On the long time behavior of the master equation in Mean Field Games

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The discounted MFG system

Given a positive discount factor $\delta > 0$, we consider the MFG system

$$(MFG-\delta) \qquad \left\{ \begin{array}{l} -\partial_t u^\delta + \delta u^\delta - \Delta u^\delta + H(x, Du^\delta) = f(x, m^\delta(t)) \qquad \text{in } (0, +\infty) \times \mathbb{T}^d \\ \partial_t m^\delta - \Delta m^\delta - \mathrm{div}(m^\delta H_p(x, Du^\delta)) = 0 \qquad \text{in } (0, +\infty) \times \mathbb{T}^d \\ m^\delta(0, \cdot) = m_0 \qquad \text{in } \mathbb{T}^d, \qquad u^\delta \text{ bounded in } (0, +\infty) \times \mathbb{T}^d \end{array} \right.$$

where

- $u^\delta = u^\delta(t, x)$ and $m^\delta = m^\delta(t, x)$ are the unknown,
- $H = H(x, p) : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is a smooth, unif. convex in p, Hamiltonian,
- $f, g : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ are “smooth" and monotone,
  ($\mathcal{P}(\mathbb{T}^d) =$ the set of Borel probability measures on $\mathbb{T}^d$)
- $m_0 \in \mathcal{P}(\mathbb{T}^d)$ is a smooth positive density

The MFG system has been introduced by Lasry-Lions and Huang-Caines-Malhamé to study optimal control problems with infinitely many controllers.

Interpretation.

If (u^δ, m^δ) solves the discounted MFG system,

- then u is the value function of a typical small player :

$$u(t,x) = \inf_\alpha \mathbb{E}\left[\int_0^{+\infty} e^{-\delta s} L(X_s,\alpha_s) + F(X_s, m^\delta(s))\, ds\right]$$

  where

$$dX_s = \alpha_s ds + \sqrt{2} dW_s \text{ for } s \in [t,+\infty), \qquad X_t = x$$

  and L is the Fenchel conjugate of H :

$$L(x,\alpha) := \sup_{p\in\mathbb{R}^d} -\alpha\cdot p - H(x,p)$$

- and m^δ is the distribution of the players when they play in an optimal way : $m^\delta := \mathcal{L}(Y_s)$ with

$$dY_s = -H_p(Y_s, Du(s,Y_s))ds + \sqrt{2} dW_s, \qquad s\in[0,+\infty), \qquad \mathcal{L}(Y_0) = m_0.$$

The limit problem

Let (u^δ, m^δ) be the solution to

$$(MFG-\delta) \qquad \begin{cases} -\partial_t u^\delta + \delta u^\delta - \Delta u^\delta + H(x, Du^\delta) = f(x, m^\delta(t)) & \text{in } (0,+\infty)\times\mathbb{T}^d \\ \partial_t m^\delta - \Delta m^\delta - \text{div}(m^\delta H_p(x, Du^\delta)) = 0 & \text{in } (0,+\infty)\times\mathbb{T}^d \\ m^\delta(0,\cdot) = m_0 \quad \text{in } \mathbb{T}^d, & u^\delta \text{ bounded in } (0,+\infty)\times\mathbb{T}^d \end{cases}$$

Study the limit as $\delta \to 0^+$ of the pair (u^δ, m^δ).

- **Motivation :** classical question in economics/game theory (players infinitely patient).
- In contrast with similar problem for HJ equation, **forward-backward** system.

One expects that (u^δ, m^δ) "converges" to the solution of the ergodic MFG problem

$$(MFG-erg) \qquad \begin{cases} \bar\lambda - \Delta \bar u + H(x, D\bar u) = f(x, \bar m) & \text{in } \mathbb{T}^d \\ -\Delta \bar m - \text{div}(\bar m H_p(x, D\bar u)) = 0 & \text{in } \mathbb{T}^d \\ \bar m \geq 0 \quad \text{in } \mathbb{T}^d, & \int_{\mathbb{T}^d} \bar m = 1 \end{cases}$$

where now the unknown are $\bar\lambda$, $\bar u = \bar u(x)$ and $\bar m = \bar m(x)$.

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$$(MFG-\delta) \qquad \left\{ \begin{array}{l} -\partial_t u^\delta + \delta u^\delta - \Delta u^\delta + H(x, Du^\delta) = f(x, m^\delta(t)) \qquad \text{in } (0,+\infty)\times\mathbb{T}^d \\ \partial_t m^\delta - \Delta m^\delta - \mathrm{div}(m^\delta H_p(x, Du^\delta)) = 0 \qquad \text{in } (0,+\infty)\times\mathbb{T}^d \\ m^\delta(0,\cdot) = m_0 \qquad \text{in } \mathbb{T}^d, \qquad u^\delta \text{ bounded in } (0,+\infty)\times\mathbb{T}^d \end{array}\right.$$

Study the limit as $\delta \to 0^+$ of the pair (u^δ, m^δ).

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$$(MFG-erg) \qquad \left\{ \begin{array}{ll} \bar\lambda - \Delta \bar u + H(x, D\bar u) = f(x, \bar m) & \text{in } \mathbb{T}^d \\ -\Delta \bar m - \mathrm{div}(\bar m H_p(x, D\bar u)) = 0 & \text{in } \mathbb{T}^d \\ \bar m \geq 0 \quad \text{in } \mathbb{T}^d, \qquad \displaystyle\int_{\mathbb{T}^d} \bar m = 1 & \end{array}\right.$$

where now the unknown are $\bar\lambda$, $\bar u = \bar u(x)$ and $\bar m = \bar m(x)$.

Classical results for decoupled problems

- For the Fokker-Plank equation driven by a vector-field V :

$$\partial_t m - \Delta m - \mathrm{div}(mV(x)) = f(x) \qquad \text{in } (0,\infty)\times\mathbb{T}^d$$

(exponential) convergence of $m(t)$ to the ergodic measure is well-known.

- For HJ equations : Let $v = v(t,x)$ and $u^\delta = u^\delta(x)$ be the solution to

$$\partial_t v - \Delta v + H(x, Dv) = f(x) \text{ in } (0,+\infty)\times\mathbb{T}^d, \qquad u(0,\cdot) = u_0 \text{ in } \mathbb{T}^d$$

and

$$\delta u^\delta - \Delta u^\delta + H(x, Du^\delta) = 0 \qquad \text{in } \mathbb{T}^d$$

  - Convergence of δu^δ as $\delta \to 0$ and $v(T)/T$ as $T \to +\infty$ to the ergodic constant $\bar\lambda$: Lions-Papanicolau-Varadhan, ...
  - (Weak-KAM theory) Limit of $v(T) - \bar\lambda T$ as $T \to +\infty$ to a corrector : Fathi, Roquejoffre, Fathi-Siconolfi, Barles-Souganidis, ...
  - Convergence of $u^\delta - \bar\lambda/\delta$ as $\delta \to 0^+$ to a corrector : Davini, Fathi, Iturriaga and Zavidovique, ...

For MFG systems

- For the MFG time-dependent system, convergence of v^T/T and m^T are known :
  - Lions (Cours in Collège de France)
  - Gomes-Mohr-Souza (discrete setting)
  - C.-Lasry-Lions-Porretta (viscous setting), C. (Hamilton-Jacobi)
  - Turnpike property (Samuelson, Porretta-Zuazua, Trélat,...)

- Similar results for δu^δ and m^δ are not known, but expected.

- Long-time behavior of $v(T,\cdot) - \bar{\lambda}T$ vs limit of $u^\delta - \bar{\lambda}/\delta$ not known so far.

General strategy of proof

- Let (u^δ, m^δ) be the solution to

$$(MFG-\delta)\left\{\begin{array}{l} -\partial_t u^\delta + \delta u^\delta - \Delta u^\delta + H(x, Du^\delta) = f(x, m^\delta(t)) \qquad \text{in } (0,+\infty)\times\mathbb{T}^d \\ \partial_t m^\delta - \Delta m^\delta - \text{div}(m^\delta H_p(x, Du^\delta)) = 0 \qquad \text{in } (0,+\infty)\times\mathbb{T}^d \\ m^\delta(0,\cdot) = m_0 \qquad \text{in } \mathbb{T}^d, \qquad u^\delta \text{ bounded in } (0,+\infty)\times\mathbb{T}^d \end{array}\right.$$

- As $(u^\delta, m^\delta) = (u^\delta(t,x), m^\delta(t,x))$, two possible limits :
  - <u>When $\delta \to 0$:</u> difficult (no obvious limit, dependence in m_0 unclear),
  - <u>When $t \to +\infty$:</u> easier.

    Expected limit : the stationary discounted problem

$$(MFG-bar-\delta) \qquad \left\{\begin{array}{l} \delta \bar u^\delta - \Delta \bar u^\delta + H(x, D\bar u^\delta) = f(x, \bar m^\delta) \qquad \text{in } \mathbb{T}^d \\ -\Delta \bar m^\delta - \text{div}(\bar m^\delta H_p(x, D\bar u^\delta)) = 0 \qquad \text{in } \mathbb{T}^d \end{array}\right.$$

General strategy of proof (continued)

- Show that $\lim_{\delta \to 0^+} \delta \bar{u}^\delta = \bar{\lambda}$ and identification of the limit of $\bar{u}^\delta - \bar{\lambda}/\delta$.

- Collect all the equations $(MFG - \delta)$ into a single equation : for $m_0 \in \mathcal{P}(\mathbb{T}^d)$, set $U^\delta(x, m_0) := u^\delta(0, x)$ where (u^δ, m^δ) solves $(MFG - \delta)$ with $m(0) = m_0$.

- Then U^δ solves the discounted master equation.
  - get Lipschitz estimate on U^δ
  - by compactness arguments, prove that $U^\delta - \bar{\lambda}/\delta$ converges to a solution $\bar{U}$ of the ergodic master equation
    (as $\delta \to 0$, up to subsequences).

- Put the previous steps together to derive the limit of $u^\delta - \bar{\lambda}/\delta$.

Outline

Detour on derivatives in the space of measures

We denote by $\mathcal{P}(\mathbb{T}^d)$ the set of Borel probability measures on $\mathbb{T}^d$, endowed for the Monge-Kantorovich distance

$$\mathbf{d}_1(m, m') = \sup_{\phi} \int_{\mathbb{T}^d} \phi(y)\, d(m - m')(y),$$

where the supremum is taken over all Lipschitz continuous maps $\phi : \mathbb{T}^d \to \mathbb{R}$ with a Lipschitz constant bounded by 1.

Given $U : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$, we consider 2 notions of derivatives :

- The directional derivative $\frac{\delta U}{\delta m}(m, y)$ (see, e.g., Mischler-Mouhot)
- The intrinsic derivative $D_m U(m, y)$ (see, e.g., Otto, Ambrosio-Gigli-Savaré, Lions)

Directional derivative

A map $U : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ is $\mathcal{C}^1$ if there exists a continuous map $\dfrac{\delta U}{\delta m} : \mathcal{P}(\mathbb{T}^d) \times \mathbb{T}^d \to \mathbb{R}$ such that, for any $m, m' \in \mathcal{P}(\mathbb{T}^d)$,

$$U(m') - U(m) = \int_0^1 \int_{\mathbb{T}^d} \frac{\delta U}{\delta m}((1-s)m + sm', y)d(m' - m)(y)ds.$$

Note that $\frac{\delta U}{\delta m}$ is defined up to an additive constant. We adopt the normalization convention

$$\int_{\mathbb{T}^d} \frac{\delta U}{\delta m}(m, y)dm(y) = 0.$$

Intrinsic derivative

If $\dfrac{\delta U}{\delta m}$ is of class $\mathcal{C}^1$ with respect to the second variable, the intrinsic derivative $D_m U : \mathcal{P}(\mathbb{T}^d) \times \mathbb{T}^d \to \mathbb{R}^d$ is defined by

$$D_m U(m, y) := D_y \frac{\delta U}{\delta m}(m, y)$$

For instance, if $U(m) = \int_{\mathbb{T}^d} g(x)dm(x)$, then $\frac{\delta U}{\delta m}(m,y) = g(y) - \int_{\mathbb{T}^d} gdm$ while $D_m U(m,y) = Dg(y)$.

Remarks.

- The directional derivative is fruitful for computations.
- The intrinsic derivative encodes the variation of the map in $\mathcal{P}(\mathbb{T}^d)$. For instance :

$$\|D_m U\|_\infty = Lip\ U$$

Standing assumptions

- $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is smooth, with :

$$C^{-1} I_d \leq D^2_{pp} H(x,p) \leq C I_d \qquad \text{for } (x,p) \in \mathbb{T}^d \times \mathbb{R}^d.$$

Moreover, there exists $\theta \in (0,1)$ and $C > 0$ such that

$$|D_{xx} H(x,p)| \leq C|p|^{1+\theta}, \qquad |D_{xp} H(x,p)| \leq C|p|^{\theta}, \qquad \forall (x,p) \in \mathbb{T}^d \times \mathbb{R}^d.$$

- the maps $f, g : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ are monotone : for any $m, m' \in \mathcal{P}(\mathbb{T}^d)$,

$$\int_{\mathbb{T}^d} (f(x,m) - f(x,m'))d(m-m')(x) \geq 0, \ \int_{\mathbb{T}^d} (g(x,m) - g(x,m'))d(m-m')(x) \geq 0$$

- the maps f, g are $\mathcal{C}^1$ in m : there exists $\alpha \in (0,1)$ such that

$$\sup_{m \in \mathcal{P}(\mathbb{T}^d)} \left(\|f(\cdot,m)\|_{3+\alpha} + \left\| \frac{\delta f(\cdot,m,\cdot)}{\delta m} \right\|_{(3+\alpha,3+\alpha)} \right) + \mathrm{Lip}_{3+\alpha}\left(\frac{\delta f}{\delta m}\right) < \infty.$$

and the same for g.

Example. If f is of the form :

$$f(x, m) = \int_{\mathbb{R}^d} \Phi(z, (\rho \star m)(z))\rho(x - z)dz,$$

where

- $\star$ denotes the usual convolution product (in $\mathbb{R}^d$),
- $\Phi = \Phi(x, r)$ is a smooth map, nondecreasing w.r. to r,
- $\rho : \mathbb{R}^d \to \mathbb{R}$ is a smooth, even function with compact support.

Then f satisfies our conditions with

$$\frac{\delta f}{\delta m}(x, m, z) = \int_{\mathbb{R}^d} \sum_{k \in \mathbb{Z}^d} \rho(y - z - k) \frac{\partial \Phi}{\partial m}(y, \rho * m(y))\rho(x - y)dy$$

Outline

The classical ergodic theory

(Lions-Papanicolau-Varadhan, Evans, Arisawa-Lions,...)

For $\delta > 0$, let u^δ solve the uncoupled HJ equation

$$\delta u^\delta - \Delta u^\delta + H(x, Du^\delta) = f(x) \qquad \text{in } \mathbb{T}^d.$$

Then

- (δu^δ) is bounded (maximum principle),
- $\|Du^\delta\|_\infty$ is bounded (growth condition on H or ellipticity)
- Thus, as $\delta \to 0^+$ and up to a subsequence, (δu^δ) and $(u^\delta - u^\delta(0))$ converge to the ergodic constant $\bar\lambda$ and a corrector $\bar u$:

$$\bar\lambda - \Delta \bar u + H(x, D\bar u) = f(x) \qquad \text{in } \mathbb{T}^d.$$

- Uniqueness of $\bar\lambda$ and of $\bar u$ (up to constants) (strong maximum principle).

The small discount behavior

For $\delta > 0$, let u^δ solve the uncoupled HJ equation

$$\delta u^\delta - \Delta u^\delta + H(x, Du^\delta) = f(x) \qquad \text{in } \mathbb{T}^d.$$

Then $u^\delta - \delta^{-1}\bar\lambda$ actually converges as $\delta \to 0$ to the unique solution $\bar u$ of the ergodic cell problem

$$\bar\lambda - \Delta \bar u + H(x, D\bar u) = f(x) \qquad \text{in } \mathbb{T}^d$$

such that $\int_{\mathbb{T}^d} \bar u \bar m = 0$, where $\bar m$ solves

$$-\Delta \bar m - \operatorname{div}(\bar m H_p(x, D\bar u)) = 0 \qquad \text{in } \mathbb{T}^d, \qquad \bar m \geq 0, \; \int_{\mathbb{T}^d} \bar m = 1.$$

Proved by

- Davini, Fathi, Iturriaga and Zavidovique for the first order problem,
- Mitake and Tran (see also Mitake and Tran —- Ishii, Mitake and Tran) for the viscous case

Outline

The stationary discounted MFG system

It takes the form

$$(MFG-bar-\delta) \qquad \left\{ \begin{array}{ll} \delta\bar{u}^\delta - \Delta\bar{u}^\delta + H(x, D\bar{u}^\delta) = f(x, \bar{m}^\delta) & \text{in } \mathbb{T}^d \\ -\Delta\bar{m}^\delta - \mathrm{div}(\bar{m}^\delta H_p(x, D\bar{u}^\delta)) = 0 & \text{in } \mathbb{T}^d \end{array} \right.$$

Proposition

There exists $\delta_0 > 0$ such that, if $\delta \in (0, \delta_0)$, there is a unique solution $(\bar{u}^\delta, \bar{m}^\delta)$ to $(MFG-bar-\delta)$.

Moreover, for any $\delta \in (0, \delta_0)$,

$$\|\delta\bar{u}^\delta - \bar{\lambda}\|_\infty + \|D(\bar{u}^\delta - \bar{u})\|_{L^2} + \|\bar{m}^\delta - \bar{m}\|_{L^2} \leq C\delta^{1/2}.$$

for some constant $C > 0$, where $(\bar{\lambda}, \bar{u}, \bar{m})$ solves the ergodic MFG system

$$(MFG-ergo) \qquad \left\{ \begin{array}{ll} \bar{\lambda} - \Delta\bar{u} + H(x, D\bar{u}) = f(x, \bar{m}) & \text{in } \mathbb{T}^d \\ -\Delta\bar{m} - \mathrm{div}(\bar{m}H_p(x, D\bar{u})) = 0 & \text{in } \mathbb{T}^d \end{array} \right.$$

Link with the discounted MFG system

The solution $(\bar{u}^\delta, \bar{m}^\delta)$ of the $(MFG - bar - \delta)$ system can be obtained the limit of the solution (u^δ, m^δ) of

$$(MFG - \delta) \qquad \begin{cases} -\partial_t u^\delta + \delta u^\delta - \Delta u^\delta + H(x, Du^\delta) = f(x, m^\delta(t)) & \text{in } (0, +\infty) \times \mathbb{T}^d \\ \partial_t m^\delta - \Delta m^\delta - \text{div}(m^\delta H_p(x, Du^\delta)) = 0 & \text{in } (0, +\infty) \times \mathbb{T}^d \\ m^\delta(0, \cdot) = m_0 \quad \text{in } \mathbb{T}^d, & u^\delta \text{ bounded in } (0, +\infty) \times \mathbb{T}^d \end{cases}$$

Theorem

Under our standing assumptions, if $\delta \in (0, \delta_0)$, then

$$\|D(u^\delta(t) - \bar{u}^\delta)\|_{L^\infty} \leq Ce^{-\gamma t} \qquad \forall t \geq 0$$

and

$$\|m^\delta(t) - \bar{m}^\delta\|_{L^\infty} \leq Ce^{-\gamma t} \qquad \forall t \geq 1,$$

where $\gamma, \delta_0 > 0$ and $C > 0$ are independent of m_0.

Towards a limit of $\bar{u}^\delta - \bar{\lambda}/\delta$

- Plugg the ansatz :

$$\bar{u}^\delta \sim \frac{\bar{\lambda}}{\delta} + \bar{u} + \bar{\theta} + \delta \bar{v}, \qquad \bar{m}^\delta \sim \bar{m} + \delta \bar{\mu},$$

into the equation for $(\bar{u}^\delta, \bar{m}^\delta)$:

$$\left\{ \begin{array}{l} \delta \bar{u}^\delta - \Delta \bar{u}^\delta + H(x, D\bar{u}^\delta) = f(x, \bar{m}^\delta) \qquad \text{in } \mathbb{T}^d \\ -\Delta \bar{m}^\delta - \text{div}(\bar{m}^\delta H_p(x, D\bar{u}^\delta)) = 0 \qquad \text{in } \mathbb{T}^d \end{array} \right.$$

- One has :

$$\left\{ \begin{array}{l} \bar{\lambda} + \delta \bar{u} + \delta \bar{\theta} + \delta^2 \bar{v} - \Delta(\bar{u} + \delta \bar{v}) + H(x, D(\bar{u} + \delta \bar{v})) = f(x, \bar{m} + \delta \bar{\mu}) \\ -\Delta(\bar{m} + \delta \bar{\mu}) - \text{div}((\bar{m} + \delta \bar{\mu}) H_p(x, D(\bar{u} + \delta \bar{v}))) = 0 \end{array} \right.$$

Towards a limit of $\bar{u}^\delta - \bar{\lambda}/\delta$

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- We recognize the equation for $(\bar{u}, \bar{m})$:

$$\left\{ \begin{array}{l} \bar{\lambda} + \delta \bar{u} + \delta \bar{\theta} + \delta^2 \bar{v} - \Delta(\bar{u} + \delta \bar{v}) + H(x, D(\bar{u} + \delta \bar{v})) = f(x, \bar{m} + \delta \bar{\mu}) \\ -\Delta(\bar{m} + \delta \bar{\mu}) - \mathrm{div}((\bar{m} + \delta \bar{\mu}) H_p(x, D(\bar{u} + \delta \bar{v}))) = 0 \end{array} \right.$$

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into the equation for $(\bar{u}^\delta, \bar{m}^\delta)$:

$$\begin{cases} \delta \bar{u}^\delta - \Delta \bar{u}^\delta + H(x, D\bar{u}^\delta) = f(x, \bar{m}^\delta) & \text{in } \mathbb{T}^d \\ -\Delta \bar{m}^\delta - \mathrm{div}(\bar{m}^\delta H_p(x, D\bar{u}^\delta)) = 0 & \text{in } \mathbb{T}^d \end{cases}$$

- Expending and simplifying :

$$\begin{cases} \delta \bar{u} + \delta \bar{\theta} + \delta^2 \bar{v} - \Delta(\delta \bar{v}) + H_p(x, D\bar{u}) \cdot (\delta \bar{v}) = \dfrac{\delta f}{\delta m}(x, \bar{m})(\delta \bar{\mu}) \\ -\Delta(\delta \bar{\mu}) - \mathrm{div}((\delta \bar{\mu}) H_p(x, D\bar{u})) - \mathrm{div}(\bar{m} H_{pp}(x, D\bar{u})(\delta \bar{v}))) = 0 \end{cases}$$

Towards a limit of $\bar u^\delta - \bar\lambda/\delta$

- Plugg the ansatz :

$$\bar u^\delta \sim \frac{\bar\lambda}{\delta} + \bar u + \bar\theta + \delta \bar v, \qquad \bar m^\delta \sim \bar m + \delta\bar\mu,$$

  into the equation for $(\bar u^\delta, \bar m^\delta)$:

$$\left\{\begin{array}{l} \delta \bar u^\delta - \Delta \bar u^\delta + H(x, D\bar u^\delta) = f(x, \bar m^\delta) \qquad \text{in } \mathbb{T}^d \\ -\Delta \bar m^\delta - \text{div}(\bar m^\delta H_p(x, D\bar u^\delta)) = 0 \qquad \text{in } \mathbb{T}^d \end{array}\right.$$

- Dividing by δ and omitting the term of lower order :

$$\left\{\begin{array}{l} \bar u + \bar\theta - \Delta \bar v + H_p(x, D\bar u)\cdot D\bar v = \dfrac{\delta f}{\delta m}(x, \bar m)(\bar\mu) \\ -\Delta\bar\mu - \text{div}(\bar\mu H_p(x, D\bar u)) - \text{div}(\bar m H_{pp}(x, D\bar u)\bar v)) = 0 \end{array}\right.$$

Proposition

There exists a unique constant $\bar{\theta}$ for which the following has a solution $(\bar{v}, \bar{\mu})$:

$$\left\{\begin{array}{ll} \bar{u} + \bar{\theta} - \Delta \bar{v} + H_p(x, D\bar{u}).D\bar{v} = \dfrac{\delta f}{\delta m}(x, \bar{m})(\bar{\mu}) & \text{in } \mathbb{T}^d \\ -\Delta \bar{\mu} - \text{div}(\bar{\mu} H_p(x, D\bar{u})) - \text{div}(\bar{m} H_{pp}(x, D\bar{u}) D\bar{v}) = 0 & \text{in } \mathbb{T}^d \\ \int_{\mathbb{T}^d} \bar{\mu} = \int_{\mathbb{T}^d} \bar{v} = 0 & \end{array}\right.$$

We can identify the limit of $\bar{u}^\delta - \bar{\lambda}/\delta$:

Proposition

Let $(\bar{\lambda}, \bar{u}, \bar{m})$, $(\bar{u}^\delta, \bar{m}^\delta)$ and $(\bar{\theta}, \bar{v}, \bar{\mu})$ be as above. Then

$$\lim_{\delta \to 0^+} \|\bar{u}^\delta - \frac{\bar{\lambda}}{\delta} - \bar{u} - \bar{\theta}\|_\infty + \|\bar{m}^\delta - \bar{m}\|_\infty = 0.$$

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Let $(\bar{\lambda}, \bar{u}, \bar{m})$, $(\bar{u}^\delta, \bar{m}^\delta)$ and $(\bar{\theta}, \bar{v}, \bar{\mu})$ be as above. Then

$$\lim_{\delta \to 0^+} \|\bar{u}^\delta - \frac{\bar{\lambda}}{\delta} - \bar{u} - \bar{\theta}\|_\infty + \|\bar{m}^\delta - \bar{m}\|_\infty = 0.$$

- This shows that $\bar{u}^\delta - \bar{\lambda}/\delta$ converges as $\delta \to 0^+$ to $\bar{u} + \bar{\theta}$, where $\bar{\theta}$ is the unique constant such that the system

$$\begin{cases} \bar{u} + \bar{\theta} - \Delta \bar{v} + H_p(x, D\bar{u}).D\bar{v} = \dfrac{\delta f}{\delta m}(x, \bar{m})(\bar{\mu}) \qquad \text{in } \mathbb{T}^d \\ -\Delta \bar{\mu} - \mathrm{div}(\bar{\mu} H_p(x, D\bar{u})) - \mathrm{div}(\bar{m} H_{pp}(x, D\bar{u}) D\bar{v}) = 0 \qquad \text{in } \mathbb{T}^d \\ \int_{\mathbb{T}^d} \bar{\mu} = \int_{\mathbb{T}^d} \bar{v} = 0 \end{cases}$$

has a solution $(\bar{v}, \bar{\mu})$.

- In the uncoupled case ($f = f(x)$), we have $\displaystyle\int_{\mathbb{T}^d} (\bar{u} + \bar{\theta})\bar{m} = 0$, because $\frac{\delta f}{\delta m} = 0$ and, if we multiply the equation for $\bar{v}$ by $\bar{m}$ and integrate, we get

$$\begin{aligned} 0 &= \int_{\mathbb{T}^d} \bar{m}(\bar{u} + \bar{\theta} - \Delta \bar{v} + H_p(x, D\bar{u}).D\bar{v}) \\ &= \int_{\mathbb{T}^d} \bar{m}(\bar{u} + \bar{\theta}) + \int_{\mathbb{T}^d} \bar{v}(-\Delta \bar{m} - \mathrm{div}(\bar{m} H_p(x, D\bar{u}))) \\ &= \int_{\mathbb{T}^d} \bar{m}(\bar{u} + \bar{\theta}) \end{aligned}$$

So one recovers the condition of Davini, Fathi, Iturriaga and Zavidovique.

Outline

The discounted master equation

In order to study the limit behavior of (u^δ, m^δ), we use the discounted master equation :

$$\left\{ \begin{array}{l} \delta U^\delta - \Delta_x U^\delta + H(x, D_x U^\delta) - f(x,m) \\ \qquad - \int_{\mathbb{T}^d} \mathrm{div}_y \left[D_m U^\delta \right] \, dm(y) + \int_{\mathbb{T}^d} D_m U^\delta \cdot H_p(y, D_x U^\delta) \, dm(y) = 0 \\ \qquad\qquad \text{in } \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \end{array} \right.$$

where $U^\delta = U^\delta(x,m) : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$.

Theorem (C.-Delarue-Lasry-Lions, 2015)

Under our assumptions, the discounted master equation has a unique classical solution U^δ.

Previous results in that direction : Lasry-Lions, Gangbo-Swiech, Chassagneux-Crisan-Delarue,...

Idea of proof : Let us set

$$U^\delta(x, m_0) := u^\delta(0, x),$$

where (u^δ, m^δ) solves

$$(MFG-\delta)\left\{\begin{array}{l} -\partial_t u^\delta + \delta u^\delta - \Delta u^\delta + H(x, Du^\delta) = f(x, m^\delta(t)) \qquad \text{in } (0, +\infty)\times \mathbb{T}^d \\ \partial_t m^\delta - \Delta m^\delta - \text{div}(m^\delta H_p(x, Du^\delta)) = 0 \qquad \text{in } (0, +\infty)\times \mathbb{T}^d \\ m^\delta(0, \cdot) = m_0 \qquad \text{in } \mathbb{T}^d, \qquad u^\delta \text{ bounded in } (0, +\infty)\times \mathbb{T}^d \end{array}\right.$$

Then one expects that U^δ solves the master equation because :

$$U^\delta(x, m^\delta(t)) = u^\delta(t, x) \qquad \forall t \geq 0.$$

Taking the derivative in $t = 0$:

$$\int_{\mathbb{T}^d} \frac{\delta U^\delta}{\delta m}(x, m_0, y)\partial_t m^\delta(0, dy) = \partial_t u(0, x),$$

so that

$$\int_{\mathbb{T}^d} \frac{\delta U^\delta}{\delta m}(x, m_0, y)(\Delta m_0 + \text{div}(m_0 H_p(y, Du^\delta(0))) = \delta u^\delta(0) - \Delta u^\delta(0) + H(x, Du^\delta(0)) - f(x, m_0).$$

Integrating by parts gives the master equation. □

The key Lipschitz estimate

Let U^δ be the solution of the discounted master equation

$$\left\{ \begin{array}{l} \delta U^\delta - \Delta_x U^\delta + H(x, D_x U^\delta) - f(x,m) \\ \qquad - \int_{\mathbb{T}^d} \mathrm{div}_y \left[D_m U^\delta \right] \, dm(y) + \int_{\mathbb{T}^d} D_m U^\delta \cdot H_p(y, D_x U^\delta) \, dm(y) = 0 \\ \qquad\qquad \text{in } \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \end{array} \right.$$

Proposition

There is a constant C, depending on the data only, such that

$$\left\| D_m U^\delta(\cdot, m, \cdot) \right\|_{2+\alpha, 1+\alpha} \leq C.$$

In particular, $U^\delta(\cdot, \cdot)$ is uniformly Lipschitz continuous.

Difficulty : equation for U^δ neither coercive nor elliptic in m.

Idea of proof

Representation formulas. Fix $m_0 \in \mathcal{P}(\mathbb{T}^d)$ a initial condition and (u^δ, m^δ) the associated solution of the discounted MFG system :

$$\left\{\begin{array}{l} -\partial_t u^\delta + \delta u^\delta - \Delta u^\delta + H(x, Du^\delta) = f(x, m^\delta(t)) \qquad \text{in } (0,+\infty)\times \mathbb{T}^d \\ \partial_t m^\delta - \Delta m^\delta - \mathrm{div}(m^\delta H_p(x, Du^\delta)) = 0 \qquad \text{in } (0,+\infty)\times \mathbb{T}^d \\ m^\delta(0,\cdot) = m_0 \text{ in } \times \mathbb{T}^d, \qquad u^\delta \text{ bounded.} \end{array}\right.$$

For any smooth map μ_0 with $\int_{\mathbb{T}^d} m_0 = 0$, one can show that

$$\int_{\mathbb{T}^d} \frac{\delta U^\delta}{\delta m}(x, m_0, y)\mu_0(y)dy = w(0,x),$$

where (w, μ) is the unique solution to the linearized system

$$\left\{\begin{array}{l} -\partial_t w + \delta w - \Delta w + H_p(x, Du^\delta).Dw = \frac{\delta f}{\delta m}(x, m^\delta(t))(\mu(t)) \qquad \text{in } (0,+\infty)\times \mathbb{T}^d \\ \partial_t \mu - \Delta\mu - \mathrm{div}(\mu H_p(x, Du^\delta)) - \mathrm{div}(m^\delta H_{pp}(x, Du^\delta)Dw) = 0 \qquad \text{in } (0,+\infty)\times \mathbb{T}^d \\ \mu(0,\cdot) = \mu_0 \text{ in } \mathbb{T}^d, \qquad w \text{ bounded.} \end{array}\right.$$

Key step for the estimate :

$$\left\| D_m U^\delta(\cdot, m, \cdot) \right\|_{2+\alpha,1+\alpha} \le C.$$

Lemma

There exist $\theta, \delta_0 > 0$ and a constant $C > 0$ such that, if $\delta \in (0, \delta_0)$, then the solution (w, μ) to the linearized system with $\int_{\mathbb{T}^d} \mu_0 = 0$ satisfies

$$\|Dw(t)\|_{L^2} \le C(1+t)e^{-\theta t}\|\mu_0\|_{L^2} \qquad \forall t \ge 0$$

and

$$\|\mu(t)\|_{L^2} \le C(1+t)e^{-\theta t}\|\mu_0\|_{L^2} \qquad \forall t \ge 1.$$

As a consequence, for any $\alpha \in (0,1)$, there is a constant C (independent of δ) such that

$$\sup_{t \ge 0} \|w(t)\|_{C^{2+\alpha}} \le C\|\mu_0\|_{(C^{2+\alpha})'}.$$

Relies on the monotonicity formula and exponential decay of some viscous transport equation.

The ergodic master equation

As in the classical framework, we have (up to a subsequence) :

- δU^δ converges to a constant λ,
- $U^\delta - U^\delta(\cdot, \bar{m})$ converges to a Lipschitz continuous map $\bar{U}$.

Proposition

The constant $\bar{\lambda}$ and the limit $\bar{U}$ satisfy the master cell-problem :

$$\lambda - \Delta_x \bar{U}(x,m) + H(x, D_x\bar{U}(x,m)) - \int_{\mathbb{T}^d} \mathrm{div}(D_m\bar{U}(x,m))dm \\ + \int_{\mathbb{T}^d} D_m\bar{U}(x,m).H_p(x,, D_x\bar{U}(x,m))dm = f(x,m) \qquad \text{in } \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$$

(in a weak sense).

Moreover, if $(\bar{\lambda}, \bar{u}, \bar{m})$ is the solution to the ergodic MFG system then

$$\bar{\lambda} = \lambda \qquad \text{and} \qquad D_x\bar{U}(x,\bar{m}) = D\bar{u}(x) \qquad \forall x \in \mathbb{T}^d.$$

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$$+ \int_{\mathbb{T}^d} D_m\bar{U}(x,m).H_p(x, , D_x\bar{U}(x,m))dm = f(x,m) \qquad \text{in } \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$$

(in a weak sense).

Moreover, if $(\bar{\lambda}, \bar{u}, \bar{m})$ is the solution to the ergodic MFG system then

$$\bar{\lambda} = \lambda \qquad \text{and} \qquad D_x\bar{U}(x,\bar{m}) = D\bar{u}(x) \qquad \forall x \in \mathbb{T}^d.$$

Remarks.

- One also shows that $\bar{U}$ is unique up to a constant.
- So the limits, up to subsequences, of $U^\delta - U^\delta(\cdot, \bar{m})$ is determined only up to a constant.
- To fix this constant, we use the identification of the limit of $\bar{u}^\delta - \bar{\lambda}/\delta$.

Outline

Link between U^δ and $\bar{u}^\delta$

Let U^δ be the solution to the discounted master equation :

$$\delta U^\delta - \Delta_x U^\delta + H(x, D_x U^\delta) - \int_{\mathbb{T}^d} \mathrm{div}(D_m U^\delta) dm \\ + \int_{\mathbb{T}^d} D_m U^\delta . H_p(x, D_x U^\delta(x,m)) dm = f(x,m) \text{ in } \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d).$$

and $(\bar{u}^\delta, \bar{m}^\delta)$ be the solution to discounted stationary problem :

$$(MFG - bar - \delta) \qquad \left\{ \begin{array}{ll} \delta \bar{u}^\delta - \Delta \bar{u}^\delta + H(x, D\bar{u}^\delta) = f(x, \bar{m}^\delta) & \text{in } \mathbb{T}^d \\ -\Delta \bar{m}^\delta - \mathrm{div}(\bar{m}^\delta H_p(x, D\bar{u}^\delta)) = 0 & \text{in } \mathbb{T}^d \end{array} \right.$$

Then, by construction of U^δ,

$$U^\delta(\cdot, \bar{m}^\delta) = \bar{u}^\delta,$$

because $(\bar{u}^\delta, \bar{m}^\delta)$ is a stationary solution of the discounted MFG system $(MFG - \delta)$.

The main result

Let U^δ be the solution to the discounted master equation :

$$\delta U^\delta - \Delta_x U^\delta + H(x, D_x U^\delta) - \int_{\mathbb{T}^d} \mathrm{div}(D_m U^\delta) dm \\ + \int_{\mathbb{T}^d} D_m U^\delta . H_p(x, D_x U^\delta(x,m)) dm = f(x,m) \text{ in } \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d).$$

Theorem

As $\delta \to 0^+$, $U^\delta - \bar\lambda/\delta$ converges uniformly to the solution $\bar U$ to the master cell problem such that $\bar U(x,\bar m) = \bar u(x) + \bar\theta$, where $\bar\theta$ is the unique constant for which the following linearized ergodic problem has a solution $(\bar v, \bar\mu)$:

$$\begin{cases} \bar u + \bar\theta - \Delta \bar v + H_p(x, D\bar u).D\bar v = \dfrac{\delta f}{\delta m}(x,\bar m)(\bar\mu) & \text{in } \mathbb{T}^d \\ -\Delta\bar\mu - \mathrm{div}(\bar\mu H_p(x,D\bar u)) - \mathrm{div}(\bar m H_{pp}(x,D\bar u)D\bar v) = 0 & \text{in } \mathbb{T}^d \\ \int_{\mathbb{T}^d} \bar\mu = \int_{\mathbb{T}^d} \bar v = 0 \end{cases}$$

The small discount behavior of v^δ

Fix $m_0 \in \mathcal{P}(\mathbb{T}^d)$ and let (u^δ, m^δ) be the solution to the discounted MFG system :

$$(MFG-\delta) \qquad \left\{ \begin{array}{l} -\partial_t u^\delta + \delta u^\delta - \Delta u^\delta + H(x, Du^\delta) = f(x, m^\delta(t)) \qquad \text{in } (0,+\infty)\times\mathbb{T}^d \\ \partial_t m^\delta - \Delta m^\delta - \text{div}(m^\delta H_p(x, Du^\delta)) = 0 \qquad \text{in } (0,+\infty)\times\mathbb{T}^d \\ m^\delta(0,\cdot) = m_0 \qquad \text{in } \mathbb{T}^d, \qquad u^\delta \text{ bounded in } (0,+\infty)\times\mathbb{T}^d \end{array}\right.$$

Corollary

We have, for any $t \geq 0$,

$$\lim_{\delta\to 0} u^\delta(t,x) - \bar\lambda/\delta = \bar U(x, m(t)),$$

uniformly with respect to x, where $\bar U$ is the solution of the ergodic cell problem given in the main Theorem and $(m(t))$ solves the McKean-Vlasov equation

$$\partial_t m - \Delta m - \text{div}(mH_p(x, D\bar U(x,m))) = 0 \text{ in } (0,+\infty)\times\mathbb{T}^d, \qquad m(0) = m_0.$$

Conclusion

We have established the small discount behavior of the discounted MFG system/master equation.
We also show in the paper the long time behavior of the time-dependent MFG system/master equation.

Open problems :

- First order setting.
- Convergence in the non-monotone setting.