y \le r^* \text{ a.e. in } Q, \quad r_* \le y_\Gamma \le r^* \text{ a.e. on } \Sigma. \tag{17}$$

Remark:

6. In view of **(A2)** and **Lemma 3**, we may assume that

$$\max_{0\le i\le 3} \left\{ \max \left\{ \|f^{(i)}(y)\|_{L^\infty(Q)}, \|g^{(i)}(y)\|_{L^\infty(\Sigma)} \right\} \right\} \le K_1^*, \tag{18}$$

whenever $(y, y_\Gamma) = \mathcal{S}(u, u_\Gamma)$ for some $(u, u_\Gamma) \in \mathcal{U}$.

Proof: There are constants $0 < r_* \le r^* < 1$ such that

$$r_* \le \min \left\{ \operatorname*{ess\,inf}_{x\in\Omega} y_0(x) \, , \, \operatorname*{ess\,inf}_{x\in\Gamma} y_{0_\Gamma}(x) \right\},$$

$$r^* \ge \max \left\{ \operatorname*{ess\,sup}_{x\in\Omega} y_0(x) \, , \, \operatorname*{ess\,sup}_{x\in\Gamma} y_{0_\Gamma}(x) \right\},$$

$$\max\{f'(r) + R, g'(r) + R\} \le 0 \quad \forall\, r \in (0, r_*),$$

$$\min\{f'(r) - R, g'(r) - R\} \ge 0 \quad \forall\, r \in (r^*, 1).$$

Now define $w := (y - r^*)^+$. Clearly, we have $w \in V$ and $w_{|\Gamma} \in V_\Gamma$. We put $w_\Gamma := w_{|\Gamma}$ and test (1) by w. We readily see that

$$0 \;=\; \frac{1}{2}\,\|w(T)\|_H^2 + \int_0^T \|\nabla w(t)\|_H^2 \,dt + \int_0^T \|\nabla_\Gamma w_\Gamma(t)\|_{H_\Gamma}^2 \,dt$$

$$+\, \frac{1}{2}\,\|w_\Gamma(T)\|_{H_\Gamma}^2 \;+\; \Phi,$$

where

$$\Phi := \int_0^T\!\!\int_\Omega (f'(y) - u)\, w \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_\Gamma (g'(y_\Gamma) - u_\Gamma)\, w_\Gamma \,\mathrm{d}\Gamma\,\mathrm{d}t \geq 0 .$$

In conclusion, $w = (y - r^*)^+ = 0$, i. e., $y \leq r^*$, almost everywhere in Q and on Σ.
The remaining inequalities follow similarly by testing (1) with $w := -(y - r_*)^-$. ∎

Remark:

7. Assume **(A2)–(A5)** are satisfied. Using arguments similar to those in the proof of (16), we are able to improve the stability estimate (12); $\exists\, K_3^* > 0$:

   whenever $(y_i, y_{i_\Gamma}) = \mathcal{S}(u_i, u_{i_\Gamma})$ for $(u_i, u_{i_\Gamma}) \in \mathcal{U}$, $i = 1, 2$, then

$$\|(y_1, y_{1_\Gamma}) - (y_2, y_{2_\Gamma})\|_{\mathcal{Y}} \leq K_3^* \,\|(u_1, u_{1_\Gamma}) - (u_2, u_{2_\Gamma})\|_{\mathcal{H}} . \qquad (19)$$

This higher Lipschitz continuity is needed to show the Fréchet differentiability of the control-to-state mapping $\mathcal{S}$.

Theorem 4: *Suppose that* **(A1)–(A4)** *are fulfilled. Then* **(CP)** *has a solution.*

Theorem 4: *Suppose that* **(A1)–(A4)** *are fulfilled. Then* **(CP)** *has a solution.*

Proof: Pick a minimizing sentence $\{(u_n, u_{n_\Gamma})\} \subset \mathcal{U}_{\mathrm{ad}}$, and let $(u_n, u_{n_\Gamma}) = \mathcal{S}(u_n, u_{n_\Gamma})$, $n \in \mathbb{N}$. By the a priori estimates, we may assume that

$$u_n \to \bar{u} \quad \text{weakly-* in } L^\infty(Q), \quad u_{n_\Gamma} \to \bar{u}_\Gamma \quad \text{weakly-* in } L^\infty(\Sigma),$$

$$y_n \to \bar{y} \quad \text{weakly-* in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)) \cap L^\infty(Q),$$

$$y_{n_\Gamma} \to \bar{y}_\Gamma \quad \text{weakly-* in } H^1(0,T;H_\Gamma) \cap L^\infty(0,T;V_\Gamma) \cap L^2(0,T;H^2(\Gamma)) \cap L^\infty(\Sigma).$$

In particular, we have (by compact embedding)

$$y_n \to \bar{y} \quad \text{strongly in } C^0([0,T];H), \quad y_{n_\Gamma} \to \bar{y}_\Gamma \quad \text{strongly in } C^0([0,T];H_\Gamma).$$

Passage to the limit $n \to \infty$ in (1)–(3) easily shows that $(\bar{y}, \bar{y}_\Gamma) = \mathcal{S}(\bar{u}, \bar{u}_\Gamma)$, and the weak sequential lower semicontinuity of J yields that $((\bar{u}, \bar{u}_\Gamma), (\bar{y}, \bar{y}_\Gamma))$ is an optimal pair.

■

Theorem 5: *Suppose that* **(A2)–(A5)** *hold. Then we have*

(i) *Let* $(\bar{u}, \bar{u}_\Gamma) \in \mathcal{U}$ *be arbitrary. Then* $\mathcal{S} : \mathcal{X} \to \mathcal{Y}$ *is Fréchet differentiable at* $(\bar{u}, \bar{u}_\Gamma)$, *and the Fréchet derivative* $D\mathcal{S}(\bar{u}, \bar{u}_\Gamma)$ *is given by* $D\mathcal{S}(\bar{u}, \bar{u}_\Gamma)(h, h_\Gamma) = (\xi, \xi_\Gamma)$, *where for any given* $(h, h_\Gamma) \in \mathcal{X}$ *the pair* $(\xi, \xi_\Gamma) \in \mathcal{Y}$ *solves the linearized system*

$$\xi_t - \Delta\xi + f''(\bar{y})\,\xi = h \quad \text{a. e. in } Q, \tag{20}$$

$$\partial_t \xi_\Gamma - \Delta_\Gamma \xi_\Gamma + \partial_{\mathbf{n}} \xi + g''(\bar{y}_\Gamma)\,\xi_\Gamma = h_\Gamma, \quad \xi_\Gamma = \xi_{|\Gamma}, \quad \text{a. e. on } \Sigma, \tag{21}$$

$$\xi(\,\cdot\,,0) = 0 \quad \text{a. e. in } \Omega, \qquad \xi_\Gamma(\,\cdot\,,0) = 0 \quad \text{a. e. on } \Gamma. \tag{22}$$

(ii) *The mapping* $D\mathcal{S} : \mathcal{U} \to \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $(\bar{u}, \bar{u}_\Gamma) \mapsto D\mathcal{S}(\bar{u}, \bar{u}_\Gamma)$, *satisfies for all* $(\bar{u}, \bar{u}_\Gamma), (\bar{u}, \bar{u}_\Gamma) \in \mathcal{U}$ *and* $(h, h_\Gamma) \in \mathcal{X}$:

$$\|(D\mathcal{S}(u, u_\Gamma) - D\mathcal{S}(\bar{u}, \bar{u}_\Gamma))(h, h_\Gamma)\|_{\mathcal{Y}} \le K_4^* \,\|(u, u_\Gamma) - (\bar{u}, \bar{u}_\Gamma)\|_{\mathcal{H}}\, \|(h, h_\Gamma)\|_{\mathcal{H}}\,. \tag{23}$$

Remarks:

8. For any $(h, h_\Gamma) \in \mathcal{X}$ the linear system (20)–(22) is of the form (13)–(15) with zero initial conditions, hence has a unique solution in $\mathcal{Y}$, and it holds $\|(\xi, \xi_\Gamma)\|_{\mathcal{Y}} \leq \widehat{C}\|h, h_\Gamma)\|_{\mathcal{H}}$.

Remarks:

8. For any $(h, h_\Gamma) \in \mathcal{X}$ the linear system (20)–(22) is of the form (13)–(15) with zero initial conditions, hence has a unique solution in $\mathcal{Y}$, and it holds $\|(\xi, \xi_\Gamma)\|_{\mathcal{Y}} \leq \widehat{C}\|h, h_\Gamma)\|_{\mathcal{H}}$.

9. By **Theorem 4** the *reduced cost functional* $\mathcal{J}(u, u_\Gamma) := J(\mathcal{S}(u, u_\Gamma), (u, u_\Gamma))$ is Fréchet differentiable at every $(u, u_\Gamma) \in \mathcal{U}$ with the dervative

$$D\mathcal{J}(\bar{u}, \bar{u}_\Gamma) = D_{(y,y_\Gamma)} J(\mathcal{S}(\bar{u}, \bar{u}_\Gamma), (\bar{u}, \bar{u}_\Gamma)) \circ D\mathcal{S}(\bar{u}, \bar{u}_\Gamma) + D_{(u,u_\Gamma)} J(\mathcal{S}(\bar{u}, \bar{u}_\Gamma), (\bar{u}, \bar{u}_\Gamma)) . \tag{24}$$

Remarks:

8. For any $(h, h_\Gamma) \in \mathcal{X}$ the linear system (20)–(22) is of the form (13)–(15) with zero initial conditions, hence has a unique solution in $\mathcal{Y}$, and it holds $\|(\xi, \xi_\Gamma)\|_{\mathcal{Y}} \leq \widehat{C}\|h, h_\Gamma)\|_{\mathcal{H}}$.

9. By **Theorem 4** the *reduced cost functional* $\mathcal{J}(u, u_\Gamma) := J(\mathcal{S}(u, u_\Gamma), (u, u_\Gamma))$ is Fréchet differentiable at every $(u, u_\Gamma) \in \mathcal{U}$ with the dervative

$$D\mathcal{J}(\bar{u}, \bar{u}_\Gamma) = D_{(y,y_\Gamma)} J(\mathcal{S}(\bar{u}, \bar{u}_\Gamma), (\bar{u}, \bar{u}_\Gamma)) \circ D\mathcal{S}(\bar{u}, \bar{u}_\Gamma) + D_{(u,u_\Gamma)} J(\mathcal{S}(\bar{u}, \bar{u}_\Gamma), (\bar{u}, \bar{u}_\Gamma)). \tag{24}$$

Now notice that $\mathcal{U}_{\mathrm{ad}}$ is convex, hence we must have

$$D\mathcal{J}(\bar{u}, \bar{u}_\Gamma)(v - \bar{u}, v_\Gamma - \bar{u}_\Gamma) \geq 0 \quad \forall (v, v_\Gamma) \in \mathcal{U}_{\mathrm{ad}}. \tag{25}$$

for any minimizer $(u, u_\Gamma) \in \mathcal{U}_{\mathrm{ad}}$ of $\mathcal{J}$.

In terms of our cost functional, this means that the following variational inequality must be satisfied: for every $(v, v_\Gamma) \in \mathcal{U}_{ad}$ it holds

$$\beta_1 \int_0^T \int_\Omega (\bar{y} - z_Q)\, \xi \, dx\, dt + \beta_2 \int_0^T \int_\Gamma (\bar{y}_\Gamma - z_\Sigma)\, \xi_\Gamma \, d\Gamma\, dt$$

$$+ \beta_3 \int_\Omega (\bar{y}(\cdot, T) - z_T)\, \xi(\cdot, T)\, dx + \beta_4 \int_\Gamma (\bar{y}_\Gamma(\cdot, T) - z_{\Gamma,T})\, \xi_\Gamma(\cdot, T)\, d\Gamma$$

$$+ \beta_5 \int_0^t \int_\Omega \bar{u}(v - \bar{u})\, dx\, dt + \beta_6 \int_0^t \int_\Gamma \bar{u}_\Gamma (v_\Gamma - \bar{u}_\Gamma)\, d\Gamma\, dt \geq 0, \tag{26}$$

where $(\xi, \xi_\Gamma) \in \mathcal{Y}$ is the unique solution to (20)–(22) with $(h, h_\Gamma) = (v - \bar{u}, v_\Gamma - \bar{u}_\Gamma)$. We aim to eliminate (ξ, ξ_Γ) by introducing the adjoint state system.

(A6) It holds $\beta_3 = \beta_4$ and $z_{\Gamma,T} = z_{T|\Gamma}$.

Theorem 6: *Let the assumptions* **(A1)–(A6)** *be satisfied, and let* $(\bar{u}, \bar{u}_\Gamma) \in \mathcal{U}_{\rm ad}$ *be optimal and* $(\bar{y}, \bar{y}_\Gamma) = \mathcal{S}(\bar{u}, \bar{u}_\Gamma) \in \mathcal{Y}$ *. Then the adjoint state system*

$$- p_t - \Delta p + f''(\bar{y})\, p = \beta_1\, (\bar{y} - z_Q) \quad \text{a.e. in } Q, \tag{27}$$

$$\partial_{\mathbf{n}} p - \partial_t p_\Gamma - \Delta_\Gamma p_\Gamma + g''(\bar{y}_\Gamma)\, p_\Gamma = \beta_2\, (\bar{y}_\Gamma - z_\Sigma) \quad \text{a.e. on } \Sigma, \tag{28}$$

$$\begin{aligned} p(\,\cdot\,, T) &= \beta_3(\bar{y}(\,\cdot\,, T) - z_T) \quad \text{a.e. in } \Omega, \\ p_\Gamma(\,\cdot\,, T) &= \beta_4\, (\bar{y}_\Gamma(\,\cdot\,, T) - z_{\Gamma,T}) \quad \text{a.e. on } \Gamma, \end{aligned} \tag{29}$$

has a unique solution $(p, p_\Gamma) \in \mathcal{Y}$ *, and for every* $(v, v_\Gamma) \in \mathcal{U}_{\rm ad}$ *we have*

$$\int_0^T \!\! \int_\Omega (p + \beta_5\, \bar{u})(v - \bar{u})\, \mathrm{d}x\, \mathrm{d}t + \int_0^T \!\! \int_\Gamma (p_\Gamma + \beta_6\, \bar{u}_\Gamma)(v_\Gamma - \bar{u}_\Gamma)\, \mathrm{d}\Gamma\, \mathrm{d}t \geq 0\,. \tag{30}$$

Remarks:

10. The compatibility condition **(A6)** was needed to guarantee the applicability of **Theorem 2** (namely, to have $p(\cdot,T)_{|\Gamma} = p_\Gamma(\cdot,T)$).

11. As usual, the Fréchet derivative $D\mathcal{J}(\bar{u},\bar{u}_\Gamma) \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ can be identified with the pair $(p+\beta_5\,\bar{u}, p_\Gamma+\beta_6\,\bar{u}_\Gamma)$. In fact, with the standard inner product $(\cdot,\cdot)_\mathcal{H}$ in $\mathcal{H}$ we have for all $(h,h_\Gamma) \in \mathcal{X}$:

$$D\mathcal{J}(\bar{u},\bar{u}_\Gamma)(h,h_\Gamma) = ((p+\beta_5\,\bar{u}, p_\Gamma+\beta_6\,\bar{u}_\Gamma),(h,h_\Gamma))_\mathcal{H}\,.$$

12. If $\beta_5 > 0$ and $\beta_6 > 0$, then it follows

$$\bar{u}(x,t) = \mathbb{P}_{[\tilde{u}_1(x,t),\tilde{u}_2(x,t)]}(-\beta_5^{-1}\,p(x,t))\,,$$
$$\bar{u}_\Gamma(x,t) = \mathbb{P}_{[\tilde{u}_{1_\Gamma}(x,t),\tilde{u}_{2_\Gamma}(x,t)]}(-\beta_6^{-1}\,p_\Gamma(x,t)) \tag{31}$$

where

$$\mathbb{P}_{[a,b]}(x) = \begin{cases} a, & x < a \\ x, & a \le x \le b \\ b, & x > b \end{cases}\,. \tag{32}$$

13. The variational inequality (30) follows from (26), since it holds the identity

$$\beta_1 \int_0^T \int_\Omega (\bar{y} - z_Q)\, \xi \, \mathrm{d}x \, \mathrm{d}t + \beta_2 \int_0^T \int_\Gamma (\bar{y}_\Gamma - z_\Sigma)\, \xi_\Gamma \, \mathrm{d}\Gamma \, \mathrm{d}t$$

$$+ \beta_3 \int_\Omega (\bar{y}(\cdot, T) - z_T)\, \xi(\cdot, T) \, \mathrm{d}x + \beta_4 \int_\Gamma (\bar{y}_\Gamma(\cdot, T) - z_{\Gamma,T})\, \xi_\Gamma(\cdot, T) \, \mathrm{d}\Gamma$$

$$= \int_0^T \int_\Omega p \, h \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_\Gamma p_\Gamma \, h_\Gamma \, \mathrm{d}\Gamma \, \mathrm{d}t \, ,$$

which follows from (20)–(22) and (27)–(29) using repeated integration by parts.

14. It is possible to derive second-order *sufficient* optimality conditions. To this end, it has to be shown that the control-to-state operator $\mathcal{S}$ is twice continuously differentiable. This requires to assume $f,g \in C^4(0,1)$. The second Fréchet derivative $D^2\mathcal{S}(\bar{u},\bar{u}_\Gamma)$ is defined as follows: if $(h,h_\Gamma),(k,k_\Gamma) \in \mathcal{X}$ are arbitrary then

$$D^2\mathcal{S}(\bar{u},\bar{u}_\Gamma)[(h,h_\Gamma),(k,k_\Gamma)] =: (\eta,\eta_\Gamma) \in \mathcal{Y}$$

is the unique solution to the IVBP

$$\eta_t - \Delta\eta + f''(\bar{y})\,\eta = -f^{(3)}(\bar{y})\,\phi\,\psi \quad \text{a.e in } Q, \tag{33}$$

$$\partial_{\mathbf{n}}\eta + \partial_t\eta_\Gamma - \Delta_\Gamma\eta_\Gamma + g''(\bar{y}_\Gamma)\,\eta_\Gamma = -g^{(3)}(\bar{y}_\Gamma)\,\phi_\Gamma\,\psi_\Gamma \quad \text{a.e. on } \Sigma, \tag{34}$$

$$\eta(\cdot,0) = 0 \quad \text{a.e. in } \Omega, \qquad \eta_\Gamma(\cdot,0) = 0 \quad \text{a.e. on } \Gamma, \tag{35}$$

where

$$\begin{aligned}&(\bar{y},\bar{y}_\Gamma) = \mathcal{S}(\bar{u},\bar{u}_\Gamma), \quad (\phi,\phi_\Gamma) = D\mathcal{S}(\bar{u},\bar{u}_\Gamma)(h,h_\Gamma),\\ &(\psi,\psi_\Gamma) = D\mathcal{S}(\bar{u},\bar{u}_\Gamma)(k,k_\Gamma).\end{aligned} \tag{36}$$

The proof is technical, but not too difficult (see Colli–Sprekels, WIAS-Preprint No. 1750).

It turns out that the mapping

$$D^2\mathcal{S} : \mathcal{U} \to \mathcal{L}(\mathcal{X}, \mathcal{L}(\mathcal{X}, \mathcal{Y})), \quad (\bar{u}, \bar{u}_\Gamma) \mapsto D^2\mathcal{S}(\bar{u}, \bar{u}_\Gamma),$$

is Lipschitz continuous on $\mathcal{U} \subset \mathcal{X}$ only in the following sense: there exists a constant $K_5^* > 0$ such that for every $(u, u_\Gamma), (\bar{u}, \bar{u}_\Gamma) \in \mathcal{U}$ and all $(h, h_\Gamma), (k, k_\Gamma) \in \mathcal{X}$ it holds

$$\begin{aligned} &\|(D^2\mathcal{S}(u, u_\Gamma) - D^2\mathcal{S}(\bar{u}, \bar{u}_\Gamma))[(h, h_\Gamma), (k, k_\Gamma)]\|_{\mathcal{Y}} \\ &\le K_5^* \, \|(u, u_\Gamma) - (\bar{u}, \bar{u}_\Gamma)\|_{\mathcal{H}} \, \|(h, h_\Gamma)\|_{\mathcal{H}} \, \|(k, k_\Gamma)\|_{\mathcal{H}} . \end{aligned} \tag{37}$$

Notice: we have to deal with a *two-norm discrepancy*.

15. The problem is considerably more difficult in the case of non-differentiability. In the paper Colli–Farshbaf-Shaker–Sprekels (to appear in Appl. Math. Optim.), we considered the same cost functional J (with $\beta_3 = \beta_4$) and the same set of control constraints $\mathcal{U}_{ad}$. The state system has the form:

$$y_t - \Delta y + \xi + f_2'(y) = u \quad \text{a.e. in } Q \tag{38}$$

$$\partial t\, y_\Gamma - \Delta_\Gamma y_\Gamma + \partial_{\mathbf{n}} y + \xi_\Gamma + g_2'(y_\Gamma) = u_\Gamma \quad \text{a.e. on } \Sigma \tag{39}$$

$$\xi \in \partial I_{[-1,1]}(y) \quad \text{a.e. in } Q\,, \quad \xi_\Gamma \in \partial I_{[-1,1]}(y) \quad \text{a.e. on } \Sigma \tag{40}$$

$$y(\cdot,0) = y_0 \quad \text{a.e. in } \Omega\,, \quad y_\Gamma(\cdot,0) = y_{0_\Gamma} \quad \text{a.e. on } \Gamma. \tag{41}$$

$$y_t - \Delta y + \xi + f_2'(y) = u \quad \text{a.e. in } Q \tag{38}$$

$$\partial t\, y_\Gamma - \Delta_\Gamma y_\Gamma + \partial_{\mathbf{n}} y + \xi_\Gamma + g_2'(y_\Gamma) = u_\Gamma \quad \text{a.e. on } \Sigma \tag{39}$$

$$\xi \in \partial I_{[-1,1]}(y) \quad \text{a.e. in } Q, \quad \xi_\Gamma \in \partial I_{[-1,1]}(y) \quad \text{a.e. on } \Sigma \tag{40}$$

$$y(\cdot,0) = y_0 \quad \text{a.e. in } \Omega, \quad y_\Gamma(\cdot,0) = y_{0_\Gamma} \quad \text{a.e. on } \Gamma. \tag{41}$$

The general idea of handling this control problem was to use a *deep quench approach* using the results of the differentiable case: one replaces the inclusions (35) by

$$\xi = \varphi(\alpha)\, h'(y), \quad \xi_\Gamma = \psi(\alpha)\, h'(y), \tag{42}$$

where $\varphi(\alpha) = \psi(\alpha) = o(\alpha)$ as $\alpha \searrow 0$ and $0 < \varphi(\alpha) \leq C\psi(\alpha)$ for $\alpha > 0$, as well as

$$h(r) = (1-r)\log(1-r) + (1+r)\log(1+r), \quad -1 \leq r \leq +1. \tag{43}$$

This approach turns out to be successful:

- **"Global" result:** If $\alpha_n \searrow 0$ and $(\bar{u}^{\alpha_n}, \bar{u}_\Gamma^{\alpha_n})$ is an optimal control of the α_n-approximating differentiable problem, $n \in \mathbb{N}$, then a subsequence converges weakly to an optimal control of the non-differentiable problem.

This approach turns out to be successful:

- **"Global" result:** If $\alpha_n \searrow 0$ and $(\bar{u}^{\alpha_n}, \bar{u}_\Gamma^{\alpha_n})$ is an optimal control of the α_n-approximating differentiable problem, $n \in \mathbb{N}$, then a subsequence converges weakly to an optimal control of the non-differentiable problem.
- **"Local" result:** For any fixed optimizer $(\bar{u}, \bar{u}_\Gamma)$ define the *"adapted"* cost functional

$$\tilde{J}((y, y_\Gamma), (u, u_\Gamma)) = J((y, y_\Gamma), (u, u_\Gamma)) + \frac{1}{2}\|u - \bar{u}\|^2_{L^2(Q)} + \frac{1}{2}\|u_\Gamma - \bar{u}_\Gamma\|^2.$$

Then consider the α-approximating problems with this functional. It holds:

This approach turns out to be successful:

- **"Global" result:** If $\alpha_n \searrow 0$ and $(\bar{u}^{\alpha_n}, \bar{u}_\Gamma^{\alpha_n})$ is an optimal control of the α_n-approximating differentiable problem, $n \in \mathbb{N}$, then a subsequence converges weakly to an optimal control of the non-differentiable problem.
- **"Local" result:** For any fixed optimizer $(\bar{u}, \bar{u}_\Gamma)$ define the *"adapted"* cost functional

$$\tilde{J}((y, y_\Gamma), (u, u_\Gamma)) = J((y, y_\Gamma), (u, u_\Gamma)) + \frac{1}{2}\|u - \bar{u}\|^2_{L^2(Q)} + \frac{1}{2}\|u_\Gamma - \bar{u}_\Gamma\|^2.$$

  Then consider the α-approximating problems with this functional. It holds:

  - $\exists$ $\alpha_n \searrow 0$ and minimizers $(\bar{u}^{\alpha_n}, \bar{u}_\Gamma^{\alpha_n})$ of the α_n-approximating problems such that $(\bar{u}^{\alpha_n}, \bar{u}_\Gamma^{\alpha_n}) \to (\bar{u}, \bar{u}_\Gamma)$ strongly in $\mathcal{H}$.

This approach turns out to be successful:

- **"Global" result:** If $\alpha_n \searrow 0$ and $(\bar{u}^{\alpha_n}, \bar{u}_\Gamma^{\alpha_n})$ is an optimal control of the α_n-approximating differentiable problem, $n \in \mathbb{N}$, then a subsequence converges weakly to an optimal control of the non-differentiable problem.

- **"Local" result:** For any fixed optimizer $(\bar{u}, \bar{u}_\Gamma)$ define the *"adapted"* cost functional

$$\tilde{J}((y, y_\Gamma), (u, u_\Gamma)) = J((y, y_\Gamma), (u, u_\Gamma)) + \frac{1}{2}\|u - \bar{u}\|^2_{L^2(Q)} + \frac{1}{2}\|u_\Gamma - \bar{u}_\Gamma\|^2.$$

  Then consider the α-approximating problems with this functional. It holds:

  - $\exists$ $\alpha_n \searrow 0$ and minimizers $(\bar{u}^{\alpha_n}, \bar{u}_\Gamma^{\alpha_n})$ of the α_n-approximating problems such that $(\bar{u}^{\alpha_n}, \bar{u}_\Gamma^{\alpha_n}) \to (\bar{u}, \bar{u}_\Gamma)$ strongly in $\mathcal{H}$.
  - Letting $\alpha_n \searrow 0$ in the first-order necessary optimality conditions for the α_n-approximating problems leads to first-order conditions for the non-differentiable case.

1. P. Colli, J. S.: *Optimal control of the Allen–Cahn equation with singular potentials and dynamic boundary condition.* WIAS Preprint No. 1750 (2012). Submitted.
2. P. Colli, M. H. Farshbaf-Shaker, J. S.: *A deep quench approach to the optimal control of an Allen–Cahn equation with dynamic boundary condition and double obstacle potentials.* WIAS Preprint No. 1838 (2013). To appear 2014 in *Appl. Math. Optim.*

1. P. Colli, J. S.: *Optimal control of the Allen–Cahn equation with singular potentials and dynamic boundary condition.* WIAS Preprint No. 1750 (2012). Submitted.

2. P. Colli, M. H. Farshbaf-Shaker, J. S.: *A deep quench approach to the optimal control of an Allen–Cahn equation with dynamic boundary condition and double obstacle potentials.* WIAS Preprint No. 1838 (2013). To appear 2014 in *Appl. Math. Optim*.

Corresponding results for the Cahn–Hilliard system (conserved case):

1. P. Colli, J. S.: *Optimal control of the Allen–Cahn equation with singular potentials and dynamic boundary condition.* WIAS Preprint No. 1750 (2012). Submitted.
2. P. Colli, M. H. Farshbaf-Shaker, J. S.: *A deep quench approach to the optimal control of an Allen–Cahn equation with dynamic boundary condition and double obstacle potentials.* WIAS Preprint No. 1838 (2013). To appear 2014 in *Appl. Math. Optim*.

Corresponding results for the Cahn–Hilliard system (conserved case):

3. P. Colli, G. Gilardi, J. S.: *A boundary control problem for the viscous Cahn–Hilliard equation with dynamic boundary conditions.* WIAS Preprint No. 1983 (2014). Submitted.

1. P. Colli, J. S.: *Optimal control of the Allen–Cahn equation with singular potentials and dynamic boundary condition.* WIAS Preprint No. 1750 (2012). Submitted.

2. P. Colli, M. H. Farshbaf-Shaker, J. S.: *A deep quench approach to the optimal control of an Allen–Cahn equation with dynamic boundary condition and double obstacle potentials.* WIAS Preprint No. 1838 (2013). To appear 2014 in *Appl. Math. Optim*.

Corresponding results for the Cahn–Hilliard system (conserved case):

3. P. Colli, G. Gilardi, J. S.: *A boundary control problem for the viscous Cahn–Hilliard equation with dynamic boundary conditions.* WIAS Preprint No. 1983 (2014). Submitted.
4. P. Colli, M. H. Farshbaf-Shaker, G. Gilardi, J. S.: *Optimal boundary control of a viscous Cahn–Hilliard system with dynamic boundary condition and double obstacle potentials.* WIAS Preprint No. 2006 (2014). Submitted.

1. P. Colli, J. S.: *Optimal control of the Allen–Cahn equation with singular potentials and dynamic boundary condition.* WIAS Preprint No. 1750 (2012). Submitted.
2. P. Colli, M. H. Farshbaf-Shaker, J. S.: *A deep quench approach to the optimal control of an Allen–Cahn equation with dynamic boundary condition and double obstacle potentials.* WIAS Preprint No. 1838 (2013). To appear 2014 in *Appl. Math. Optim*.

Corresponding results for the Cahn–Hilliard system (conserved case):

3. P. Colli, G. Gilardi, J. S.: *A boundary control problem for the viscous Cahn–Hilliard equation with dynamic boundary conditions.* WIAS Preprint No. 1983 (2014). Submitted.
4. P. Colli, M. H. Farshbaf-Shaker, G. Gilardi, J. S.: *Optimal boundary control of a viscous Cahn–Hilliard system with dynamic boundary condition and double obstacle potentials.* WIAS Preprint No. 2006 (2014). Submitted.
5. E. Rocca, J. S.: *Optimal distributed control of a nonlocal convective Cahn–Hilliard equation by the velocity in 3D.* WIAS Preprint No. 1942 (2014). Submitted.

1. P. Colli, J. S.: *Optimal control of the Allen–Cahn equation with singular potentials and dynamic boundary condition.* WIAS Preprint No. 1750 (2012). Submitted.
2. P. Colli, M. H. Farshbaf-Shaker, J. S.: *A deep quench approach to the optimal control of an Allen–Cahn equation with dynamic boundary condition and double obstacle potentials.* WIAS Preprint No. 1838 (2013). To appear 2014 in *Appl. Math. Optim*.

Corresponding results for the Cahn–Hilliard system (conserved case):

3. P. Colli, G. Gilardi, J. S.: *A boundary control problem for the viscous Cahn–Hilliard equation with dynamic boundary conditions.* WIAS Preprint No. 1983 (2014). Submitted.
4. P. Colli, M. H. Farshbaf-Shaker, G. Gilardi, J. S.: *Optimal boundary control of a viscous Cahn–Hilliard system with dynamic boundary condition and double obstacle potentials.* WIAS Preprint No. 2006 (2014). Submitted.
5. E. Rocca, J. S.: *Optimal distributed control of a nonlocal convective Cahn–Hilliard equation by the velocity in 3D.* WIAS Preprint No. 1942 (2014). Submitted.
6. S. Frigeri, E. Rocca, J. S.: *Optimal control of a nonlocal Cahn–Hilliard/Navier–Stokes system in 2D.* In preparation.