Dirichlet problems with singular convection terms and applications to some elliptic systems

Lucio Boccardo (Roma 1)

1 Equations

In the paper ([2]), dedicated to the memory of Guido Stampacchia, I improved some of his results (see [7]) concerning the Dirichlet problem

$$(1)\qquad \begin{cases} -\operatorname{div}(M(x)\nabla u) = -\operatorname{div}(u\,E(x)) + f(x) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega. \end{cases}$$

Here Ω is a bounded, open subset of $\mathbb{R}^N$, $N>2$,

$$(2)\qquad E \in (L^N(\Omega))^N,$$

$$(3)\qquad f \in L^m(\Omega),\ \ m \geq 1,$$

and $M(x)$ is a bounded and measurable matrix such that

$$(4)\qquad \alpha|\xi|^2 \leq M(x)\xi\xi, \quad |M(x)| \leq \beta, \quad \text{a.e. } x \in \Omega, \quad \forall\, \xi \in \mathbb{R}^N.$$

To be more precise, in [2] is proved the existence of u

$$(5)\qquad \begin{cases} \text{weak solution belonging to } W_0^{1,2}(\Omega)\cap L^{m^{**}}(\Omega),\ \text{if } m \geq \frac{2N}{N+2} \text{ ("Stampacchia" theory)};\\ \text{distributional solution belonging to } W_0^{1,m^*}(\Omega),\ \text{if } 1<m<\frac{2N}{N+2} \text{ ("Calderon-Zygmund" theory)}; \end{cases}$$

where $m^* = \frac{mN}{N-m}$ $(1 \leq m < N)$ and $m^{**} = \frac{mN}{N-2m}$ $(1 \leq m < \frac{N}{2})$. Note that the above existence results are exactly the results proved with $E=0$ in [7] and [5]. The starting point is a nonlinear approach to a linear noncoercive problem.

Then, in a more recent paper ([3]), dedicated to Thierry Gallouet, differential problems with vector fields E which do not belong to $(L^N(\Omega))^N$ are considered. The first step is the study of the boundary value problem (1) if the main assumption of [2] is not satisfied and we assume

$$(6)\qquad |E(x)| \leq \frac{A}{|x|},\ A>0, \quad 0\in\Omega,$$

which is slightly weaker than (2).

An important feature on E is that we do not make any assumption on the divergence of the field E.

The most important aim (both for the importance and the difficulty) is the study of the case $E \in (L^2(\Omega))^N$ (see also Remark 2.2), where a key point is the definition of solution. It is possible to give a meaning to solution for problem (1), using the concept of *entropy solutions* introduced in [1]. In order to use the functional framework of [1] an important point is the observation that even u does not belong to $W_0^{1,2}(\Omega)$, where u is a solution, nevertheless $\|T_k(u)\|_{W_0^{1,2}(\Omega)} \leq C(k)$, where T_k is the truncature at levels $\pm k$ and $C(k)$ is an unbounded function of k. To be more precise, we have

$$\frac{\alpha}{2}\int_\Omega |\nabla T_k(u)|^2 \leq \frac{k^2}{2\alpha}\,\|E\|^2_{(L^2(\Omega))^N} + k\|f\|_{L^1(\Omega)}$$

and

$$\left[\int_\Omega |\log(1+|u|)|^{2^*}\right]^{\frac{2}{2^*}} \leq \frac{1}{\mathcal{S}^2\alpha^2}\|E\|^2_{(L^2(\Omega))^N} + \frac{2}{\mathcal{S}^2\alpha}\|f\|_{L^1(\Omega)},$$

where $\mathcal{S}$ is the Sobolev constant. Then an important case is (equations with lower order terms) the following boundary value problem

$$\text{(7)} \qquad \lambda > 0, \quad \begin{cases} -\operatorname{div}(M(x)\nabla u) + \lambda u = -\operatorname{div}(u\,E(x)) + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and we prove the existence simple, but important estimate

$$\text{(8)} \qquad \int_\Omega |u| \leq \frac{1}{\lambda}\int_\Omega |f|,$$

which is very useful in the applications.

2 Systems

The situation is even more unusual if we study systems of two equations: the functions $u = \frac{1}{|x|^2} - 1$, $z = -2\log(|x|) - \frac{3}{2}|x|^2 + \frac{3}{2}$ are unbounded solution of the system

$$\begin{cases} -\Delta u + (N-2)u = -\operatorname{div}(u\nabla z) + 2(N+1) & \text{in } B(0,1), \\ -\Delta z = 2(N-2)u + 5N - 4, & \text{in } B(0,1), \\ u = z = 0, & \text{on } \partial B(0,1), \end{cases}$$

where the data $2(N+1)$ and $5N-4$ are bounded (ed even constant). Then one of the existence theorems concerning systems (the existence result uses a duality method in a nonlinear problem) is the following one.

THEOREM 2.1 (ENTROPY-WEAK SOLUTIONS) *Let* $0 \le \rho \le 1$, $\gamma \in \mathbb{R}^+$ *and* f *be a positive function in* $L^m(\Omega)$, *with*

$$m = \frac{(\gamma+1)N}{2(\gamma+1)+N}, \quad \frac{2}{N-2} < \gamma < \frac{N+2}{N-2}, \tag{9}$$

or

$$m = 1, \quad 0 < \gamma < \frac{2}{N-2}, \tag{10}$$

or

$$f \log(1+f) \in L^1(\Omega), \quad \gamma = \frac{2}{N-2}. \tag{11}$$

Assume that the matrix M *is symmetric and satisfies the ellipticity and boundedness assumption (4) and that the matrix* A *is bounded and*

$$A(x)\xi\xi \le 0. \tag{12}$$

Then there exist an entropy solution $u \ge 0$ *and a weak solution* $z \ge 0$, *which belongs to* $W_0^{1,2}(\Omega)$, *of the following system*

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) + u = -\operatorname{div}(u\,A(x)\nabla z) + f(x) & \textit{in } \Omega, \\ -\operatorname{div}(M(x)\nabla z) + \rho\, z = u^\gamma & \textit{in } \Omega, \\ u = z = 0 & \textit{on } \partial\Omega, \end{cases} \tag{13}$$

Furthermore, depending on the summability of $f(x)$, u *belongs respectively to* $L^{m^{**}}(\Omega)$, $L^\sigma(\Omega)$ *with* $\sigma < \frac{N}{N-2}$, $L^{\frac{N}{N-2}}(\Omega)$.

REMARK 2.2 *Note that* $A(x)\nabla z$ *in the first equation of (13) plays the role of* E *in (1) and, in general,* $A(x)\nabla z$ *does not belong to* L^N, *but only belongs to* L^2.

Then, in a joint (with Luigi Orsina and Alessio Porretta) work [1] in progress we study existence of solutions for the following elliptic problem, related to mean field games systems:

$$\begin{cases} -\operatorname{div}(M(x)\nabla\zeta) + \zeta - \operatorname{div}(\zeta\, A(x)\nabla u) = f, \text{in } \Omega, \\ -\operatorname{div}(M(x)\nabla u) + u + \theta\, A(x)\nabla u \cdot \nabla u = \zeta^p, \text{in } \Omega, \\ \qquad \zeta = 0 = u \qquad\qquad , \text{on } \partial\Omega, \end{cases}$$

where $p > 0$, $0 < \theta < 1$, and $f \ge 0$ is a function in some Lebesgue space.

[1] inspired by a talk by Italo Capuzzo Dolcetta

References

[1] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, J.L. Vazquez: An L^1 theory of existence and uniqueness of solutions of nonlinear elliptic equations; Annali Sc. Norm. Sup. Pisa, 22 (1995), 241–273.

[2] L. Boccardo: Some developments on Dirichlet problems with discontinuous coefficients; Boll. Unione Mat. Ital. (9) 2 (2009), 285–297.

[3] L. Boccardo: Dirichlet problems with singular convection term and applications; J. Differential Equations 258 (2015), 2290–2314.

[4] L. Boccardo, T. Gallouët: Nonlinear elliptic and parabolic equations involving measure data; J. Funct. Anal. 87 (1989), 149–169.

[5] L. Boccardo, T. Gallouët: Nonlinear elliptic equations with right hand side measures; Comm. P.D.E. 17 (1992), 641–655.

[6] L. Boccardo, T. Gallouët: $W_0^{1,1}$ solutions in some borderline cases of Calderon-Zygmund theory; J. Differential Equations, 253 (2012), 2698–2714.

[7] G. Stampacchia: Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus; Ann. Inst. Fourier (Grenoble) 15 (1965), 189–258.