A Linear-Quadratic Mean Field Team with Mixed Players

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Outline of talk

- Mean Field Teams (MFTs) – Cooperative (i.e., social optimization)
  - Recall mean field games (MFG) – noncooperative
    - with peers (i.e., comparably small players)
    - with mixed players (i.e., with a major player)
  - Motivation for cooperation
  - Parallel development (peers, mixed players, etc)
- This talk: MFTs with **mixed players** and dynamic coupling
  - **Method**: person-by-person optimality; two-scale variations due to dynamic coupling (neither usually used in MFG)
  - **Result**: Social optimality theorem

Mean Field Game: A major player $\mathcal{A}_0$ and minor players $\mathcal{A}_i$, $1 \le i \le N$

Dynamics (Huang'10):

$$dx_0(t) = \left[A_0 x_0(t) + B_0 u_0(t) + F_0 x^{(N)}(t)\right] dt + D_0 dW_0(t), \quad t \ge 0,$$
$$dx_i(t) = \left[A(\theta_i) x_i(t) + B u_i(t) + F x^{(N)}(t) + G x_0(t)\right] dt + D dW_i(t),$$

Costs:

$$J_0(u_0, ..., u_N) = E \int_0^\infty e^{-\rho t} \left\{ \left| x_0 - \Phi(x^{(N)}) \right|^2_{Q_0} + u_0^T R_0 u_0 \right\} dt,$$
$$J_i(u_0, ..., u_N) = E \int_0^\infty e^{-\rho t} \left\{ \left| x_i - \Psi(x_0, x^{(N)}) \right|^2_{Q} + u_i^T R u_i \right\} dt,$$

$x^{(N)} = \frac{1}{N} \sum_{i=1}^N x_i$, $\Phi(x^{(N)}) = H_0 x^{(N)} + \eta_0$, $\Psi(x_0, x^{(N)}) = H x_0 + \hat{H} x^{(N)} + \eta$

- Different variants are possible

MFGs: 1 major and N minor players:

1) Simultaneous strategy selection (Nash game and variants)

- Huang (2010); Nguyen and Huang (2012) – LQG
- Nourian and Caines (2013); Carmona and Zhu (2015); Bensoussan et al. (2015) – Nonlinear diffusion; conditional mean field
- Buckdahn, Li, and Peng (2014)– nonlinear diffusion, minor players have coordination
- Sen and Caines (2016) – noisy information

2) Strategy selection with leadership

- Wang and Zhang (2014) – Discrete time
- Moon and Basar (2015); Bensoussan et al (2016) – Continuous time
- Kolokoltsov (2015) – Principal agent

Motivation for cooperative decision (mean field team)

- Manage space heaters in large buildings (hotel, apartment building, etc); they can run cooperatively to maintain comfort and good average load (as a mean field)
- Kizilkale and Malhame (2016) considered related collective target tracking reflecting partial cooperation; linear SDE temperature dynamics

LQ Mean Field Team (social optimization) with peers:

- Dynamics and costs (Huang, Caines, Malhame, 2012):

$$dx_i = A(\theta_i)x_i dt + Bu_i dt + DdW_i, \qquad 1 \le i \le N,$$

$$J_i = E\int_0^\infty e^{-\rho t}\left\{|x_i - \Phi(x^{(N)})|_Q^2 + u_i^T R u_i\right\} dt,$$

where $\Phi(x^{(N)}) = \Gamma x^{(N)} + \eta$, $x^{(N)} = (1/N)\sum_{i=1}^N x_i$.

- **Objective**: minimize **social cost**: $J_{\mathrm{soc}}^{(N)} = \sum_{i=1}^N J_i$.
- **Main Results**:

$$|(1/N)J_{\mathrm{soc}}^{(N)}(\hat{u}) - \inf_{u\in\mathcal{U}_o}(1/N)J_{\mathrm{soc}}^{(N)}(u)| = O(1/\sqrt{N} + \bar{\epsilon}_N),$$

where $\hat{u} = (\hat{u}_1, \ldots, \hat{u}_N)$, $\hat{u}_i = -R^{-1}B^T(\Pi_{\theta_i}\hat{x}_i + s_{\theta_i}(t))$;
$\mathcal{U}_o$: centralized controls.

Nonlinear extension: Sen, Huang and Malhamé (CDC'16)

MFT: 1 major and N minor players:

Simultaneous strategy selection to minimize social cost $J_0 + \frac{\lambda}{N}\sum_{i=1}^{N} J_i$

- Huang and Nguyen (IFAC'2011) – LQ
  - Method and result: Uses state space augmentation; Only partial solution
- Huang and Nguyen (IEEE CDC'16)

$$dx_0 = (A_0 x_0 + B_0 u_0)dt + D_0 dW_0,$$
$$dx_i = (Ax_i + Bu_i)dt + DdW_i, \quad 1 \le i \le N.$$

(Decoupled dynamics; players are coupled via the social cost.)

  - Method and result: Person-by-person optimality; existence under standard positive (semi-)definiteness assumption for cost weight matrices

This talk considers the LQ MFT with

- a major player
- coupled dynamics

Example:

$$dx_{i,t}^N = (Ax_{i,t}^N + Bu_{i,t}^N + Fx_t^{(N)} + Gx_{0,t}^N)dt + DdW_{i,t}, \quad 1 \leq i \leq N.$$

The dynamic coupling causes some very delicate difficulties

- This generates small but important perturbations
- Different from MFG

Dynamics of the major player $\mathcal{A}_0$, and N minor players $\mathcal{A}_i$:

$$dx_{0,t}^N = (A_0 x_{0,t}^N + B_0 u_{0,t}^N + F_0 x_t^{(N)})dt + D_0 dW_{0,t},$$
$$dx_{i,t}^N = (A x_{i,t}^N + B u_{i,t}^N + F x_t^{(N)} + G x_{0,t}^N)dt + D dW_{i,t}, \quad 1 \le i \le N.$$

(A1) The initial states $x_{j,0}^N = x_j(0)$ for $j \ge 0$. $\{x_j(0), 0 \le j \le N\}$ are independent, and for all $1 \le i \le N$, $Ex_i(0) = \mu_0$. $\sup_i E|x_i(0)|^2 \le c$ for a constant c independent of N.

Note: The condition of equal initial means can be generalized.

The cost for $\mathcal{A}_0$ and $\mathcal{A}_i$, $1 \le i \le N$:

$$J_0(u_0^N, u_{-0}^N) = E\int_0^T \{|x_0^N - \Phi(x^{(N)})|_{Q_0}^2 + (u_0^N)^T R_0 u_0^N\}dt,$$
$$J_i(u_i^N, u_{-i}^N) = E\int_0^T \{|x_i^N - \Psi(x_0^N, x^{(N)})|_{Q}^2 + (u_i^N)^T R u_i^N\}dt,$$

where $Q_0 \ge 0$, $Q \ge 0$ and $R_0 > 0$, $R > 0$,

- $u_{-j}^N = (u_0^N, \ldots, u_{j-1}^N, u_{j+1}^N, \ldots, u_N^N)$, $\quad x^{(N)} = \frac{1}{N}\sum_{i=1}^N x_i^N$,
- $\Phi(x^{(N)}) = H_0 x^{(N)} + \eta_0$, $\quad \Psi(x_0^N, x^{(N)}) = H x_0^N + \hat{H} x^{(N)} + \eta$.

The social cost:

$$J_{\mathrm{soc}}^{(N)}(u^N) = J_0 + \frac{\lambda}{N}\sum_{k=1}^N J_k,$$

where $u^N = (u_0^N, u_1^N, \ldots, u_N^N)$ and $\lambda > 0$.

Recall the social cost:

$$J_{\mathrm{soc}}^{(N)}(u^N) = J_0 + \frac{\lambda}{N}\sum_{k=1}^{N} J_k,$$

where $u^N = (u_0^N, u_1^N, \ldots, u_N^N)$ and $\lambda > 0$.

- Give a big share to $\mathcal{A}_0$
- If λ/N were replaced by 1, the limiting control problem would be too insensitive to the performance of the major player and become inappropriate.

Notation:

- $u^N = (u_0^N, u_1^N, \cdots, u_N^N)$
- $u_{-j}^N = (u_0^N, \ldots, u_{j-1}^N, u_{j+1}^N, \ldots, u_N^N)$, $j \geq 0$
- $x^{(N)} = \frac{1}{N}\sum_{i=1}^N x_i^N$, $\hat{x}^{(N)} = \frac{1}{N}\sum_{i=1}^N \hat{x}_i^N$, etc
- $\hat{u}^{(N)} = \frac{1}{N}\sum_{i=1}^N \hat{u}_i^N$
- $\hat{x}_{-i}^{(N)} = \frac{1}{N}\sum_{j\neq i}^N \hat{x}_j^N$.
- $\tilde{x}_{-i}^{(N)} = \frac{1}{N}\sum_{j\neq i}^N \tilde{x}_j^N$.
- x_0^∞, x_i^∞, u_i^∞, etc. for the limiting model
- m, $\hat{m}$, $\tilde{m}$ for the mean field

Existence of (centralized) social optimum

Fact: Since the optimal control problem minimizing $J_{\mathrm{soc}}^{(N)}$ is strictly convex and coercive, there exists a unique optimal control

$$\hat{u}^N = (\hat{u}_0^N, \hat{u}_1^N, \ldots, \hat{u}_N^N),$$

where each $\hat{u}_j^N$ belongs to $L^2_{\mathcal{F}}(0, T; \mathbb{R}^{n_1})$.

What to do next?

- Use person-by-person optimality; perturb one component in $\hat{u}^N$; for instance (similarly for a minor player)

$$J_{\mathrm{soc}}^{(N)}(\hat{u}_0^N, \hat{u}_1^N, \ldots, \hat{u}_N^N) \le J_{\mathrm{soc}}^{(N)}({\color{red}u_0^N}, \hat{u}_1^N, \ldots, \hat{u}_N^N)$$

- Construct **two limiting variational problems**: $P_{\mathcal{A}_0}$ for the major player and $P_{\mathcal{A}_i}$ for a representative minor player

1. – The major player's variational problem

Consider variation $\tilde{u}_0^N \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{n_1})$. Let $(\hat{x}_j^N)_{j=0}^N$ correspond to $(\hat{u}_j^N)_{j=0}^N$, $(\hat{x}_j^N+\tilde{x}_j^N)_{j=0}^N$ correspond to $(\hat{u}_0^N+\tilde{u}_0^N, \hat{u}_1^N, \dots, \hat{u}_N^N)$. Then

$$d\tilde{x}_0^N = [A_0\tilde{x}_0^N + F_0\tilde{x}^{(N)} + B_0\tilde{u}_0^N]dt, \quad \tilde{x}_0^N(0)=0,$$

$$d\tilde{x}^{(N)} = [(A+F)\tilde{x}^{(N)} + G\tilde{x}_0^N]dt, \quad \tilde{x}^{(N)}(0)=0,\ \tilde{x}^{(N)} = \frac{1}{N}\sum_{i=1}^N \tilde{x}_i^N$$

The first variation of the social cost

$$\frac{1}{2}\delta J_0 + \frac{\lambda}{2N}\sum_{i=1}^N \delta J_i = E\int_0^T L_0^N(t)dt, \quad \text{linear functional of } \tilde{u}_0^N$$

where

$$\begin{aligned} L_0^N =&[\hat{x}_0^N - (H_0\hat{x}^{(N)}+\eta_0)]^T Q(\tilde{x}_0^N - H_0\tilde{x}^{(N)}) + (\hat{u}_0^N)^T R_0\tilde{u}_0^N \\ &+ \lambda[(I-\hat{H})\hat{x}^{(N)} - H\hat{x}_0^N - \eta]^T Q[(I-\hat{H})\tilde{x}^{(N)} - H\tilde{x}_0^N] \end{aligned}$$

Recall $E\int_0^T L_0^N dt$ is a linear functional of $\tilde{u}_0^N$.

Lemma (The first order variational condition) We have

$$E\int_0^T L_0^N(t)dt = 0$$

for all $\tilde{u}_0^N \in L_{\mathcal{F}}(0, T; \mathbb{R}^{n_1})$.

Proof. Use person-by-person optimality.

1.1. The major player's limiting variational problem

$$d\hat{x}_0^\infty = (A_0\hat{x}_0^\infty + B_0\hat{u}_0^\infty + F_0\hat{m})dt + D_0 dW_0(t),$$
$$d\hat{m} = ((A+F)\hat{m} + B\bar{u} + G\hat{x}_0^\infty)dt.$$

$$d\tilde{x}_0^\infty = (A_0\tilde{x}_0^\infty + B_0\tilde{u}_0^\infty + F_0\tilde{m})dt, \quad \tilde{x}_{0,0}^\infty = 0,$$
$$d\tilde{m} = ((A+F)\tilde{m} + G\tilde{x}_0^\infty)dt, \quad \tilde{m}_0 = 0.$$

Problem $P_{\mathcal{A}_0}$: Find $\hat{u}_0^\infty$ to satisfy the variational condition

$$E\int_0^T L_0^\infty dt = 0, \qquad \forall\ \tilde{u}_0^\infty \in L_{\mathcal{F}}(0,T;\mathbb{R}^{n_1})$$

where

$$\begin{aligned} L_0^\infty =& [\hat{x}_0^\infty - (H_0\hat{m} + \eta_0)]^T Q(\tilde{x}_0^\infty - H_0\tilde{m}) + (\hat{u}_0^\infty)^T R_0 \tilde{u}_0^\infty \\ &+ \lambda[(I-\hat{H})\hat{m} - H\hat{x}_0^\infty - \eta]^T Q[(I-\hat{H})\tilde{m} - H\tilde{x}_0^\infty] \end{aligned}$$

Method: construct an appropriate adjoint process (p_0, p), The $\mathcal{A}_0$-FBSDE($\bar{u}$):

$$\begin{aligned}
d\hat{x}_0^\infty &= (A_0\hat{x}_0^\infty + B_0\hat{u}_0^\infty + F_0\hat{m})dt + D_0 dW_0,\\
d\hat{m} &= [(A+F)\hat{m} + G\hat{x}_0^\infty + B\bar{u}]dt,\\
dp_0 &= \{-A_0^T p_0 - G^T p + Q_0[\hat{x}_0^\infty - (H_0\hat{m} + \eta_0)]\\
&\quad - H^T \lambda Q[(I-\hat{H})\hat{m} - H\hat{x}_0^\infty - \eta]\}dt + \xi_0 dW_0,\\
dp &= \{-F_0^T p_0 - (A+F)^T p - H_0^T Q_0[\hat{x}_0^\infty - (H_0\hat{m} + \eta_0)]\\
&\quad + (I-\hat{H})^T \lambda Q[(I-\hat{H})\hat{m} - H\hat{x}_0^\infty - \eta]\}dt + \xi dW_0,
\end{aligned}$$

where $\hat{x}_0(0) = x_0(0)$, $\hat{m}(0) = \mu_0$, $p_0(T) = p(T) = 0$.

The optimal control (critical point) is

$$\hat{u}_0^\infty = R_0^{-1} B_0^T p_0.$$

Lemma The $\mathcal{A}_0$-FBSDE($\bar{u}$) has a unique solution.

Proof: Identify a Hamiltonian with nonnegative state weight matrix; see next page.

Lemma.

$$\mathbf{Q} = \begin{bmatrix} Q_0 & -Q_0 H_0 \\ -H_0^T Q_0 & H_0^T Q_0 H_0 \end{bmatrix} + \lambda \begin{bmatrix} H^T Q H & -H^T Q \big(I - \hat{H}\big) \\ -\big(I - \hat{H}\big)^T Q H & \big(I - \hat{H}\big)^T Q \big(I - \hat{H}\big) \end{bmatrix}$$

is positive semi-definite.

2. – The minor player's variational problem.
Suppose $\hat{u}^N$ yields the state processes $\hat{x}_j^N$, $j = 0, \ldots, N$.

Consider $(u_i^N, \hat{u}_{-i}^N)$ for a fixed $i \geq 1$, which generates

$$dx_0^N = (A_0 x_0^N + B_0 \hat{u}_0^N + F_0 x^{(N)})dt + D_0 dW_0,$$
$$dx_j^N = (A x_j^N + B \hat{u}_j^N + F x^{(N)} + G x_0^N)dt + D dW_j, \quad 1 \leq j \neq i,$$
$$dx_i^N = (A x_i^N + B u_i^N + F x^{(N)} + G x_0^N)dt + D dW_i.$$

The variations of the state processes (0 initial conditions)

$$d\tilde{x}_0^N = (A_0 \tilde{x}_0^N + F_0 \tilde{x}_{-i}^{(N)} + \tfrac{1}{N} F_0 \tilde{x}_i^N)dt,$$
$$d\tilde{x}_j^N = (A \tilde{x}_j^N + F \tilde{x}_{-i}^{(N)} + \frac{1}{N} F \tilde{x}_i^N + G \tilde{x}_0^N)dt, \quad 1 \leq j \neq i$$
$$d\tilde{x}_i^N = (A \tilde{x}_i^N + B \tilde{u}_i^N + F \tilde{x}_{-i}^{(N)} + \frac{1}{N} F \tilde{x}_i^N + G \tilde{x}_0^N)dt.$$

The first variations of the costs:

$$\frac{1}{2}\delta J_0 = E\int_0^T \chi_0 dt, \qquad \frac{\lambda}{2N}\delta J_i = E\int_0^T \chi_i dt,$$

$$\frac{\lambda}{2N}\sum_{j\neq i}\delta J_j = E\int_0^T \chi_{-i} dt$$

where

$$\chi_0 = [\hat{x}_0^N - (H_0\hat{x}^{(N)} + \eta_0)]^T Q_0[\tilde{x}_0^N - H_0\tilde{x}_{-i}^{(N)} - \frac{1}{N}H_0\tilde{x}_i^N]$$

$$\chi_i = (\hat{x}_i^N - (H\hat{x}_0^N + \hat{H}\hat{x}^{(N)} + \eta))^T\frac{1}{N}\lambda Q$$

$$\cdot(\tilde{x}_i^N - H\tilde{x}_0^N - \hat{H}\tilde{x}_{-i}^{(N)} - \frac{1}{N}\hat{H}\tilde{x}_i^N) + (\hat{u}_i^N)^T\frac{1}{N}\lambda R\tilde{u}_i^N$$

$$\chi_{-i} = [(I-\hat{H})\hat{x}^{(N)} - H\hat{x}_0^N - \eta]^T\lambda Q$$

$$\cdot[(I-\hat{H})\tilde{x}_{-i}^{(N)} - H\tilde{x}_0^N - \frac{1}{N}\hat{H}\tilde{x}_i^N] + O(\frac{1}{N^2})$$

PbP optimality implies the variational condition:

$$E\int_0^T L_i^N dt = 0, \quad \forall \tilde{u}_i^N,$$

where

$$\begin{aligned}
L_i^N &= \chi_0 + \chi_i + \chi_{-i} \\
&= [\hat{x}_0^N - (H_0\hat{x}^{(N)} + \eta_0)]^T Q_0(\tilde{x}_0^N - H_0\tilde{x}_{-i}^{(N)} - \frac{1}{N}H_0\tilde{x}_i^N) \\
&\quad + [\hat{x}_i^N - (H\hat{x}_0^N + \hat{H}\hat{x}^{(N)} + \eta)]^T \frac{1}{N}\lambda Q\tilde{x}_i^N + (\hat{u}_i^N)^T \frac{1}{N}\lambda R\tilde{u}_i^N \\
&\quad + [(I-\hat{H})\hat{x}^{(N)} - H\hat{x}_0^N - \eta]^T \lambda Q[(I-\hat{H})\tilde{x}_{-i}^{(N)} - H\tilde{x}_0^N - \frac{1}{N}\hat{H}\tilde{x}_i^N] \\
&\quad + O(\frac{1}{N^2})
\end{aligned}$$

For the minor player, we introduce a limiting problem:

- Use a limiting model below to produce approximations of $(\hat{x}_0^N, \hat{x}^{(N)}, \hat{x}_i^N)$.
- Further approximate $(\tilde{x}_0^N, \tilde{x}_{-i}^{(N)}, \tilde{x}_i^N)$ appropriately.

Consider

$$\begin{aligned} d\hat{x}_0^\infty &= (A_0\hat{x}_0^\infty + B_0\hat{u}_0^\infty + F_0\hat{m})dt + D_0 dW_0 \\ d\hat{m} &= ((A+F)\hat{m} + B\bar{u} + G\hat{x}_0^\infty)dt \\ dx_i^\infty &= (Ax_i^\infty + Bu_i^\infty + F\hat{m} + G\hat{x}_0^\infty)dt + DdW_i, \end{aligned}$$

where $\hat{x}_0^\infty(0) = x_0(0)$, $\hat{m}(0) = \mu_0$, $x_i^\infty(0) = x_i(0)$, and $\hat{u}_0^\infty$ has been determined by the variational problem of $\mathcal{A}_0$.

Let

$$d\tilde{x}_0^\infty = (A_0\tilde{x}_0^\infty + F_0\tilde{m} + \tfrac{1}{N}F_0\tilde{x}_i^\infty)dt, \quad \tilde{x}_{0,0}^\infty = 0,$$
$$d\tilde{m} = [(A+F)\tilde{m} + \frac{1}{N}F\tilde{x}_i^\infty + G\tilde{x}_0^\infty]dt, \quad \tilde{m}_0 = 0,$$
$$d\tilde{x}_i^\infty = (A\tilde{x}_i^\infty + B\tilde{u}_i^\infty)dt, \quad \tilde{x}_{0,0}^i = 0$$

$$\begin{aligned} L_i^\infty = {} & [\hat{x}_0^\infty - (H_0\hat{m} + \eta_0)]^T Q_0(\tilde{x}_0^\infty - H_0\tilde{m} - \frac{1}{N}H_0\tilde{x}_i^\infty) \\ & + [\hat{x}_i^\infty - (H\hat{x}_0^\infty + \hat{H}\hat{m} + \eta)]^T \frac{1}{N}\lambda Q\tilde{x}_i^\infty + (\hat{u}_i^\infty)^T \frac{1}{N}\lambda R\tilde{u}_i^\infty \\ & + [(I-\hat{H})\hat{m} - H\hat{x}_0^\infty - \eta]^T \lambda Q[(I-\hat{H})\tilde{m} - H\tilde{x}_0^\infty - \frac{1}{N}\hat{H}\tilde{x}_i^\infty] \end{aligned}$$

The new variational problem $P_{\mathcal{A}_i}$: Find $\hat{u}_i^\infty$ such that

$$E\int_0^T L_i^\infty dt = 0, \qquad \forall\ \tilde{u}_i^\infty.$$

Solution Method: identify adjoint processes.

Remark: non-commutativity!

Fact: $(\tilde{x}_0^\infty, \tilde{m})$ is not determined as the variations of the limiting dynamics in $P_{\mathcal{A}_i}$ since the control variation does not affect the first two equations.

Recall:

$$\begin{aligned} d\hat{x}_0^\infty &= (A_0\hat{x}_0^\infty + B_0\hat{u}_0^\infty + F_0\hat{m})dt + D_0 dW_0 \\ d\hat{m} &= ((A+F)\hat{m} + B\bar{u} + G\hat{x}_0^\infty)dt \\ dx_i^\infty &= (Ax_i^\infty + Bu_i^\infty + F\hat{m} + G\hat{x}_0^\infty)dt + DdW_i, \end{aligned}$$

and

$$\begin{aligned} d\tilde{x}_0^\infty &= (A_0\tilde{x}_0^\infty + F_0\tilde{m} + \tfrac{1}{N}F_0\tilde{x}_i^\infty)dt, \\ d\tilde{m} &= [(A+F)\tilde{m} + \frac{1}{N}F\tilde{x}_i^\infty + G\tilde{x}_0^\infty]dt, \\ d\tilde{x}_i^\infty &= (A\tilde{x}_i^\infty + B\tilde{u}_i^\infty)dt \end{aligned}$$

Now, for the limiting variational equations, we introduce the adjoint equations ($(\hat{x}_0^\infty, \hat{m})$ solved from $P_{\mathcal{A}_0}$):

$$\begin{aligned}
dq_0 = & \{-A_0^T q_0 - G^T q + Q_0[\hat{x}_0^\infty - (H_0\hat{m} + \eta_0)] \\
& - H^T \lambda Q[(I - \hat{H})\hat{m} - H\hat{x}_0^\infty - \eta]\}dt + \zeta_0^a dW_0 + \zeta_0^b dW_i, \\
dq = & \{-F_0^T q_0 - (A + F)^T q - H_0^T Q_0[\hat{x}_0^\infty - (H_0\hat{m} + \eta_0)] \\
& + (I - \hat{H})^T \lambda Q[(I - \hat{H})\hat{m} - H\hat{x}_0^\infty - \eta]\}dt + \zeta^a dW_0 + \zeta^b dW_i, \\
dq_i = & \{-F_0^T q_0 - F^T q - A^T q_i - H_0^T Q_0[\hat{x}_0^\infty - (H_0\hat{m} + \eta_0)] \\
& + \lambda Q[\hat{x}_i^\infty - (H\hat{x}_0^\infty + \hat{H}\hat{m} + \eta)] \\
& - \hat{H}\lambda Q[(I - \hat{H})\hat{m} - H\hat{x}_0^\infty - \eta]\}dt + \zeta_i^a dW_0 + \zeta_i^b dW_i,
\end{aligned}$$

where $q_0(T) = q(T) = q_i(T) = 0$. We have $P_{\mathcal{A}_i}$'s solution

$$\hat{u}_i^\infty = (\lambda R)^{-1} B^T q_i.$$

Lemma We have $(q_0, q) = (p_0, p)$.

Remark: Somehow unexpected. Good for reducing dimension.

Recall

$$
\begin{aligned}
d\tilde{x}_0^\infty &= (A_0\tilde{x}_0^\infty + F_0\tilde{m} + \tfrac{1}{N}F_0\tilde{x}_i^\infty)dt,\\
d\tilde{m} &= [(A+F)\tilde{m} + \frac{1}{N}F\tilde{x}_i^\infty + G\tilde{x}_0^\infty]dt,\\
d\tilde{x}_i^\infty &= (A\tilde{x}_i^\infty + B\tilde{u}_i^\infty)dt
\end{aligned}
$$

Construction of the adjoint processes (q_0, q, q_i):

- Suppose $\tilde{u}_i^\infty = O(1)$. In the variational dynamics of $(\tilde{x}_0^\infty, \tilde{m}, \tilde{x}_i^\infty)$, the first two entries have magnitude $O(1/N)$, and $\tilde{x}_i^\infty = O(1)$.
- Two scales
- Homogenize by using the equation of $\tilde{x}_i^\infty/N$.

So using **Lemma** $(q_0, q) = (p_0, p)$ where the RHS is from $\mathcal{A}_0$-FBSDE$(\bar{u})$, the adjoint equations for the minor player are:

$$\begin{aligned} dp_0 &= \{-A_0^T p_0 - G^T p + Q_0[\hat{x}_0^\infty - (H_0 \hat{m} + \eta_0)] \\ &\qquad - H^T \lambda Q[(I - \hat{H})\hat{m} - H\hat{x}_0^\infty - \eta]\}dt + \xi_0 dW_0, \\ dp &= \{-F_0^T p_0 - (A + F)^T p - H_0^T Q_0[\hat{x}_0^\infty - (H_0 \hat{m} + \eta_0)] \\ &\qquad + (I - \hat{H})^T \lambda Q[(I - \hat{H})\hat{m} - H\hat{x}_0^\infty - \eta]\}dt + \xi dW_0, \\ dq_i &= \{-F_0^T p_0 - F^T p - A^T q_i - H_0^T Q_0[\hat{x}_0^\infty - (H_0 \hat{m} + \eta_0)] \\ &\qquad + \lambda Q[\hat{x}_i^\infty - (H\hat{x}_0^\infty + \hat{H}\hat{m} + \eta)] \\ &\qquad - \hat{H}\lambda Q[(I - \hat{H})\hat{m} - H\hat{x}_0^\infty - \eta]\}dt + \zeta_i^a dW_0 + \zeta_i^b dW_i, \end{aligned}$$

where $p_0(T) = p(T) = q_i(T) = 0$. Recall we have $P_{\mathcal{A}_i}$'s solution

$$\hat{u}_i^\infty = (\lambda R)^{-1} B^T q_i.$$

Remainder: Still need to determine $\bar{u}$!

Question: how to determine $\bar{u}$?

Recall

$$\begin{aligned} dp_0 &= \{\cdots\}dt + \xi_0 dW_0, \\ dp &= \{\cdots\}dt + \xi dW_0, \\ dq_i &= \{-F_0^T p_0 - F^T p - A^T q_i - H_0^T Q_0[\hat{x}_0^\infty - (H_0 \hat{m} + \eta_0)] \\ &\quad + \lambda Q[\hat{x}_i^\infty - (H\hat{x}_0^\infty + \hat{H}\hat{m} + \eta)] \\ &\quad - \hat{H}\lambda Q[(I - \hat{H})\hat{m} - H\hat{x}_0^\infty - \eta]\}dt + \zeta_i^a dW_0 + \zeta_i^b dW_i, \end{aligned}$$

where $q_i(T) = 0$. And $\hat{u}_i^\infty = (\lambda R)^{-1} B^T q_i$.

Fact: $\bar{u} \approx \frac{1}{N}\sum_i \hat{u}_i^\infty = (\lambda R)^{-1} B^T \frac{1}{N}\sum_i q_i$.

Lemma. Averaging the equations of q_i, the resulting SDE is equivalent to that of $p(= q)$.

Consistency condition: Take $\bar{u}$ to satisfy

$$\bar{u} = (\lambda R)^{-1} B^T p$$

Now the "closed-loop" FBSDE for the major player:

$$\begin{aligned}
d\hat{x}_0^\infty &= (A_0 \hat{x}_0^\infty + B_0 R_0^{-1} B_0^T p_0 + F_0 \hat{m})dt + D_0 dW_0, \\
d\hat{m} &= [(A+F)\hat{m} + G\hat{x}_0^\infty + B(\lambda R)^{-1} B^T p]dt, \\
dp_0 &= \{-A_0^T p_0 - G^T p + Q_0[\hat{x}_0^\infty - (H_0 \hat{m} + \eta_0)] \\
&\quad - H^T \lambda Q[(I - \hat{H})\hat{m} - H\hat{x}_0^\infty - \eta]\}dt + \xi_0 dW_0, \\
dp &= \{-F_0^T p_0 - (A+F)^T p - H_0^T Q_0[\hat{x}_0^\infty - (H_0 \hat{m} + \eta_0)] \\
&\quad + (I - \hat{H})^T \lambda Q[(I - \hat{H})\hat{m} - H\hat{x}_0^\infty - \eta]\}dt + \xi dW_0,
\end{aligned}$$

where $\hat{x}_0^\infty(0) = x_0(0)$, $\hat{m}(0) = \mu_0$, $p_0(T) = p(T) = 0$.

Theorem: This FBSDE has a unique solution.
Proof: Use a nice Hamiltonian matrix structure.

The two extra equations of the minor player:

$$\begin{aligned} d\hat{x}_i^\infty &= [A\hat{x}_i^\infty + B(\lambda R)^{-1}B^T q_i + F\hat{m} + G\hat{x}_0]dt + DdW_i, \\ dq_i &= \{-F_0^T p_0 - F^T p - A^T q_i - H_0^T Q_0[\hat{x}_0^\infty - (H_0\hat{m} + \eta_0)] \\ &\quad + \lambda Q[\hat{x}_i^\infty - (H\hat{x}_0^\infty + \hat{H}\hat{m} + \eta)] \\ &\quad - \hat{H}\lambda Q[(I - \hat{H})\hat{m} - H\hat{x}_0^\infty - \eta]\}dt + \zeta_i^a dW_0 + \zeta_i^b dW_i, \end{aligned}$$

which can be uniquely solved.

The whole FBSDE of the minor player:

$$
\begin{aligned}
d\hat{x}_0^\infty &= (A_0\hat{x}_0^\infty + B_0R_0^{-1}B_0^Tp_0 + F_0\hat{m})dt + D_0dW_0,\\
d\hat{m} &= [(A+F)\hat{m} + G\hat{x}_0^\infty + B(\lambda R)^{-1}B^Tp]dt,\\
dp_0 &= \{-A_0^Tp_0 - G^Tp + Q_0[\hat{x}_0^\infty - (H_0\hat{m} + \eta_0)]\\
&\quad - H^T\lambda Q[(I-\hat{H})\hat{m} - H\hat{x}_0^\infty - \eta]\}dt + \xi_0dW_0,\\
dp &= \{-F_0^Tp_0 - (A+F)^Tp - H_0^TQ_0[\hat{x}_0^\infty - (H_0\hat{m} + \eta_0)]\\
&\quad + (I-\hat{H})^T\lambda Q[(I-\hat{H})\hat{m} - H\hat{x}_0^\infty - \eta]\}dt + \xi dW_0,\\
d\hat{x}_i^\infty &= [A\hat{x}_i^\infty + B(\lambda R)^{-1}B^Tq_i + F\hat{m} + G\hat{x}_0]dt + DdW_i,\\
dq_i &= \{-F_0^Tp_0 - F^Tp - A^Tq_i - H_0^TQ_0[\hat{x}_0^\infty - (H_0\hat{m} + \eta_0)]\\
&\quad + \lambda Q[\hat{x}_i^\infty - (H\hat{x}_0^\infty + \hat{H}\hat{m} + \eta)]\\
&\quad - \hat{H}\lambda Q[(I-\hat{H})\hat{m} - H\hat{x}_0^\infty - \eta]\}dt + \zeta_i^a dW_0 + \zeta_i^b dW_i.
\end{aligned}
$$

Theorem. This FBSDE has a unique solution.

Remark 1: General FBSDEs do not always have a solution.

Remark 2: We expect it is easy to have existence (as happens here) due to optimal control nature; different from games; even a two player LQ game may have no solution

Key error estimates

Proposition. Take a fixed $v \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{n_1})$ and let $\tilde{u}_i^N = \tilde{u}_i^\infty = v$ for both the $N+1$ player model and the limiting variational problem. Then for some constant C we have

$$\sup_{t \leq T} E[|\tilde{x}_0^\infty - \tilde{x}_0^N|^2 + |\tilde{m} - \tilde{x}_{-i}^{(N)}|^2 + |\frac{1}{N}\tilde{x}_i^\infty - \frac{1}{N}\tilde{x}_i^N|^2] \leq \frac{C}{N^4}.$$

Performance gap

Social Optimality Theorem We have

$$|J_{\mathrm{soc}}^{(N)}(\hat{u}) - \inf_u J_{\mathrm{soc}}^{(N)}(u)| = O(1/\sqrt{N}),$$

where each u_j^N, $0 \le j \le N$ within u is in $L^2_{\mathcal{F}}(0, T; \mathbb{R}^{n_1})$, and

$$\hat{u}_0^N = \hat{u}_0^\infty = R_0^{-1} B_0^T p_0, \quad \hat{u}_i^N = \hat{u}_i^\infty = (\lambda R)^{-1} B^T p_i,$$

where (p_0, p_i) are solved from $P_{\mathcal{A}_0}$ and $P_{\mathcal{A}_i}$. $\square$

We can further show that p_0 is a linear function of $(\hat{x}_0^\infty, \hat{m})$.

We may choose $\mathcal{F}_t$ as the σ-algebra
$\mathcal{F}_t^{x.(0), W.} \triangleq \sigma(x_j(0), W_j(\tau), 0 \le j \le N, \tau \le t)$.

Thank you!