Mean field games with congestion: weak solutions

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Outline

1 Introduction

2 Weak solutions in the non singular case: $\mu > 0$

3 Weak solutions in the singular case: $\mu = 0$

4 Mean field type control with congestion
MFG with Congestion

The dynamics of a representative agent is

\[ dX_t = \sqrt{2\nu}dW_t + \gamma_t dt \]

where

- \((W_t)\) is a \(d\)-dimensional Brownian motion
- \((\gamma_t)\) is the control of the agent.

1. **Individual optimal control problem:** the representative agent minimizes

\[
\mathbb{E}_{t,x} \left( \int_t^T \mathcal{L}(X_s, \gamma_s; m_s) ds + G(X_T; m_T) \right),
\]

where \(m_s\) is the distribution of states (a single agent is assumed to have no influence on \(m_s\)).
Dynamic programming yields an optimal feedback \(\gamma_t^*\) and an optimal trajectory \(X_t^*\).

2. **Nash equilibria:**

\[ m_t = \text{law of } X_t^*. \]
Introduction

**Congestion**

- The cost of motion at $x$ depends on $m(x)$ in an increasing manner.
- A typical example was introduced by P-L. Lions (lectures at Collège de France):
  \[ \mathcal{L}(x, \gamma; m) \sim (\mu + m(x))^{\sigma} |\gamma|^q' + F(x, m(x)) \]
  where $\mu \geq 0$, $\sigma > 0$ and $q' > 1$.

  The corresponding Hamiltonian is of the form
  \[ \mathcal{H}(x, p; m) = \frac{|p|^q}{(\mu + m(x))^{\alpha}} - F(x, m(x)), \]
  with $\alpha = \sigma(q - 1)$.

**Remarks**

- Degeneracy of the Hamiltonian $H$ as $m \rightarrow +\infty$
- This model is named “Soft Congestion” by Santambrogio and his coauthors. Their “Hard Congestion” models include inequality contraints on $m$: $m \leq \bar{m}$
The system of PDEs and the main assumptions

\begin{align*}
\begin{cases}
- \partial_t u - \nu \Delta u + \frac{1}{q} \frac{|Du|^q}{(m+\mu)^\alpha} = F(m), & (t, x) \in (0, T) \times \Omega \\
\partial_t m - \nu \Delta m - \text{div} \left( m \frac{|Du|^{q-2} Du}{(m+\mu)^\alpha} \right) = 0, & (t, x) \in (0, T) \times \Omega \\
m(0, x) = m_0(x), & u(T, x) = G(x, m(T)), & x \in \Omega.
\end{cases}
\end{align*}
The system of PDEs and the main assumptions

\[
\begin{aligned}
-\partial_t u - \nu \Delta u + \frac{1}{q} \frac{|Du|^q}{(m+\mu)^\alpha} &= F(m), \quad (t, x) \in (0, T) \times \Omega \\
\partial_t m - \nu \Delta m - \text{div} \left( m \frac{|Du|^{q-2} Du}{(m+\mu)^\alpha} \right) &= 0, \quad (t, x) \in (0, T) \times \Omega \\
m(0, x) = m_0(x), \quad u(T, x) = G(x, m(T)) \quad x \in \Omega.
\end{aligned}
\]  

(1)

Main assumptions

For simplicity, \( \Omega = \mathbb{R}^d / \mathbb{Z}^d \): no difficulty from boundary conditions
The system of PDEs and the main assumptions

\[
\begin{aligned}
-\partial_t u - \nu \Delta u + \frac{1}{q} \frac{|Du|^q}{(m+\mu)^\alpha} &= F(m), \quad (t, x) \in (0, T) \times \Omega \\
\partial_t m - \nu \Delta m - \text{div} \left( m \frac{|Du|^{q-2} Du}{(m+\mu)^\alpha} \right) &= 0, \quad (t, x) \in (0, T) \times \Omega \\
m(0, x) = m_0(x), \quad u(T, x) = G(x, m(T)), \quad x \in \Omega.
\end{aligned}
\]

Main assumptions

\( F \) and \( G \) are bounded from below.

\[ \exists \lambda > 0, \kappa \geq 0, \text{ and a nondecreasing function } f \text{ such that } s \mapsto f(s)s \text{ is convex s.t.} \]

\[ \lambda f(m) - \kappa \leq F(t, x, m) \leq \frac{1}{\lambda} f(m) + \kappa, \quad \forall m \geq 0. \]

Remark: no restriction on the growth

Same kind of assumption for \( G \).
The system of PDEs and the main assumptions

\[
\begin{cases}
-\partial_t u - \nu \Delta u + \frac{1}{q} \frac{|Du|^q}{(m+\mu)^\alpha} = F(m), & (t, x) \in (0, T) \times \Omega \\
\partial_t m - \nu \Delta m - \text{div} \left( m \frac{|Du|^q - 2Du}{(m+\mu)^\alpha} \right) = 0, & (t, x) \in (0, T) \times \Omega \\
m(0, x) = m_0(x), & u(T, x) = G(x, m(T)), & x \in \Omega.
\end{cases}
\]  

Main assumptions

\[m_0 \in C(\Omega) \quad \text{and} \quad m_0 \geq 0.\]
The system of PDEs and the main assumptions

\[
\begin{cases}
-\partial_t u - \nu \Delta u + \frac{1}{q} \frac{|Du|^q}{(m+\mu)^\alpha} = F(m), & (t, x) \in (0, T) \times \Omega \\
\partial_t m - \nu \Delta m - \text{div} \left( m \frac{|Du|^{q-2}Du}{(m+\mu)^\alpha} \right) = 0, & (t, x) \in (0, T) \times \Omega \\
m(0, x) = m_0(x), & u(T, x) = G(x, m(T)), & x \in \Omega.
\end{cases}
\]  

(1)

Main assumptions

- \(1 < q \leq 2\)
- Either \(\mu > 0\) (non singular case) or \(\mu = 0\) (singular case)
- \(0 < \alpha \leq 4 \frac{q-1}{q} = \frac{4}{q'}\)
The condition $\alpha \leq 4(q - 1)/q$

General MFG with local coupling: for systems of the form

$$\begin{cases}
-\partial_t u - \nu \Delta u + H(x, p, m) = F(m) & (t, x) \in (0, T) \times \Omega, \\
\partial_t m - \nu \Delta m - \operatorname{div}(m H_p(x, p, m)) = 0 , & (t, x) \in (0, T) \times \Omega, \\
m(0, x) = m_0(x), \ u(T, x) = G(m(T, x)) , & x \in \Omega ,
\end{cases}$$

P-L. Lions proved that a sufficient condition for the uniqueness of classical solutions is that $F$ and $G$ be non decreasing and that

$$\begin{pmatrix}
-H_m(x, p, m) & \frac{1}{2} m H^T_{m,p}(x, p, m) \\
\frac{1}{2} m H_{m,p}(x, p, m) & m H_{p,p}(x, p, m)
\end{pmatrix} > 0,$$

for all $x \in \Omega$, $m > 0$ and $p \in \mathbb{R}^d$.

In the present congestion model, this condition is equivalent to $\alpha \leq 4 \frac{q-1}{q}$. 
Some references

- P-L. Lions [∼ 2011]: lectures at Collège de France. In particular, the condition for uniqueness of classical solutions.
- Gomes-Mitake [2015]: existence of classical solutions in a specific stationary case: purely quadratic Hamiltonian, i.e. $H(x, p, m) = \frac{|p|^2}{m^\alpha}$, with a very special trick
- Gomes-Voskanyan[2015] and Graber[2015]: short-time existence results of classical solutions for evolutive MFG with congestion

In general, for the existence of classical solutions, restrictive assumptions (e.g. on the growth of $F$ and $G$) are needed.

In particular, if $H(x, p, m) = \frac{|p|^q}{m^\alpha}$, one needs to prove that $m$ does not vanish.

It seems more feasible to work with weak solutions.
Weak solutions

- Weak solutions of the MFG systems were introduced by Lasry and Lions in 2007.
- For Hamiltonians with separate dependencies: $H(x, p, m) = H(x, p) - F(m)$, Porretta, [ARMA 2015], showed that weak solutions allow to build a very general well-posed setting.
- Allow to prove general convergence results for numerical schemes [A.-Porretta 2016].
- If the MFG system of PDEs can be rephrased as the optimality conditions of an optimal control problem driven by some PDE, then
  - weak solutions are the minima of a relaxed functional
  - variational methods can be used
  This occurs often when the Hamiltonian depends separately on $p$ and $m.$
- Can be used for degenerate diffusion [Cardaliaguet-Graber-Porretta-Tonon 2015]
- The variational approach leads to robust (but often slow) numerical methods [Benamou-Carlier], [A.-Laurière].
- Difficulty with the present congestion model: it is not possible to use a variational approach.
Outline

1. Introduction

2. Weak solutions in the non singular case: $\mu > 0$

3. Weak solutions in the singular case: $\mu = 0$

4. Mean field type control with congestion
The main result

Consider the model problem:

\[
\begin{align*}
-\partial_t u - \nu \Delta u + \frac{1}{q} \frac{|Du|^q}{(m+\mu)^\alpha} &= F(m), & (t, x) \in (0, T) \times \Omega \\
\partial_t m - \nu \Delta m - \text{div} \left( m \frac{|Du|^q - 2Du}{(m+\mu)^\alpha} \right) &= 0, & (t, x) \in (0, T) \times \Omega \\
m(0, x) &= m_0(x), \quad u(T, x) = G(x, m(T, x)), & x \in \Omega.
\end{align*}
\]

Definition

A weak solution \((u, m)\) is a distributional solution of the system such that

\[
\begin{align*}
mF(m) &\in L^1, & m_TG(m_T) &\in L^1(\Omega), \\
m\frac{|Du|^q}{(\mu+m)^\alpha} &\in L^1, & \frac{|Du|^q}{(\mu+m)^\alpha} &\in L^1.
\end{align*}
\]

Theorem

Under the previous assumptions and if \(F\) and \(G\) are non decreasing, then there exists a unique weak solution.
Weak solutions in the non singular case: $\mu > 0$

**Extension**

Existence and uniqueness of weak solutions holds for

\[
\begin{aligned}
-\partial_t u - \nu \Delta u + H(t, x, p, m) &= F(m) \quad (t, x) \in (0, T) \times \Omega \\
\partial_t m - \nu \Delta m - \text{div}(m H_p(t, x, p, m)) &= 0, \quad (t, x) \in (0, T) \times \Omega \\
m(0, x) &= m_0(x), \quad u(T, x) = G(m(T, x)), \quad x \in \Omega,
\end{aligned}
\]

under the structure conditions

\[
H(t, x, 0, m) \leq 0,
\]

\[
H(t, x, p, m) \geq c_0 \frac{|p|^q}{(m + \mu)^\alpha} - c_1 (1 + m^{\frac{q-1}{q}}),
\]

\[
|H_p(t, x, p, m)| \leq c_2 (1 + \frac{|p|^{q-1}}{(m + \mu)^\alpha}),
\]

\[
H_p(t, x, p, m) \cdot p \geq (1 + \sigma) H(t, x, p, m) - c_3 (1 + m^{\frac{q-1}{q}}),
\]

for a.e. $(t, x) \in Q_T$ and every $p \in \mathbb{R}^N$, where $\sigma, c_0, \ldots, c_3$ are positive constants,

and the same assumptions on $F, G, \alpha$ and $q$. 

Y. Achdou

Roma, 15-6-2017
Main arguments in the proof
A regularized problem and energy estimates

\[- \partial_t u^\epsilon - \nu \Delta u^\epsilon + H(t, x, T_{1/\epsilon}m^\epsilon, Du^\epsilon) = F^\epsilon(t, x, m^\epsilon), \quad (t, x) \in (0, T) \times \Omega \]
\[\partial_t m^\epsilon - \nu \Delta m^\epsilon - \text{div}(m^\epsilon H_p(t, x, T_{1/\epsilon}m^\epsilon, Du^\epsilon)) = 0, \quad (t, x) \in (0, T) \times \Omega \]

\[m^\epsilon(0, x) = m_0^\epsilon(x), \quad u^\epsilon(T, x) = G^\epsilon(x, m^\epsilon(T)), \quad x \in \Omega \]

where

\[T_{1/\epsilon}m = \min(m, 1/\epsilon), \]
\[F^\epsilon(t, x, m) = \rho^\epsilon \ast F(t, \cdot, \rho^\epsilon \ast m))(x), \]
\[G^\epsilon(x, m) = \rho^\epsilon \ast G(\cdot, \rho^\epsilon \ast m))(x), \]
\[m_0^\epsilon = \rho^\epsilon \ast m_0, \]

and \(\rho^\epsilon\) is a standard symmetric mollifier in \(\mathbb{R}^d\).
A regularized problem and energy estimates

\[ -\partial_t u^\epsilon - \nu \Delta u^\epsilon + H(t, x, T_{1/\epsilon} m^\epsilon, D u^\epsilon) = F^\epsilon(t, x, m^\epsilon), \quad (t, x) \in (0, T) \times \Omega \]

\[ \partial_t m^\epsilon - \nu \Delta m^\epsilon - \text{div}(m^\epsilon H_p(t, x, T_{1/\epsilon} m^\epsilon, D u^\epsilon)) = 0, \quad (t, x) \in (0, T) \times \Omega \]

\[ m^\epsilon(0, x) = m_0^\epsilon(x), \quad u^\epsilon(T, x) = G^\epsilon(x, m^\epsilon(T)), \quad x \in \Omega \]

where

\[ T_{1/\epsilon} m = \min(m, 1/\epsilon), \]

\[ F^\epsilon(t, x, m) = \rho^\epsilon \ast F(t, \cdot, \rho^\epsilon \ast m)(x), \]

\[ G^\epsilon(x, m) = \rho^\epsilon \ast G(\cdot, \rho^\epsilon \ast m)(x), \]

\[ m_0^\epsilon = \rho^\epsilon \ast m_0, \]

and \( \rho^\epsilon \) is a standard symmetric mollifier in \( \mathbb{R}^d \).

Standard energy estimates:

\[ u^\epsilon(t, x) \geq C, \quad \|u^\epsilon\|_{L^\infty(0, T; L^1(\Omega))} \leq C, \]

\[ \int_\Omega G^\epsilon(x, m^\epsilon(T))m^\epsilon(T)dx + \int_0^T \int_\Omega F^\epsilon(t, x, m^\epsilon)m^\epsilon dxdt + \|T_{1/\epsilon}m^\epsilon\|^\frac{\alpha}{q-1} + 1 \|N+2\|^\frac{N+2}{N} \leq C, \]

\[ \int_0^T \int_\Omega \frac{|Du^\epsilon|^q}{(T_{1/\epsilon} m^\epsilon + \mu)^\alpha} dxdt + \int_0^T \int_\Omega m^\epsilon \frac{|Du^\epsilon|^q}{(T_{1/\epsilon} m^\epsilon + \mu)^\alpha} dxdt \leq C \]
Properties of Fokker-Planck equations with $L^2$ drifts. [Porretta, ARMA 2015]

Set $Q_T = (0, T) \times \Omega$. For any $b \in L^2(Q_T; \mathbb{R}^d)$, consider the Fokker-Planck equation

$$\begin{cases}
\partial_t m - \nu \Delta m - \text{div}(mb) = 0 & \text{in } (0,T) \times \Omega, \\
m(t = 0) = m_0.
\end{cases}$$

(2)
Weak solutions in the non singular case: $\mu > 0$

Properties of Fokker-Planck equations with $L^2$ drifts. [Porretta, ARMA 2015]

Set $Q_T = (0, T) \times \Omega$. For any $b \in L^2(Q_T; \mathbb{R}^d)$, consider the Fokker-Planck equation

$$\begin{cases}
\partial_t m - \nu \Delta m - \text{div} (mb) & = 0 \quad \text{in } (0, T) \times \Omega, \\
m(t = 0) & = m_0.
\end{cases}$$

(2)

- A weak solution is a nonnegative distributional sol. $m \in L^1(Q_T)$ of (2) s.t.

$$m|b|^2 \in L^1(Q_T).$$

(3)

- A renormalized solution of (2) is a nonnegative function $m \in L^1(Q_T)$ s.t.

- for any $k > 0$, $T_k(m) \in L^2(0, T, H^1(\Omega))$, where $(T_k(m)) = \max(-k, \min(k, m))$

$$\lim_{n \to +\infty} \frac{1}{n} \int_{\{n < m(t,x) < 2n\}} |Dm|^2 dx dt = 0$$

- for all $S \in W^{2,\infty}(\mathbb{R})$ such that $S'$ has compact support,

$$\begin{cases}
\partial_t S(m) - \nu \Delta S(m) - \text{div} (mbS'(m)) + \nu S''(m)|Dm|^2 + S''(m)mb \cdot Dm & = 0, \\
S(m(t = 0)) & = S(m_0).
\end{cases}$$

Y. Achdou

Roma, 15-6-2017
Weak solutions in the non singular case: $\mu > 0$

Properties of Fokker-Planck equations with $L^2$ drifts. [Porretta, ARMA 2015]

Set $Q_T = (0, T) \times \Omega$. For any $b \in L^2(Q_T; \mathbb{R}^d)$, consider the Fokker-Planck equation

$$\begin{cases} \partial_t m - \nu \Delta m - \text{div}(mb) &= 0 \quad \text{in } (0, T) \times \Omega, \\ m(t = 0) &= m_0. \end{cases} \tag{2}$$

1. **Uniqueness:** there exists at most one weak solution of (2)

2. **Weak sol. $\Leftrightarrow$ renormalized sol. and $m|b|^2 \in L^1(Q_T)$:** any weak solution $m$ belongs to $C([0, T]; L^1(\Omega))$ and is a renormalized solution

3. **Compactness:** if $(b, m_0)$ lies in a bounded subset of $L^2(Q_T) \times L^1(\Omega)$, then $m$ lies in a relatively compact subset of $L^1(Q_T)$

4. **Stability:** consider a sequence $m^\epsilon$ of weak solutions of the F.P equation associated to $b^\epsilon \in L^2(Q_T; \mathbb{R}^d)$.
   If $m^\epsilon \to m$ a.e. in $Q_T$ and if $m^\epsilon|b^\epsilon|^2 \to m|b|^2$ in $L^1(Q_T)$,
   then $m^\epsilon \to m$ in $C([0, T]; L^1(\Omega))$ and $m$ is a weak solution of the F.P. equation associated to $b$. 

Y. Achdou Roma, 15-6-2017
Passage to the limit if $1 < q < 2$: main steps

- Energy estimates $\Rightarrow -\partial_t u^\epsilon - \nu \Delta u^\epsilon$ is bounded in $L^1(Q_T)$: $\Rightarrow$ for subsequences, $u^\epsilon \to u$ and $Du^\epsilon \to Du$ in $L^1(Q_T)$ and a.e.
Passage to the limit if $1 < q < 2$: main steps

- $u^\epsilon \rightarrow u$ and $Du^\epsilon \rightarrow Du$ in $L^1(Q_T)$ and a.e.

- $\partial_t m^\epsilon - \nu \Delta m^\epsilon - \text{div}(m^\epsilon b^\epsilon) = 0$, with $|b^\epsilon| \approx \frac{|Du^\epsilon|^{q-1}}{(\mu + T_1/m^\epsilon)^\alpha}$.

  - Energy estimates: $|b^\epsilon|_{L^{q'}(Q_T)} \leq C$ with $q' \geq 2$
    $\Rightarrow m^\epsilon$ is compact in $L^1(Q_T) \Rightarrow m^\epsilon \rightarrow m$ in $L^1(Q_T)$ and a.e.

- $b^\epsilon \rightarrow b = H_p(x, Du, m)$ a.e.
Weak solutions in the non singular case: \( \mu > 0 \)

Passage to the limit if \( 1 < q < 2 \): main steps

- \( u^\epsilon \to u \) and \( Du^\epsilon \to Du \) in \( L^1(Q_T) \) and a.e.

- \( \partial_t m^\epsilon - \nu \Delta m^\epsilon - \text{div}(m^\epsilon b^\epsilon) = 0 \), with \( |b^\epsilon| \approx \frac{|Du^\epsilon|^{q-1}}{\mu + T_1/\epsilon m^\epsilon} \).

- \( m^\epsilon \to m \) in \( L^1(Q_T) \) and a.e.

- \( b^\epsilon \to b = H_p(x, Du, m) \) a.e.

- Energy estimates and \( (1 - \alpha)q' \geq -\alpha \Rightarrow \int_{Q_T} m^\epsilon |b^\epsilon|^{q'} \leq C \), for a constant \( C \) independent of \( \mu \).

- Since \( q' > 2 \), \( m^\epsilon |b^\epsilon|^2 \) is compact in \( L^1(Q_T) \).

- Stability result: \( m^\epsilon \to m \) in \( C([0, T], L^1(\Omega)) \) and \( m \) is a weak sol. of the Fokker Planck eq. related to \( b \).
Passage to the limit if $1 < q < 2$: main steps

- $u^\epsilon \rightarrow u$ and $Du^\epsilon \rightarrow Du$ in $L^1(Q_T)$ and a.e.
- $\partial_t m^\epsilon - \nu \Delta m^\epsilon - \text{div}(m^\epsilon b^\epsilon) = 0$, with $|b^\epsilon| \approx \frac{|Du^\epsilon|^{q-1}}{(\mu + T_1/\epsilon m^\epsilon)^\alpha}$.
- $m^\epsilon \rightarrow m$ in $L^1(Q_T)$ and a.e.
- $b^\epsilon \rightarrow b = H_p(x, Du, m)$ a.e.
- $m^\epsilon \rightarrow m$ in $C([0, T], L^1(\Omega))$ and $m$ is a weak sol. of the Fokker Planck eq. related to $b$.
- $F^\epsilon(m^\epsilon) \rightarrow F(m)$ and $G^\epsilon(x, m^\epsilon(T)) \rightarrow G(x, m(T))$ in $L^1$
- Passage to the limit in the Bellman equation: OK from the steps above and from stability results for HJB eq. ([Porretta 99]) because the Hamiltonian has natural growth and the good sign
- The proof is achieved for $q < 2$. 

Y. Achdou
Roma, 15-6-2017
The case $q = 2$

What remains from the steps above?

- $u_\epsilon \to u$, $Du_\epsilon \to Du$, $m_\epsilon \to m$ in $L^1(Q_T)$ and a.e.,
  $b^\epsilon = H_p(t, x, T_1/\epsilon m^\epsilon, Du^\epsilon) \to b = H_p(t, x, m, Du)$ a.e.

- $u$ is a subsolution of the Bellman equation (no terminal condition yet) (from Fatou lemma and the equi-integrability of $F^\epsilon(t, x, m^\epsilon)$)

$$\lim_{\epsilon \to 0} \int_{\Omega} u^\epsilon|_{t=0}(x)m_0(x)dx \leq \langle u|_{t=0}, m_0 \rangle,$$

where both members of the inequality are well defined
Weak solutions in the non singular case: $\mu > 0$

The case $q = 2$

What remains from the steps above?

- $u_\epsilon \to u$, $Du_\epsilon \to Du$, $m_\epsilon \to m$ in $L^1(Q_T)$ and a.e.,
  $b_\epsilon = H_p(t, x, T_{1/\epsilon} m_\epsilon, Du_\epsilon) \to b = H_p(t, x, m, Du)$ a.e.

- $u$ is a subsolution of the Bellman equation (no terminal condition yet) and
  $$\lim_{\epsilon \to 0} \int_\Omega u_\epsilon|_{t=0}(x)m_0(x)dx \leq \langle u|_{t=0}, m_0 \rangle$$

- $$m^\epsilon b_\epsilon = m^\epsilon H_p(t, x, T_{1/\epsilon} m^\epsilon, Du^\epsilon) \leq c_2 m^\epsilon + w^\epsilon + \sqrt{m^\epsilon} z^\epsilon$$

where $w^\epsilon$ and $z^\epsilon$ are bounded in $L^2(Q_T)$. Therefore, $m^\epsilon b^\epsilon$ is equi-integrable
  $\Rightarrow m$ is a distribution sol. of the F.P. related to $b$
Weak solutions in the non singular case: $\mu > 0$

The case $q = 2$

What remains from the steps above?

- $u_\epsilon \to u$, $Du_\epsilon \to Du$, $m_\epsilon \to m$ in $L^1(Q_T)$ and a.e.,
  $b^\epsilon = H_p(t, x, T_1/\epsilon m^\epsilon, Du^\epsilon) \to b = H_p(t, x, m, Du)$ a.e.

- $u$ is a subsolution of the Bellman equation (no terminal condition yet) and
  \[
  \lim_{\epsilon \to 0} \int_{\Omega} u^\epsilon|_{t=0}(x)m_0(x)dx \leq \langle u|_{t=0}, m_0 \rangle
  \]

- $m$ is a distribution sol. of the F.P. related to $b$

\[
\int_0^T \int_{\Omega} F(t, x, m)m\,dx\,dt + \int_0^T \int_{\Omega} \frac{|Du|^2}{(m + \mu)^\alpha} \,dx\,dt + \int_0^T \int_{\Omega} m \frac{|Du|^2}{(m + \mu)^\alpha} \,dx\,dt \leq C.
\]

- $m$ is a weak solution of the Fokker-Planck equation, so $m \in C([0, T], L^1(\Omega))$
Weak solutions in the non singular case: $\mu > 0$

The case $q = 2$

What remains from the steps above?

- $u_\epsilon \to u$, $Du_\epsilon \to Du$, $m_\epsilon \to m$ in $L^1(Q_T)$ and a.e.

- $u$ is a subsolution of the Bellman equation (no terminal condition yet) and
  $$\lim_{\epsilon \to 0} \int_{\Omega} u_\epsilon|_{t=0}(x) m_0(x) dx \leq \langle u|_{t=0}, m_0 \rangle$$

- $m$ is a weak solution of the Fokker-Planck eq., and $m \in C([0, T], L^1(\Omega))$

- Finer arguments related to the F.P. equation imply that $m_\epsilon(t) \rightharpoonup m(t)$ in $L^1(\Omega)$ weak for all $t$, and
  $$\int_{\Omega} G(m(T)) m(T) dx \leq C.$$
The case $q = 2$

What remains from the steps above?

- $u_\epsilon \to u$, $Du_\epsilon \to Du$, $m_\epsilon \to m$ in $L^1(Q_T)$ and a.e.
- $u$ is a subsolution of the Bellman equation (no terminal condition yet) and
  \[
  \lim_{\epsilon \to 0} \int_{\Omega} u_\epsilon|_{t=0}(x)m_0(x)dx \leq \langle u|_{t=0}, m_0 \rangle
  \]
- $m$ is a weak solution of the Fokker-Planck eq., and $m \in C([0, T], L^1(\Omega))$
- $m_\epsilon(t) \rightharpoonup m(t)$ in $L^1(\Omega)$ weak for all $t$, and
  \[
  \int_{\Omega} G(m(T))m(T) dx \leq C.
  \]

One needs to work more, because $m_\epsilon|b_\epsilon|^2$ is no longer equi-integrable, so we do not obtain the convergence of $m_\epsilon$ to $m$ in $C([0, T]; L^1(\Omega))$, and the convergence of $G_\epsilon(m_\epsilon(T))$ is more difficult to prove.
Weak solutions in the non singular case: $\mu > 0$

A crossed energy inequality

**Theorem**

Consider $(u, m)$ such that

1. $mF(m) \in L^1$, $m|_{t=T}G(m|_{t=T}) \in L^1(\Omega)$,
2. $m\frac{|Du|^q}{(\mu+m)^\alpha} \in L^1$, $\frac{|Du|^q}{(\mu+m)^\alpha} \in L^1$,
3. $m$ is a weak sol. of \( \left( \text{F.P. equation } + m|_{t=0} = m_0 \right) \)
4. $u$ is a distrib. subsol. of \( \left( \text{the Bellman equation } + u|_{t=T} \leq G(m|_{t=T}) \right) \)

For any pair $(\tilde{u}, \tilde{m})$ with the same properties as $(u, m)$, we have the crossed-integrability:

$$\tilde{m} \frac{|Du|^q}{(\mu+m)^\alpha} \in L^1, \quad m \frac{|D\tilde{u}|^q}{(\mu+\tilde{m})^\alpha} \in L^1, \ldots$$

and the energy inequality:

$$\langle \tilde{m}_0, u(0) \rangle \leq \int_{\Omega} G(x, m(T)) \tilde{m}(T) \, dx + \int_0^T \int_{\Omega} F(t, x, m) \tilde{m} \, dx \, dt$$
$$+ \int_0^T \int_{\Omega} [\tilde{m} H_P(t, x, \tilde{m}, D\tilde{u}) \cdot Du - \tilde{m} H(t, x, m, Du)] \, dx \, dt$$

Y. Achdou Roma, 15-6-2017
Passage to the limit using the crossed energy inequality

We start from the energy identity for \((u^\varepsilon, m^\varepsilon)\):

\[
\int_0^T \int_\Omega m^\varepsilon \left( H_p(t,x,T_1/\varepsilon m^\varepsilon, \text{Du}^\varepsilon) \cdot \text{Du}^\varepsilon - H(t,x,T_1/\varepsilon m^\varepsilon, \text{Du}^\varepsilon) \right) \, dx \, dt \\
+ \int_\Omega m^\varepsilon(T)G^\varepsilon(x,m^\varepsilon(T)) \, dx + \int_0^T \int_\Omega m^\varepsilon F^\varepsilon(t,x,m^\varepsilon) \, dx \, dt = \int_\Omega u^\varepsilon(0)m_0^\varepsilon \, dx.
\]

By Fatou lemma:

\[
\limsup_{\varepsilon \to 0} \int_\Omega m^\varepsilon(T)G^\varepsilon(x,m^\varepsilon(T)) \, dx \leq \langle u(0), m_0 \rangle - \int_0^T \int_\Omega mF(t,x,m) \, dx \, dt \\
- \int_0^T \int_\Omega m \left( H_p(t,x,m, \text{Du}) \cdot \text{Du} - H(t,x,m, \text{Du}) \right) \, dx \, dt
\]

But thanks to the crossed energy inequality applied to \((\tilde{u}, \tilde{m}) = (u, m)\),

\[
\langle u(0), m_0 \rangle - \int_0^T \int_\Omega mF(t,x,m) \, dx \, dt \\
- \int_0^T \int_\Omega m \left( H_p(t,x,m, \text{Du}) \cdot \text{Du} - H(t,x,m, \text{Du}) \right) \, dx \, dt \leq \int_\Omega m(T)G(x,m(T)) \, dx
\]

Using the monotonicity of \(G\), we then get \(G^\varepsilon(x,m^\varepsilon(T)) \to G(x,m(T))\) in \(L^1(\Omega)\).

Conclusion as in the case \(q < 2\).
To be more correct ...

I cheated a bit in the previous slides, because I did not prove that $u$ is a subsolution of the boundary value problem with the terminal condition $u(T) \leq G(x, m(T))$.

The rigorous argument goes through the parametrized Young measure generated by the sequence $\rho^\epsilon \star m^\epsilon(T)$:

$$f(x, \rho^\epsilon \star m^\epsilon(T)) \rightharpoonup \int_\mathbb{R} f(x, \lambda) d\nu_x(\lambda) \quad \text{weakly in } L^1(\Omega)$$

for every Carathéodory function $f(x, s)$ such that $f(x, \rho^\epsilon \star m^\epsilon(T))$ is equi-integrable.

We get, with the previous argument,

$$\lim_{\epsilon \to 0} \int_\Omega m^\epsilon(T) G^\epsilon(x, m^\epsilon(T)) dx = \int_\Omega \int_\mathbb{R} m(T) G(x, \lambda) d\nu_x(\lambda) dx,$$

which allows to conclude that $G^\epsilon(x, m^\epsilon(T)) \to G(x, m(T))$. 
Weak solutions in the non singular case: $\mu > 0$

Uniqueness

Needs the following lemma:

**Lemma**

A weak solution $(u, m)$ satisfies the energy identity:

$$
\langle m_0, u(0) \rangle = \int_{\Omega} G(x, m(T)) m(T) \, dx + \int_0^T \int_{\Omega} F(t, x, m) m \, dx \, dt \\
+ \int_0^T \int_{\Omega} [m H_p(t, x, m, Du) \cdot Du - m H(t, x, m, Du)] \, dx \, dt
$$

Then, take two weak solutions $(u, m)$ and $(\tilde{u}, \tilde{m})$. Use

1. the 2 energy identities for $(u, m)$ and $(\tilde{u}, \tilde{m})$
2. the 2 crossed energy inequalities for
   1. $(u, m)$ and $(\tilde{u}, \tilde{m})$
   2. $(u, m)$ and $(\tilde{u}, \tilde{m})$

Adding all the identities/inequalities, we conclude as P-L. Lions for classical solutions, using the monotonicity of $F$ and $G$ and $\alpha \leq 4/q'$. 
Outline

1. Introduction

2. Weak solutions in the non singular case: $\mu > 0$

3. Weak solutions in the singular case: $\mu = 0$

4. Mean field type control with congestion
Weak solutions in the singular case: \( \mu = 0 \)

The case \( \mu = 0 \): weak solutions (1)

\[
\begin{align*}
-\partial_t u - \nu \Delta u + \frac{1}{q} \frac{|D u|^q}{m^\alpha} &= F(m), & (t, x) \in (0, T) \times \Omega \\
\partial_t m - \nu \Delta m - \text{div}(m \frac{|D u|^{q-2} D u}{m^\alpha}) &= 0, & (t, x) \in (0, T) \times \Omega \\
m(0, x) &= m_0(x), & u(T, x) = G(x, m(T)), & x \in \Omega.
\end{align*}
\]
**The case $\mu = 0$: weak solutions (2)**

**Definition** The pair $(u,m)$ is a weak solution if

1. \[ mF(m) \in L^1, \quad m_T G(m_T) \in L^1(\Omega), \]
\[ m1\{m>0\} \frac{|Du|^q}{m^\alpha} \in L^1, \quad 1\{m>0\} \frac{|Du|^q}{m^\alpha} \in L^1. \]

2. $u$ is a subsolution of the Bellman equation: for any $0 \leq \varphi \in C^\infty_c((0,T] \times \Omega)$,
\[
\int_0^T \int_\Omega u \varphi_t \, dxdt - \nu \int_0^T \int_\Omega u \Delta \varphi \, dxdt + \int_0^T \int_\Omega H(t,x,m,Du)1\{m>0\} \varphi \, dxdt
\leq \int_0^T \int_\Omega F(t,x,m) \varphi \, dxdt + \int_\Omega G(x,m(T)) \varphi(T) \, dx
\]

3. $m$ is a weak solution of
\[
\partial_t m - \nu \Delta m - \text{div} \left( m1\{m>0\} \frac{|Du|}{m^\alpha} \right) = 0
\]
\[
m(t = 0) = m_0
\]

4. The energy identity holds:
\[
\int_\Omega m_0 u(0) \, dx = \int_\Omega G(x,m(T)) m(T) \, dx + \int_0^T \int_\Omega F(t,x,m) m \, dxdt
\]
\[
+ \int_0^T \int_\Omega m \left[ H_p(t,x,m,Du) \cdot Du - H(t,x,m,Du) \right] 1\{m>0\} \, dxdt
\]
where the first term is understood as the trace of $\int_\Omega u(t) m_0 \, dx$ in $BV(0,T)$. 

Y. Achdou
Roma, 15-6-2017
Existence and uniqueness

Theorem

If either \((q < 2 \text{ and } \alpha \leq \frac{4}{q'})\) or \((q = 2 \text{ and } \alpha < \frac{4}{q'} = 2)\), and if \(F\) and \(G\) are nondecreasing, then there exists a unique weak solution.

Remark We miss the limit case \(q = 2\) and \(\alpha = 2\).

Proof

- Consists of passing to the limit as \(\mu \to 0^+\) in the non singular case discussed previously.

- It goes through a careful adaption of the previous arguments with a special attention to the regions where \(m = 0\).
Outline

1. Introduction

2. Weak solutions in the non singular case: \( \mu > 0 \)

3. Weak solutions in the singular case: \( \mu = 0 \)

4. Mean field type control with congestion
MFG vs. Mean field type control (MFTC) (1)

- MFG: look for Nash equilibria with $N$ identical agents, then let $N \to \infty$

- Carmona and Delarue / Bensoussan et al have studied the control of McKean-Vlasov dynamics:
  - Assume that the all $N$ agents use the same feedback law $\gamma$
  - The perturbations of $\gamma$ impact the empirical distribution
  - Pass to the limit as $N \to \infty$ first, then minimize the asymptotic cost
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- MFTC models consists of an optimal control problem driven by a Fokker-Planck equation:

  Find a feedback $\gamma_s = \gamma(s, X_s; m_s)$ which minimizes

  \[
  J(t) = \mathbb{E} \left\{ \int_t^T \mathcal{L}(X_s, \gamma_s; m_s) \, ds + \mathcal{G}(X_T; m_T) \right\}
  \]

  subject to

  \[
  dX_t = \sqrt{2\nu} dW_t + \gamma_t \, dt
  \]

  $m_t$ is the law of $X_t$,

  therefore

  \[
  \frac{\partial m}{\partial t} - \nu \Delta m + \text{div} (m \gamma) = 0
  \]

  $m|_{t=0} = m_0$. 

Y. Achdou Roma, 15-6-2017
Assume local dependency: \( \mathcal{L}(x, \gamma; m) = L(x, \gamma, m(x)) \) and \( \mathcal{G}(x; m) = G(x, m(x)) \): the cost can be expressed as

\[
J(t) = \int_t^T \int_{\Omega} L(x, \gamma(s, x, m_s) m(s, x)) m(s, x) ds dx + \int_{\Omega} G(x; m(T, x)) m(T, x) dx
\]

and the optimality conditions read

\[
\frac{\partial u}{\partial t} + \nu \Delta u - H(x, \nabla u, m(t, x)) - m(t, x) \frac{\partial H}{\partial m}(x, \nabla u(x, t), m(t, x)) = 0
\]

\[
\frac{\partial m}{\partial t} - \nu \Delta m - \text{div} \left( m \frac{\partial H}{\partial p}(x, \nabla u; m) \right) = 0
\]

with the terminal and initial conditions

\[
u(t = T, x) = G(x, m(T, x)) + m(T, x) \frac{\partial G}{\partial m}(x, m(T, x))
\]

\[
m(0, x) = m_0(x)
\]
The latter PDE system enjoys uniqueness if
\[
\begin{pmatrix}
-(mH)_{m,m} & 0 \\
0 & mH_{p,p}
\end{pmatrix} > 0
\]
for all \(x \in \Omega, \ m > 0\) and \(p \in \mathbb{R}^d\).

If \(H(x, p, m) = \frac{|p|^q}{(\mu + m)^\alpha}\), then the latter condition holds if \(\alpha \leq 1\), (while the condition is \(\alpha \leq \frac{4q}{q-1}\) for the MFG system with the same Hamiltonian).

Existence and uniqueness for weak solutions of the mean field type control problem with congestion and possibly degenerate diffusion were proved in [A., Laurière 2015], using a variational approach.

Numerical methods using the variational approach were studied in [A., Laurière 2016].