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Joint work with Luciano Campi, LSE

MFG-4, Rome, June 14-16, 2017

Mean field games with absorption

N-player games

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Mean field game with absorption

Given a flow of probability measures, solve optimal control problem for the one representative player; control until first exit from set of non-absorbing states or final time. Flow of measures here flow of conditional probabilities.

Aim and scope

Simple class of systems. Evolution of players' states described by controlled Itô equations with constant diffusion coefficient, performance in terms of expected costs over a finite time horizon; set of non-absorbing states open and bounded.

Start from N-player games. Define mean field game through formal passage to the limit.

Justify definition in the usual way (cf. [Huang et al.(2006)], ..., [Carmona and Lacker(2015)], ...): Show that solution of the mean field game induces approximate Nash equilibria for the N-player games. Works if solution is continuous "almost everywhere" and diffusion coefficient non-degenerate.

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Here, probabilistic approach. Alternatively, PDE approach.

Some related works

- Systems of interacting firms (loss from default): Dai Pra et al. [2009], Cvitanić - Ma - Zhang [2012], Giesecke, Spiliopoulos et al. [2013–2015];
- Neuronal networks: Delarue et al. [2015]
- Interacting diffusions with absorption on the half-line: Hambly & Ledger [2017+]
- Bertrand oligopoly mean field game model: Chan & Sircar [2015], Bensoussan & Graber [2016+]
- Games with varying number of players: Bensoussan - Frehse - Grün [2014]

Let $T > 0$ be the time horizon, O the set of non-absorbing states, $\Gamma \subset \mathbb{R}^d$ the set of control actions.
Given a vector $\boldsymbol{u} = (u_1, \dots, u_N)$ of Γ-valued progressive feedback strategies, the players' states evolve according to

(1)

$$X_i^N(t) = X_i^N(0) + \int_0^t \left(u_i(s, \boldsymbol{X}^N) + \bar{b}\left(s, X_i^N(s), \int_{\mathbb{R}^d} w(y)\pi^N(s, dy) \right) \right) ds + \sigma W_i^N(t), \quad t \in [0, T],\ i \in \{1, \dots, N\},$$

where $\pi^N(t, \cdot)$ is the renormalized empirical measure of the states of the players still in O at time t:

$$\pi_\omega^N(t, \cdot) \doteq \begin{cases} \frac{1}{\bar{N}_\omega^N} \sum_{j=1}^N \mathbf{1}_{[0, \tau^{X_j^N}(\omega))}(t) \cdot \delta_{X_j^N(t,\omega)}(\cdot) & \text{if } \bar{N}_\omega^N > 0, \\ \delta_0(\cdot) & \text{if } \bar{N}_\omega^N = 0, \end{cases}$$

$\bar{N}_\omega^N \doteq \sum_{j=1}^N \mathbf{1}_{[0, \tau^{X_j^N}(\omega))}(t)$, $\tau^{X_j^N}(\omega) \doteq \inf\{t \geq 0 : X_j^N(t, \omega) \notin O\}$, $\omega \in \Omega$.

Initial distribution $\nu_N \doteq \text{Law}(X_1^N(0), \dots, X_N^N(0))$ fixed and symmetric.

N-player costs

Let $\mathcal{U}_{fb}^N$ be the set of all strategy vectors $\boldsymbol{u} \in \times^N \mathcal{U}_N$ such that Eq. (1) under $\boldsymbol{u}$ with initial distribution ν_N possesses a solution unique in law.

Player i evaluates $\boldsymbol{u} = (u_1, \ldots, u_N) \in \mathcal{U}_{fb}^N$ according to

$$J_i^N(\boldsymbol{u}) \doteq \mathbf{E}\left[\int_0^{\tau_i^N} f\left(s, X_i^N(s), \int_{\mathbb{R}^d} w(y)\pi^N(s, dy), u_i\left(s, \boldsymbol{X}^N\right)\right) ds \right. \\ \left. + F\left(\tau_i^N, X_i^N(\tau_i^N)\right)\right],$$

where $\boldsymbol{X}^N = (X_1^N, \ldots, X_N^N)$ is a solution of Eq. (1) under $\boldsymbol{u}$ with initial distribution ν_N,

$$\tau_i^N(\omega) \doteq \tau^{X_i^N}(\omega) \wedge T, \quad \omega \in \Omega,$$

the random time horizon for player $i \in \{1, \ldots, N\}$, and $\pi^N(\cdot)$ the conditional empirical measure process induced by $(X_1^N, \ldots, X_N^N)$.

Assumptions

(H1) Boundedness and measurability: w, $\bar{b}$, f, F are Borel measurable functions uniformly bounded by some constant $K > 0$.

(H2) Continuity: w, f, F are continuous.

(H3) Lipschitz continuity: $\bar{b}(t, \cdot, \cdot)$ Lipschitz with constant L uniformly in t.

(H4) Action space: $\Gamma \subset \mathbb{R}^d$ is compact (and non-empty).

(H5) State space: $O \subset \mathbb{R}^d$ is non-empty, open, and bounded such that ∂O is a C^2-manifold.

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For main results, additional non-degeneracy assumption:

- σ is a matrix of full rank.

Under non-degeneracy assumption, $\mathcal{U}_{fb}^N = \times^N \mathcal{U}_N$.

Nash equilibria

Given a strategy vector $\boldsymbol{u} = (u_1, \ldots, u_N)$ and an individual strategy $v \in \mathcal{U}_N$, indicate by

$$[\boldsymbol{u}^{-i}, v] \doteq (u_1, \ldots, u_{i-1}, v, u_{i+1}, \ldots, u_N)$$

the strategy vector obtained from $\boldsymbol{u}$ by replacing u_i with v.

Definition.

Let $\varepsilon \geq 0$. A strategy vector $\boldsymbol{u} = (u_1, \ldots, u_N) \in \mathcal{U}_{fb}^N$ is called an ε-Nash equilibrium for the N-player game if for every $i \in \{1, \ldots, N\}$, every $v \in \mathcal{U}_N$ such that $[\boldsymbol{u}^{-i}, v] \in \mathcal{U}_{fb}^N$,

$$J_i^N(\boldsymbol{u}) \leq J_i^N\left([\boldsymbol{u}^{-i}, v]\right) + \varepsilon.$$

If $\boldsymbol{u}$ is an ε-Nash equilibrium with $\varepsilon = 0$, then $\boldsymbol{u}$ is called a Nash equilibrium.

Nash equilibria in full information feedback strategies.

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Mean field limit suggests to consider the equation

$$X(t) = X(0) + \int_0^t \left(u(s, X) + \bar{b}\left(s, X(s), \int_{\mathbb{R}^d} w(y)\mathfrak{p}(s, dy) \right) \right) ds + \sigma W(t), \quad t \in [0, T], \tag{2}$$

where $\mathfrak{p} \in \mathcal{M} \doteq \mathbf{M}([0, T], \mathcal{P}(\mathbb{R}^d))$ is a flow of probability measures, $u \in \mathcal{U}_1$ a Γ-valued progressive feedback strategy, and W a d-dimensional Wiener process.

In view of N-player game, $\mathfrak{p}$ should correspond to a flow of conditional probabilities.

Limit costs

Let $\mathcal{U}_{fb}$ denote the set of all feedback strategies $u \in \mathcal{U}_1$ such that Eq. (2) possesses a solution unique in law given any initial distribution with support in O. Under non-degeneracy assumption, $\mathcal{U}_{fb} = \mathcal{U}_1$.

Costs associated with a strategy $u \in \mathcal{U}_{fb}$, a flow of measures $\mathfrak{p} \in \mathcal{M}$, and an initial distribution $\nu \in \mathcal{P}(\mathbb{R}^d)$ with support in O:

$$J(\nu, u; \mathfrak{p}) \doteq \mathbf{E}\left[\int_0^\tau f\left(s, X(s), \int_{\mathbb{R}^d} w(y)\mathfrak{p}(s, dy), u(s, X)\right) ds \right.$$
$$\left. + F(\tau, X(\tau))\right],$$

where X is a solution of Eq. (2) under u with initial distribution ν, and $\tau \doteq \tau^X \wedge T$ the random time horizon.

Minimal costs

Minimal costs associated with $\mathfrak{p} \in \mathcal{M}$ and $\nu \in \mathcal{P}(\mathbb{R}^d)$ with respect to stochastic open-loop strategies:

$$V(\nu; \mathfrak{p}) \doteq \inf_{((\Omega,\mathcal{F},(\mathcal{F}_t),\mathrm{P}),\xi,\alpha,W)\in\mathcal{A}:\mathrm{P}\circ\xi^{-1}=\nu} \mathbf{E}\left[\int_0^\tau f\left(s, X(s), \int_{\mathbb{R}^d} w(y)\mathfrak{p}(s,dy), \alpha(s)\right) ds + F(\tau, X(\tau))\right],$$

where X is the unique solution of

$$X(t) = \xi + \int_0^t \left(\alpha(s) + \bar{b}\left(s, X(s), \int_{\mathbb{R}^d} w(y)\mathfrak{p}(s,dy)\right)\right) ds + \sigma W(t), \quad t \in [0,T], \tag{3}$$

and $\mathcal{A}$ set of all quadruples $((\Omega,\mathcal{F},(\mathcal{F}_t),\mathrm{P}),\xi,\alpha,W)$ such that $(\Omega,\mathcal{F},(\mathcal{F}_t),\mathrm{P})$ is a filtered probability space, ξ an O-valued $\mathcal{F}_0$-measurable random variable, α a Γ-valued $(\mathcal{F}_t)$-progressively measurable process, and W a d-dimensional $(\mathcal{F}_t)$-Wiener process.

Notice that

$$\inf_{u\in\mathcal{U}_{fb}} J(\nu, u; \mathfrak{p}) \geq V(\nu; \mathfrak{p}).$$

Mean field game

Definition.

A feedback solution of the mean field game is a triple $(\nu, u, \mathfrak{p})$ *such that*

(i) $\nu \in \mathcal{P}(\mathbb{R}^d)$ *with* $\mathrm{supp}(\nu) \subset O$, $u \in \mathcal{U}_{fb}$, *and* $\mathfrak{p} \in \mathcal{M}$;

(ii) *optimality property: strategy* u *is optimal for* $\mathfrak{p}$ *and initial distribution* ν *in the sense that*

$$J(\nu, u; \mathfrak{p}) = V(\nu; \mathfrak{p});$$

(iii) *conditional mean field property: if* X *is a solution of Eq.* (2) *with flow of measures* $\mathfrak{p}$, *strategy* u, *and initial distribution* ν, *then* $\mathfrak{p}(t) = \mathrm{P}\left(X(t) \in \cdot \mid \tau^X > t\right)$ *for every* $t \in [0, T]$ *such that* $\mathrm{P}\left(\tau^X > t\right) > 0$.

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Approximate Nash equilibria from the mean field game

Set $\mathcal{X} \doteq \mathbf{C}([0,T],\mathbb{R}^d)$. For $\nu \in \mathcal{P}(\mathbb{R}^d)$, let $\Theta_\nu \in \mathcal{P}(\mathcal{X})$ denote the law of $X(t) = \xi + \sigma W(t)$, $t \in [0,T]$, where $\mathrm{Law}(\xi) = \nu$.

Theorem 1.

Grant the non-degeneracy assumption in addition to (H1)-(H5). Suppose $(\nu_N)_{N\in\mathbb{N}}$ is ν-chaotic for some $\nu \in \mathcal{P}(\mathbb{R}^d)$ with support in O.
If $(\nu, u, \mathfrak{p})$ is a feedback solution of the mean field game regular in the sense that

$$\Theta_\nu\left(\{\varphi \in \mathcal{X} \ : \ u(t,\cdot) \text{ is discontinuous at } \varphi\}\right) = 0, \ \text{a.e. } t \in [0,T],$$

then $(\boldsymbol{u}^N)_{N\in\mathbb{N}} \subset \mathcal{U}^N_{fb}$ with $\boldsymbol{u}^N = (u^N_1,\dots,u^N_N)$ defined by

$$u^N_i(t,\boldsymbol{\varphi}) \doteq u(t,\varphi_i), \quad (t,\boldsymbol{\varphi}) \in [0,T]\times\mathcal{X}^N,$$

yields a sequence of approximate Nash equilibria: for every $\varepsilon > 0$, there exists $N_0(\varepsilon) \in \mathbb{N}$ such that $\boldsymbol{u}^N$ is an ε-Nash equilibrium for the N-player game whenever $N \geq N_0(\varepsilon)$.

Proof (sketch)

Let $\varepsilon > 0$. By symmetry, enough to let player one deviate. Thus, show that there exists $N_0 = N_0(\varepsilon) \in \mathbb{N}$ such that for all $N \geq N_0$,

$$J_1^N(\boldsymbol{u}^N) \leq \inf_{v \in \mathcal{U}_N} J_1^N\left([\boldsymbol{u}^{N,-1}, v]\right) + \varepsilon.$$

First step. Rewrite dynamics using unconditional measures on path space: for $(t, \varphi, \theta) \in [0, T] \times \mathcal{X} \times \mathcal{P}(\mathcal{X})$,

$$\begin{aligned} \hat{b}(t, \varphi, \theta) &\doteq b\left(t, \varphi, \theta, u(t, \varphi)\right) \\ &= \begin{cases} u(t, \varphi) + \bar{b}\left(t, \varphi(t), \frac{\int w(\tilde{\varphi}(t)) \mathbf{1}_{[0,\tau(\tilde{\varphi}))}(t)\theta(d\tilde{\varphi})}{\int \mathbf{1}_{[0,\tau(\tilde{\varphi}))}(t)\theta(d\tilde{\varphi})}\right) & \text{if } \theta(\tau > t) > 0, \\ u(t, \varphi) + \bar{b}\left(t, w(0)\right) & \text{if } \theta(\tau > t) = 0, \end{cases} \end{aligned}$$

where $\tau(\varphi) \doteq \inf\{t \geq 0 : \varphi(t) \notin O\}$.
Set $\theta_* \doteq \mathrm{Law}(X)$ where X solution of Eq. (2) with flow of measures $\mathfrak{p}$, feedback strategy u, and initial distribution ν. Then θ_* unique McKean-Vlasov solution of dynamics associated with $(\hat{b}, \sigma)$.

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Proof (sketch cont.)

Second step. For $N \in \mathbb{N}$, let $\boldsymbol{X}^N$ be solution of Eq. (1) under strategy vector $\boldsymbol{u}^N$ with initial distribution ν_N. Denote by μ^N the associated empirical measure on $\mathcal{X}$. Then

$$\mathrm{Law}(\mu^N) \overset{N\to\infty}{\longrightarrow} \delta_{\theta_*} \text{ in } \mathcal{P}(\mathcal{P}(\mathcal{X})),$$

where θ_* is the measure identified in Step One.
Use Tanaka-Sznitman theorem, chaoticity of initial distributions, and symmetry of coefficients to conclude that

$$J_1^N(\boldsymbol{u}^N) \overset{N\to\infty}{\longrightarrow} = J(\nu, u; \mathfrak{p}).$$

Difficulty here: built in discontinuity due to absorption; non-degeneracy of σ provides "sufficient" continuity.

Proof (sketch cont.)

Third step. For $N \in \mathbb{N} \setminus \{1\}$, choose $v_1^N \in \mathcal{U}_N$ such that

$$J_1^N\left([\boldsymbol{u}^{N,-1}, v_1^N]\right) \leq \inf_{v \in \mathcal{U}_N} J_1^N\left([\boldsymbol{u}^{N,-1}, v]\right) + \varepsilon/2.$$

Let $\tilde{\boldsymbol{X}}^N$ be a solution of Eq. (1) under strategy vector $[\boldsymbol{u}^{N,-1}, v_1^N]$ with initial distribution ν_N. Denote by $\tilde{\mu}^N$ the associated empirical measure. Then

$$\text{Law}\left(\tilde{\mu}^N\right) \overset{N\to\infty}{\longrightarrow} \delta_{\theta_*} \text{ in } \mathcal{P}(\mathcal{P}(\mathcal{X})),$$

where θ_* unique McKean-Vlasov solution found in Steps One and Two. Re-express cost functional in terms of unconditional measure and interpret $v_1^N(\cdot, \tilde{\boldsymbol{X}}^N)$ as stochastic open-loop control. Using the convergence of $(\tilde{\mu}^N)$, conclude that

$$\liminf_{N\to\infty} J_1^N([\boldsymbol{u}^{N,-1}, v_1^N]) \geq V(\nu; \mathfrak{p}).$$

Proof (sketch cont.)

Fourth step. For every $N \in \mathbb{N} \setminus \{1\}$,

$$J_1^N(\boldsymbol{u}^N) - \inf_{v \in \mathcal{U}_N} J_1^N([\boldsymbol{u}^{N,-1}, v])$$
$$\leq J_1^N(\boldsymbol{u}^N) - J(\nu, u; \mathfrak{p}) + J(\nu, u; \mathfrak{p}) - J_1^N([\boldsymbol{u}^{N,-1}, v_1^N]) + \varepsilon/2.$$

By Steps Two and Three, there exists $N_0 = N_0(\varepsilon)$ such that for all $N \geq N_0$,

$$J_1^N(\boldsymbol{u}^N) - J(\nu, u; \mathfrak{p}) + V(\nu; \mathfrak{p}) - J_1^N([\boldsymbol{u}^{N,-1}, v_1^N]) \leq \varepsilon/2.$$

Since $(\nu, u; \mathfrak{p})$ is a solution of the mean field game, $J(\nu, u; \mathfrak{p}) = V(\nu; \mathfrak{p})$. It follows that for all $N \geq N_0$,

$$J_1^N(\boldsymbol{u}^N) - \inf_{v \in \mathcal{U}_N} J_1^N([\boldsymbol{u}^{N,-1}, v]) \leq \varepsilon.$$

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Existence of regular solutions to the MFG

Theorem 2.

In addition to the hypotheses of Theorem 1, assume that

$$\Gamma \ni \gamma \mapsto f(t, x, m, \gamma) + \gamma \cdot z$$

has a unique minimizer given any (t, x, m, z) *(plus some mild technical assumptions).*
Then there exists a feedback solution of the mean field game $(\nu, u, \mathfrak{p})$ *such that*

$$u(t, \varphi) = \alpha\,(t, \varphi(t))$$

for some continuous $\alpha\colon [0, T] \times \mathbb{R}^d \to \Gamma$*; thus, u Markov feedback strategy.*

Proof (idea): Existence of feedback solution through Brouwer-Schauder fixed point theorem following [Carmona and Lacker(2015)].
Continuity and Markov property of strategy from classical regularity result due to [Fleming and Rishel(1975)].

PDE approach

Let $(\nu, \alpha, \mathfrak{p})$ be a solution according to Theorem 2 with $\nu(dx) = m_0(x)dx$. For simplicity, assume that

$$\sigma \equiv \sigma \, \mathrm{Id}_d, \quad \bar{b}(t,x,m) \equiv 0, \quad f(t,x,m,\gamma) = f_0(t,x,\gamma) + f_1(t,x,m).$$

Set $H(t,x,z) \doteq \max_{\gamma \in \Gamma} \{-\gamma \cdot z - f_0(t,x,\gamma)\}$. Let V be the unique solution of the Hamilton-Jacobi Bellman equation

$$-\partial_t V - \frac{\sigma^2}{2}\Delta V + H(t,x,\nabla V) = f_1\left(t, x, \int w(y)\mathfrak{p}(t,dy)\right) \text{ in } [0,T) \times O$$

with boundary condition $V(t,x) = F(t,x)$ in $\{T\} \times \mathrm{cl}(O) \cup [0,T) \times \partial O$, and let m be the unique solution of the Kolmogorov forward equation

$$\partial_t m - \frac{\sigma^2}{2}\Delta m + \mathrm{div}\left(m(t,x) \cdot \alpha(t,x)\right) = 0 \text{ in } (0,T] \times O$$

with $m(0,x) = m_0(x)$ and $m(t,x) = 0$ in $(0,T] \times \partial O$. Then

$$\alpha(t,x) = -D_z H\left(t,x,\nabla V(t,x)\right), \qquad \mathfrak{p}(t,dx) = \frac{m(t,x)}{\int_O m(t,y)dy}dx.$$

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Counterexample: system data

- Dimension $d = 3$, time horizon $T = 2$, dispersion coefficient $\sigma \equiv 0$.
- Initial distributions: $\nu_N \doteq \otimes^N \nu$ with $\nu \doteq \rho \otimes \rho \otimes \delta_0$, ρ Rademacher.
- Set of control actions $\Gamma \doteq \left\{\gamma \in \mathbb{R}^3 : \gamma_1 \in [-1,1], \gamma_2 = 0 = \gamma_3\right\}$;
- Set of non-absorbing states

$$O \doteq \left\{x \in \mathbb{R}^3 : -4 < x_1 < 1 + e^{x_3 - 1},\ -2 < x_2 < 2,\ -1 < x_3 < \tfrac{11}{5}\right\};$$

- Drift coefficient: w bounded Lipschitz with $w(x) = x_2$ if $x \in \mathrm{cl}(O)$,

$$\bar{b}(t,x,y) \doteq \begin{pmatrix} -|y| \wedge \frac{1}{4} \\ 0 \\ 1 \end{pmatrix}, \quad (t,x,y) \in [0,2] \times \mathbb{R}^3 \times \mathbb{R}.$$

- Cost coefficients: $f \equiv 1$, F non-negative bounded Lipschitz with

$$F(t,x) = 1 + \frac{x_3}{12} \cdot x_1 \text{ for all } (t,x) \in [0,2] \times \mathrm{cl}(O).$$

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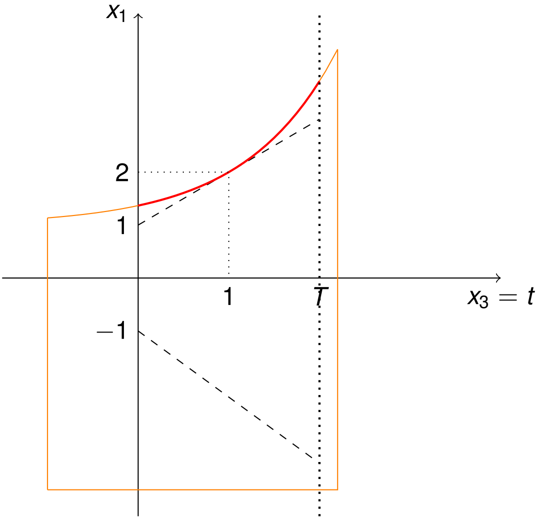
Counterexample: boundary of O

Boundary of the set of non-absorbing states O on the x_3-x_1-plane:



Counterexample: N-player game (N odd)

Since $u_{i,1}$ takes values in $[-1,1]$ and $\xi^N_{i,1}$ values in $\{-1,1\}$,

$$-1-\frac{5}{4}t \leq X^N_{i,1}(t) \leq 1+t \quad \text{for all } t\in[0,2] \text{ with probability one.}$$

X^N_i can leave O before $T=2$ only if $X^N_{i,1}(1)=2$; possible only if $\sum_{j=1}^N \xi^N_{j,2}=0$. Probability of this event equal to zero if N odd, hence $\tau^N_i=2$ for every i. Dynamics of the N-player game therefore

$$\begin{pmatrix} X^N_{i,1}(t) \\ X^N_{i,2}(t) \\ X^N_{i,3}(t) \end{pmatrix} = \begin{pmatrix} \xi^N_{i,1} + \int_0^t u_{i,1}(s,\boldsymbol{X}^N)ds - t\cdot\left(\left|\frac{1}{N}\sum_{j=1}^N \xi^N_{j,2}\right| \wedge \frac{1}{4}\right) \\ \xi^N_{i,2} \\ t \end{pmatrix},$$

where $\xi^N_{i,k}$ i.i.d. Rademacher. Costs for player i:

$$J^N_i(\boldsymbol{u}) = 2 + \mathbf{E}_N\left[1+\frac{1}{6}\int_0^2 u_{i,1}(s,\boldsymbol{X}^N)ds - \frac{1}{3}\left(\left|\frac{1}{N}\sum_{j=1}^N \xi^N_{j,2}\right| \wedge \frac{1}{4}\right)\right].$$

Nash equilibrium: $\boldsymbol{u}$ such that $u_{i,1}\equiv -1$ for all $i\in\{1,\ldots,N\}$.

Counterexample: limit system

Dynamics, given $\mathfrak{p} \in \mathcal{M}$, $(\xi, \alpha) \in \mathcal{A}$ with $\mathrm{Law}(\xi) = \nu$:

$$\begin{pmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} \alpha_1(s) - \left| \int_{\mathbb{R}^3} w(y)\mathfrak{p}(s, dy) \right| \wedge \frac{1}{4} \\ 0 \\ 1 \end{pmatrix} ds.$$

Suppose $\mathfrak{p}$ is such that $\mathrm{supp}(\mathfrak{p}(t)) \subseteq \mathrm{cl}(O)$ and $\int_{\mathbb{R}^3} w(y)\mathfrak{p}(t, dy) = 0$ for all t. Dynamics then reduce to

$$\text{(4)} \qquad \begin{pmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{pmatrix} = \begin{pmatrix} \xi_1 + \int_0^t \alpha_1(s)ds \\ \xi_2 \\ t \end{pmatrix},$$

while costs are equal to

$$J((\xi, \alpha); \mathfrak{p}) = \mathbf{E}\left[\tau^X \wedge 2 + 1 + \frac{\tau^X \wedge 2}{12} \cdot X_1(\tau^X \wedge 2) \right].$$

Counterexample: mean field game

If (ξ, α) is such that

$$\alpha(t, \omega) = \begin{cases} (1, 0, 0)^{\mathsf{T}} & \text{if } \xi_1(\omega) = 1 \text{ and } t \in [0, 1], \\ (-1, 0, 0)^{\mathsf{T}} & \text{otherwise,} \end{cases}$$

then $J((\xi, \alpha); \mathfrak{p}) = V(\nu; \mathfrak{p})$. Let X be the unique solution of Eq. (4) under such a control, and set

$$\mathfrak{p}_*(t, \cdot) \doteq \mathrm{P}(X \in \cdot \mid \tau^X > t), \quad t \in [0, 2].$$

Then $\mathrm{supp}(\mathfrak{p}_*(t)) \subseteq \mathrm{cl}(O)$ and $\int_{\mathbb{R}^3} w(y)\mathfrak{p}_*(t, dy) = 0$ for all t.

Define the feedback strategy u^* in $\mathcal{U}_1$ by

$$u^*(t, \varphi) \doteq \begin{cases} (1, 0, 0)^{\mathsf{T}} & \text{if } \varphi_1(t) \geq 1 \text{ and } t \in [0, 1], \\ (-1, 0, 0)^{\mathsf{T}} & \text{if } \varphi_1(t) \leq -1, \\ \text{arbitrarily} & \text{otherwise.} \end{cases}$$

Then $(\nu, u^*, \mathfrak{p}_*)$ is a feedback solution of the mean field game.

Counterexample: approximate Nash equilibria?

In analogy with Theorem 1, define $\boldsymbol{u}^N = (u_1^N, \ldots, u_N^N)$ by

$$u_i^N(t, \varphi) \doteq u^*(t, \varphi_i), \quad (t, \varphi) \in [0, T] \times \mathcal{X}^N.$$

Then $\boldsymbol{u}^N \in \mathcal{U}_{fb}^N$ and, if N is odd,

$$J_i^N(\boldsymbol{u}^N) = 3 - \frac{1}{2} \cdot \frac{2}{6} - \frac{1}{3}\,\mathbf{E}\left[\left|\frac{1}{N}\sum_{j=1}^{N} \xi_{j,2}^N\right| \wedge \frac{1}{4}\right] \geq \frac{33}{12}.$$

Suppose player one deviates from $\boldsymbol{u}^N$ by always playing -1. Then

$$J_1^N([\boldsymbol{u}^{N,-1}, -1]) = 3 - \frac{1}{3} - \frac{1}{3}\,\mathbf{E}_N\left[\left|\frac{1}{N}\sum_{j=1}^{N} \xi_{j,2}^N\right| \wedge \frac{1}{4}\right] \leq \frac{32}{12}.$$

Player one thus saves costs of $1/12$ by deviating from $\boldsymbol{u}^N$ for every N odd (asymptotically, also for N even).
Strategy vectors induced by solution $(\nu, u^*, \mathfrak{p}_*)$ do not yield approximate Nash equilibria with vanishing error!

Conclusions and open questions

- Class of mean field games and N-player games with absorption.
- Construction of approximate Nash equilibria from the mean field game under non-degeneracy condition. Counter-example in the degenerate case.
- Sufficient conditions for existence of solutions. What about uniqueness / non-uniqueness?

Conclusions

Thank you.

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