Mean Field Games on networks

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Outline

- Brief introduction
- Definition of networks
- Formal derivation of the MFG system on networks
- Study of the MFG system on networks
- A numerical scheme

Model example on the torus $\mathbb{T}^n$ [Lasry-Lions,'06]

Consider a game with N rational and indistinguishable players. The i-th player's dynamics is

$$dX_t^i = -\alpha_t^i dt + \sqrt{2\nu}\, dW_t^i, \qquad X_0^i = x^i \in \mathbb{T}^n$$

where $\nu > 0$, W^i are independent Brownian motions and α^i is the control chosen so to minimize the cost functional

$$\liminf_{T \to +\infty} \frac{1}{T} \mathbb{E}_x \left\{ \int_0^T \left[L(X_s^i, \alpha_s^i) + V\left(\frac{1}{N-1} \sum_{j \neq i} \delta_{X_s^j} \right) \right] ds \right\}.$$

The Nash equilibria are characterized by a system of $2N$ equations. As $N \to +\infty$, this system reduces to the following one:

$$
(\text{MFG-}\mathbb{T}^n) \qquad \begin{cases} -\nu\Delta u + H(x,Du) + \rho = V([m]) & \text{in } \mathbb{T}^n \\ \nu\Delta m + \operatorname{div}\left(m\frac{\partial H}{\partial p}(x,Du)\right) = 0 & \text{in } \mathbb{T}^n \\ \int_{\mathbb{T}^n} m\,dx = 1, \qquad m > 0 \\ \int_{\mathbb{T}^n} u\,dx = 0 \end{cases}
$$

- $H(x,p) := \sup_{q\in\mathbb{R}^n}\{-p\cdot q - L(x,q)\}$;
- "$[m]$" means that V depends on m in a local or in a nonlocal way.

Theorem [Lasry-Lions '06]

- There exists a smooth solution (u,m,ρ) to the above problem;
- Assume
  - either V is strictly monotone in m (i.e. $\int_{\mathbb{T}^n}(V([m_1]) - V([m_2]))(m_1 - m_2)dx \leq 0$ implies $m_1 = m_2$)
  - or V is monotone in m (i.e. $\int_{\mathbb{T}^n}(V([m_1]) - V([m_2]))(m_1 - m_2)dx \geq 0$) and H is strictly convex in p.

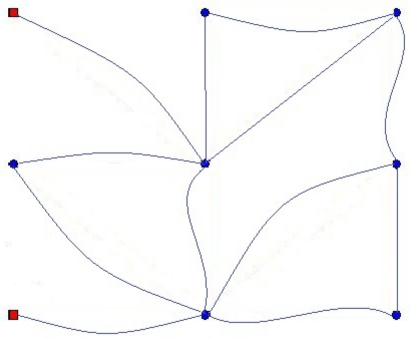
Then the solution is unique.

Basic References for MFG theory:

- Lasry-Lions, C.R. Math. Acad. Sci. Paris 343 (2006), 619-625.
- Lasry-Lions, C.R. Math. Acad. Sci. Paris 343 (2006), 679-684.
- Lasry-Lions, Jpn. J. Math. 2 (2007), 229-260.
- Huang-Malhamé-Caines, Commun. Inf. Syst. 6 (2006), 221-251.
- Lions' course at College de France '06-'12 and '16-'17 `www.college-de-france.fr`
- Cardaliaguet, Notes on MFG (from Lions' lectures at College de France), `www.ceremade.dauphine.fr/~cardalia/`
- Achdou-Capuzzo Dolcetta, SIAM J. Num. Anal. 48 (2010), 1136-1162.
- Achdou-Camilli-Capuzzo Dolcetta, SIAM J. Num. Anal. 51 (2013),2585-2612.
- MFG on graphs (i.e., agents have a finite number of states)
  - Discrete time, finite state space:
    Gomes-Mohr-Souza, J. Math. Pures Appl. 93 (2010), 308-328.
  - Continuous time, finite state space:
    Gomes-Mohr-Souza, Appl. Math. Optim., 68 (2013), 99-143.
    Guéant, Appl. Math. Optim. 72 (2015), 291-303.

Network

A network is a connected set Γ consisting of vertices $V := \{v_i\}_{i \in I}$ and edges $E := \{e_j\}_{j \in J}$ connecting the vertices. We assume that the network is embedded in the Euclidian space $\mathbb{R}^n$ and that any two edges can only have intersection at a vertex.



(a) An example of network

Some Notations

- $Inc_i := \{j \in J : \; e_j \text{ incident to } v_i\}$ is the set of edges incident to the vertex v_i.
- A vertex v_i is a transition vertex if it has more than one incident edge. We denote by $\Gamma_T = \{v_i,\; i \in I_T\}$ the set of transition vertices. A vertex v_i is a boundary vertex if it has only one incident edge. For simplicity, we assume that the set of boundary vertices is empty.
- Any edge e_j is parametrized by a smooth function $\pi_j : [0, l_j] \to \mathbb{R}^n$. For a function $u : \Gamma \to \mathbb{R}$ we denote by $u_j : [0, l_j] \to \mathbb{R}$ its restriction to e_j, i.e. $u(x) = u_j(y)$ for $x \in e_j$, $y = \pi_j^{-1}(x)$.
- The derivative are considered w.r.t. the parametrization.
- The oriented derivative of a function u at a transition vertex v_i is

$$\partial_j u(v_i) := \begin{cases} \lim_{h \to 0^+} (u_j(h) - u_j(0))/h, & \text{if } v_i = \pi_j(0) \\ \lim_{h \to 0^+} (u_j(l_j - h) - u_j(l_j))/h, & \text{if } v_i = \pi_j(l_j). \end{cases}$$

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Some functional spaces

- $u \in C^{q,\alpha}(\Gamma)$, for $q \in \mathbb{N}$ and $\alpha \in (0,1]$, when $u \in C^0(\Gamma)$ and $u^j \in C^{q,\alpha}([0,l_j])$ for each $j \in J$. We set
$$\|u\|_\Gamma^{(q+\alpha)} = \max_{j\in J} \|u_j\|_{[0,l_j]}^{(q+\alpha)}.$$
- $u \in L^p(\Gamma)$, $p \geq 1$ if $u^j \in L^p(0,l_j)$ for each $j \in J$. We set
$$\|u\|_{L^p} = (\sum_{j\in J} \|u_j\|_{L^p(e_j)}^p)^{1/p}.$$
- $u \in W^{k,p}(\Gamma)$, for $k \in \mathbb{N}$, $k \geq 1$ and $p \geq 1$ if $u \in C^0(\Gamma)$ and $u^j \in W^{k,p}(0,l_j)$ for each $j \in J$. We set
$$\|u\|_{W^{k,p}} = (\sum_{j\in J} \|u_j\|_{W^{k,p}(e_j)}^p)^{1/p}.$$

Formal derivation of MFG systems on networks

Dynamics of a generic player.

Inside each edge e_j, the dynamics of a generic player is

$$dX_t = -\alpha_t dt + \sqrt{2\nu_j} dW_t$$

where α is the control, $\nu_j > 0$ and W is an independent Brownian motions.

At any internal vertex v_i, the player spends zero time a.s. at v_i and it enters in one of the incident edges, say e_j, with probability β_{ij} with

$$\beta_{ij} > 0, \qquad \sum_{j \in Inc_i} \beta_{ij} = 1.$$

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Discussion on the transition condition: probabilistic approach

We consider the uncontrolled case. Fix a vertex v_i.

- **Rigorous definition of "enters in one of the incident edges..."** For $\delta > 0$, consider $\theta_\delta := \inf\{t > 0 \mid \ dist(X_t, v_i) = \delta\}$. Then,

$$\lim_{\delta \to 0^+} P\{X_{\theta_\delta} \in e_j\} = \beta_{ij}.$$

- **Fattening interpretation.** Let M_ε be the set in $\mathbb{R}^n$ obtained "enlarging" each edge e_j by a ball of radius $\varepsilon\beta_{ij}$. One can obtain these dynamics as the limit as $\varepsilon \to 0^+$ of a Brownian motion in M_ε with normal reflection at the boundary.
- **Itô's formula** still holds true.

See: [Freidlin-Wentzell, Ann. Prob.'93], [Freidlin-Sheu, PTRF'00].

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Discussion on the transition condition: analytical approach

Consider the operator $\mathcal{A}$ defined on $C^0(\Gamma)$, defined for $x \in e_j$ by

$$A_j u := \nu_j \frac{d^2 u}{dy^2}(y) + \bar{\alpha} \frac{du}{dy}(y), \qquad y = \pi^{-1}(x)$$

with domain

$$D(\mathcal{A}) := \Big\{ u \in C^2(\Gamma) \mid \underbrace{\sum_{j \in Inc_i} \beta_{ij} \partial_j u(v_i) = 0}_{\text{(weighted) Kirchhoff condition}} \Big\}.$$

- $\mathcal{A}$ generates on Γ the Markov process X_t described before.
- $\mathcal{A}$ fulfills the Maximum Principle.
- **Fattening interpretation.** The solution of $\mathcal{A}u = 0$ is the $\lim_{\varepsilon \to 0^+}$ of u_ε, solution of a "similar" problem in M_ε with $\frac{\partial u_\varepsilon}{\partial n} = 0$ on ∂M_ε.

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Formal derivation of the MFG system on the network

We formally derive the MFG system on the network: the HJB equation is obtained through the dynamic programming principle while the FP equation is obtained as adjoint of the linearized HJB one.

Hence, the HJB equation is

$$\begin{cases} -\nu_j \partial^2 u + H_j(x, \partial u) + \rho = V[m] & x \in e_j,\ j \in J \\ \sum_{j \in Inc_i} \beta_{ij} \partial_j u(v_i) = 0 & i \in I_T \\ u_j(v_i) = u_k(v_i) & j, k \in Inc_i. \end{cases}$$

The linearized equation is

$$\begin{cases} -\nu_j \partial^2 w + \partial_p H_j(x, \partial u) \partial w = 0 & x \in e_j,\ j \in J \\ \sum_{j \in Inc_i} \beta_{ij} \partial_j w(v_i) = 0 & i \in I_T \\ w_j(v_i) = w_k(v_i) & j, k \in Inc_i. \end{cases}$$

Writing the weak formulation for a test function m, we get

$$
\begin{aligned}
0 &= \sum_{j\in J}\int_{e_j}\big(-\nu_j\partial^2 w + \partial_p H_j(x,\partial u)\partial w\big)\, m\, dx \\
&= \sum_{j\in J}\int_{e_j}\big[-\nu_j\partial^2 m - \partial(m\partial_p H_j(x,\partial u))\big]\, w\, dx \\
&\quad + \sum_{i\in I_T}\sum_{j\in Inc_i}\big(\nu_j\partial_j m(v_i) + \partial_p H(v_i,\partial u)m_j(v_i)\big)\, w(v_i) \\
&\quad - \underbrace{\sum_{i\in I_T}\sum_{j\in Inc_i}\nu_j m_j(v_i)\partial_j w(v_i)}_{=0 \quad \text{if } \frac{m_j(v_i)\nu_j}{\beta_{ij}} = \frac{m_k(v_i)\nu_k}{\beta_{ik}}}.
\end{aligned}
$$

By the integral term, we obtain

$$
\nu_j\partial^2 m + \partial(m\,\partial_p H_j(x,\partial u)) = 0 \qquad x\in e_j,\ j\in J.
$$

The MFG system on a network

Assume $\frac{\nu_j}{\beta_{ij}} = \frac{\nu_k}{\beta_{ik}}$ $\forall j, k \in Inc_i$, $i \in I_T$. The MFG systems is

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(MFG_\Gamma) \quad \begin{cases}
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\int_\Gamma u(x)dx = 0 & \\
\int_\Gamma m(x)dx = 1, \qquad m \geq 0. &
\end{cases}
$$

The MFG systems on networks

Theorem (Camilli-M., SJCO '16)

We assume

- $H_j \in C^2(e_j \times \mathbb{R})$, convex, with $\delta|p|^2 - C \leq H_j(x,p) \leq \delta|p|^2 + C$,
- $\nu_j > 0$,
- $V \in C^1([0,+\infty))$.

Then, there exists a solution $(u,m,\rho) \in C^2(\Gamma) \times C^2(\Gamma) \times \mathbb{R}$ to (MFG_Γ).

Moreover, assume

- either V is strictly monotone in m
- or V is monotone in m and H is strictly convex in p.

Then the solution is unique.

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Sketch of the proof

Step 1: On the HJB equation.

For $f \in C^{0,\alpha}(\Gamma)$, $\exists!(u,\rho) \in C^2(\Gamma) \times \mathbb{R}$ solution to

$$\text{(HJB)} \quad \begin{cases} -\nu\partial^2 u + H(x,\partial u) + \rho = f(x), & x \in \Gamma \\ \sum_{j\in Inc_i} \nu_j \partial_j u(v_i) = 0 & i \in I_T \\ u_j(v_i) = u_k(v_i) & j,k \in Inc_i, \quad i \in I_T \\ \int_\Gamma u(x) = 0. & \end{cases}$$

Moreover $u \in C^{2,\alpha}(\Gamma)$ and: $\|u\|_{C^{2,\alpha}(\Gamma)} \le C$, $|\rho| \le \max_\Gamma |H(\cdot,0) - f(\cdot)|$.

The proof is based on

- $\exists u_\lambda \in W^{1,2}(\Gamma)$, weak solution to the *discounted approximation*

$$-\nu\partial^2 u_\lambda + H(x,\partial u_\lambda) + \lambda u_\lambda = f(x) \qquad x \in \Gamma$$

  as in [Boccardo-Murat-Puel,'83]; the Comparison Principle applies;
- $u_\lambda \in C^{2,\alpha}(\Gamma)$ by the 1-d of the problem and Sobolev theorem;
- as $\lambda \to 0^+$, $\lambda u_\lambda \to \rho$ and $(u_\lambda - \min u_\lambda) \to u$.

Step 2: On the FP equation.

For $b \in C^1(\Gamma)$, there exists a unique weak solution $m \in W^{1,2}(\Gamma)$ to

$$(FP) \quad \begin{cases} \nu\partial^2 m + \partial(b(x)\, m) = 0 & x \in \Gamma \\ \sum_{j \in Inc_i}[b(v_i)m_j(v_i) + \nu_j \partial_j m(v_i)] = 0 & i \in I_T \\ m_j(v_i) = m_k(v_i) & j,k \in Inc_i, \quad i \in I_T \\ m \geq 0, \qquad \int_\Gamma m(x)dx = 1. & \end{cases}$$

Moreover, m is a classical solution with $\|m\|_{H^1} \leq C$, $0 < m(x) \leq C$ (for some $C > 0$ depending only on $\|b\|_\infty$ and ν).

The proof is based on

- the existence of a weak solution is based on the theory of bilinear forms;
- the adjoint problem (both equation and transition condition) fulfills the Maximum Principle;
- $m \in C^2(\Gamma)$ by the 1-d of the problem and Sobolev theorem.

Step 3: Fixed point argument.

We set $\mathcal{K} := \{\mu \in C^{0,\alpha}(\Gamma) : \mu \geq 0, \int_\Gamma \mu dx = 1\}$ and we define an operator $T : \mathcal{K} \to \mathcal{K}$ according to the scheme

$$\mu \to u \to m$$

as follows:

- given $\mu \in \mathcal{K}$, solve (HJB) with $f(x) = V(\mu(x))$ for the unknowns $u = u_\mu$ and ρ, which are uniquely defined by Step 1;
- given u_μ, solve (FP) with $b(x) = \partial_p H(x, \partial u_\mu)$ for the unknown m which is uniquely defined by Step 2;
- set $T(\mu) := m$.

Since T is continuous with compact image, Schauder's fixed point theorem ensures the existence of a solution.

Step 4: Uniqueness.

Cross-testing the equations in (MFG_Γ), by the transition conditions, we get

$$\sum_{j\in J}\int_{e_j}\underbrace{(m_1-m_2)(V(m_1)-V(m_2))dx}_{\geq 0 \text{ by monotonicity}}+$$

$$\sum_{j\in J}\int_{e_j} m_1\underbrace{\left[H_j(x,\partial_j u_2)-H_j(x,\partial_j u_1)-\partial_p H_j(x,\partial_j u_1)\partial_j(u_2-u_1)\right]}_{\geq 0 \text{ by convexity}}dx+$$

$$\sum_{j\in J}\int_{e_j} m_2\underbrace{\left[(H_j(x,\partial_j u_1)-H_j(x,\partial_j u_2)-\partial_p H_j(x,\partial_j u_2)\partial_j(u_1-u_2)\right]}_{\geq 0 \text{ by convexity}}dx=0.$$

Therefore, each one of these three lines must vanish and we conclude as in [Lasry-Lions '06].

A finite difference scheme for MFG on network

- We introduce a grid on Γ. For the parametrization $\pi_j : [0, l_j] \to \mathbb{R}^n$ of e_j, let $y_{j,k} = kh_j$ ($k = 0, \ldots, N_j^h$) be an uniform partition of $[0, l_j]$:

$$\mathcal{G}_h = \{x_{j,k} = \pi_j(y_{j,k}),\ j \in J,\ k = 0, \ldots, N_j^h\}.$$

- $Inc_i^+ := \{j \in Inc_i : v_i = \pi_j(0)\}, \quad Inc_i^- := \{j \in Inc_i : v_i = \pi_j(N_j^h h_j)\}.$
- We introduce the (1-dimensional) finite difference operators

$$(D^+U)_{j,k} = \frac{U_{j,k+1} - U_{j,k}}{h_j}, \qquad [D_hU]_{j,k} = \left((D^+U)_{j,k}, (D^+U)_{j,k-1}\right)^T,$$

$$(D_h^2U)_{j,k} = \frac{U_{j,k+1} + U_{j,k-1} - 2U_{j,k}}{h_j^2}.$$

- We introduce the inner product. For $U, W : \mathcal{G}_h \to \mathbb{R}$, set

$$(U, W)_2 = \sum_{j\in J} \sum_{k=1}^{N_j^h-1} h_j U_{j,k} W_{j,k} + \sum_{i\in I} \Big(\sum_{j\in Inc_i^+} \frac{h_j}{2} U_{j,0} W_{j,0} + \sum_{j\in Inc_i^-} \frac{h_j}{2} U_{j,N_j^h} W_{j,N_j^h} \Big).$$

- We introduce the numerical Hamiltonian $g_j : [0, l_j] \times \mathbb{R}^2 \to \mathbb{R}$, s.t.:

  ($\mathbf{G_1}$) monotonicity: $g_j(x, q_1, q_2)$ is nonincreasing with respect to q_1 and nondecreasing with respect to q_2.

  ($\mathbf{G_2}$) consistency: $g_j\,(x, q, q) = H_j(x, q), \quad \forall x \in [0, l_j], \forall q \in \mathbb{R}$.

  ($\mathbf{G_3}$) differentiability: g_j is of class C^1.

  ($\mathbf{G_4}$) superlinear growth : $g_j(x, q_1, q_2) \geq \alpha((q_1^-)^2 + (q_2^+)^2)^{\gamma/2} - C$ for some $\alpha > 0$, $C \in \mathbb{R}$ and $\gamma > 1$.

  ($\mathbf{G_5}$) convexity : $(q_1, q_2) \to g_j\,(x, q_1, q_2)$ is convex.

- We introduce a continuous numerical potential V_h such that $\exists C$ independent of h such that

$$\max_{j,k} |(V_h[M])_{j,k}| \leq C, \qquad |(V_h[M])_{j,k} - (V_h[M])_{j,\ell}| \leq C|y_{j,k} - y_{j,\ell}|.$$

  for all $M \in \mathcal{K}_h := \{M : M \text{ is continuous},\ M_{j,k} \geq 0,\ (M, 1)_2 = 1\}$.

We get the following system in the unknown (U, M, R)

$$
\begin{cases}
-\nu_j(D_h^2 U)_{j,k} + g(x_{j,k}, [D_h U]_{j,k}) + R = V_h(M_{j,k}) \\
\nu_j(D_h^2 M)_{j,k} + \mathcal{B}^h(U, M)_{j,k} = 0, \\
\sum\limits_{j \in Inc_i^+} \left[\nu_j(D^+U)_{j,0} + \frac{h_j}{2}(V_{j,0} - R)\right] - \sum\limits_{j \in Inc_i^-} \left[\nu_j(D^+U)_{j,N_j^h - 1} - \frac{h_j}{2}(V_{j,N_j^h} - R)\right] = 0 \\
\sum\limits_{j \in Inc_i^+} \left[\nu_j(D^+M)_{j,0} + M_{j,1}\frac{\partial g}{\partial q_2}(x_{j,1}, [D_h U]_{j,1})\right] - \\
\qquad \sum\limits_{j \in Inc_i^-} \left[\nu_j(D^+M)_{j,N_j^h - 1} + M_{j,N_j^h - 1}\frac{\partial g}{\partial q_1}(x_{j,N_j^h - 1}, [D_h U]_{j,N_j^h - 1})\right] = 0 \\
U,\ M \text{ continuous at } v_i, \quad i \in I, \\
(M, 1)_2 = 1, \quad (U, 1)_2 = 0,
\end{cases}
$$

Theorem (Cacace-Camilli-M., M2AN'17)

- For any $h = \{h_j\}_{j\in J}$, the discrete problem has at least a solution (U_h, M_h, ρ_h). Moreover

$$|\rho_h| \le C_1, \qquad \|U_h\|_\infty + \|D_h U_h\|_\infty \le C_2$$

  for some constants C_1, C_2 independent of h.
- Moreover, if V_h is strictly monotone, then the solution is unique.
- If (u, m, ρ) is the solution of the MFG system (MFG_Γ), then

$$\lim_{|h|\to 0} \Big[\|U^h - u\|_\infty + \|M^h - m\|_\infty + |\rho_h - \rho| \Big] = 0.$$

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The solution of the MFG discrete system is usually performed by means of some regularizations, e.g. large time approximation or ergodic approximation. We propose a different method:

- We collect the unknowns in a vector $X = (U, M, R)$ of length $2N^h + 1$;
- we consider the nonlinear map $\mathcal{F} : \mathbb{R}^{2N^h+1} \to \mathbb{R}^{2N^h+2}$ defined by

$$\mathcal{F}(X) = \begin{cases} -\nu_j(D_h^2 U)_{j,k} + g(x_{j,k}, [D_h U]_{j,k}) + R - V_h(M_{j,k}), \\ \nu_j(D_h^2 M)_{j,k} + \mathcal{B}^h(U,M)_{j,k}, \\ \sum_{j\in Inc_i^+} \left[\nu_j(D^+U)_{j,0} + \frac{h_j}{2}(V_{j,0} - R)\right] - \sum_{j\in Inc_i^-} \left[\nu_j(D^+U)_{j,N_j^h-1} - \frac{h_j}{2}(V_{j,N_j^h} - R)\right] \\ \sum_{j\in Inc_i^+} \left[\nu_j(D^+M)_{j,0} + M_{j,1}\frac{\partial g}{\partial q_2}(x_{j,1}, [D_h U]_{j,1})\right] \\ -\sum_{j\in Inc_i^-} \left[\nu_j(D^+M)_{j,N_j^h-1} + M_{j,N_j^h-1}\frac{\partial g}{\partial q_1}(x_{j,N_j^h-1}, [D_h U]_{j,N_j^h-1})\right] \\ (M,1)_2 - 1 \\ (U,1)_2. \end{cases}$$

- The solution of the discrete MFG is the unique $X^\star$ s.t. $\mathcal{F}(X^\star) = 0$.

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- The solution of the discrete MFG is the unique $X^\star$ s.t. $\mathcal{F}(X^\star) = 0$.

The system $\mathcal{F}(X^\star) = 0$ is formally overdetermined ($2N^h + 2$ equations in $2N^h + 1$ unknowns), hence the solution is meant in the following nonlinear-least-squares sense:

$$X^\star = \arg\min_X \frac{1}{2} \|\mathcal{F}(X)\|_2^2 \,.$$

The previous optimization problem is solved by means of the Gauss-Newton method

$$J_{\mathcal{F}}(X^k)\delta_X = -\mathcal{F}(X^k), \qquad X^{k+1} = X^k + \delta_X \,.$$

via the QR factorization of the Jacobian $J_{\mathcal{F}}(X^k)$.

Numerical experiments

We consider a network with 2 vertices and 3 edges (boundary vertices are identified!). Each edge has unit length and connects $(0,0)$ to $(cos(2\pi j/3), sin(2\pi j/3))$ with $j = 0, 1, 2$.

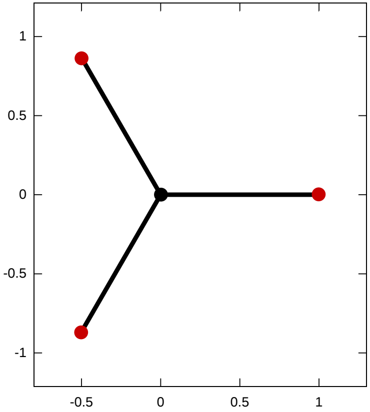
Data:

- Uniform diffusion $\nu_j \equiv \nu$
- $H_j(x,p) = \frac{1}{2}|p|^2 + f(x)$

$f(x) = s_j\Big(1 + \cos(2\pi(x + \frac{1}{2}))\Big)$

$s_j \in \{0, 1\}$

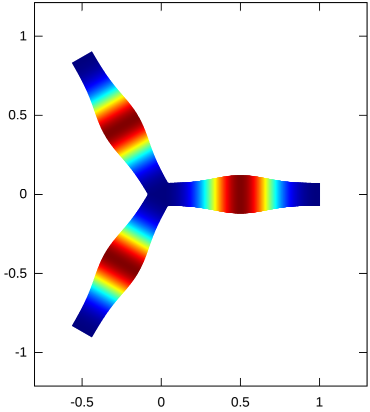
- $V[m] = m^2$



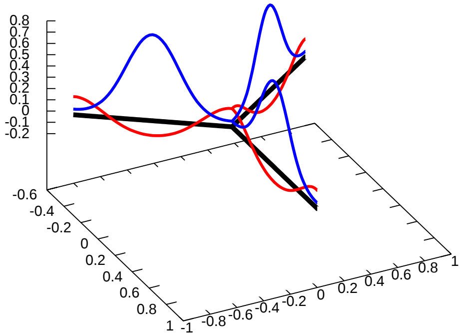
Computational time for $N^h \sim 5000$ is of the order of seconds!

Cost active on all edges

$\nu = 0.1, \qquad s_0 = 1, s_1 = 1, s_2 = 1, \qquad V[m] = m^2$



M

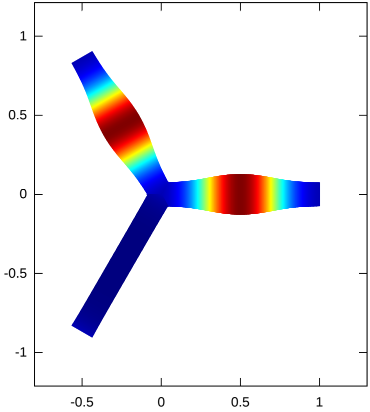


M (blu) and U (red)

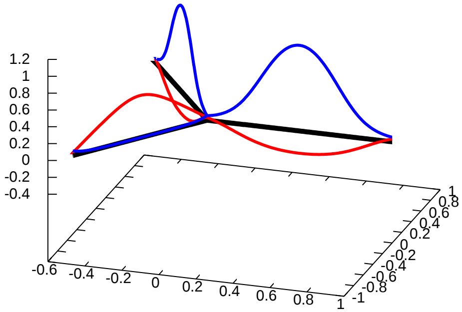
Computed $\rho = -1.066667$

Cost active on two edges

$\nu = 0.1, \quad s_0 = 1, s_1 = 1, s_2 = 0, \quad V[m] = m^2$



M

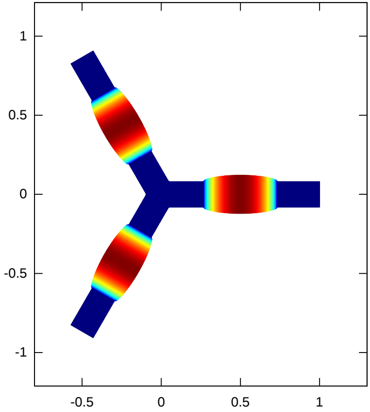


M (blu) and U (red)

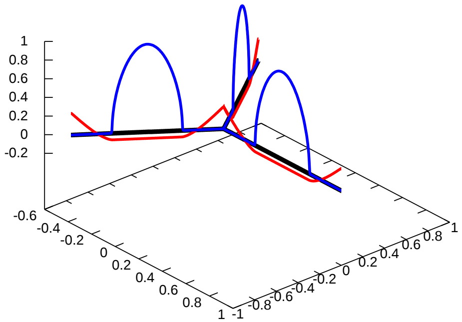
Computed $\rho = -0.741639$

Small viscosity, cost active on all edges

$\nu = 10^{-4}$, $\quad s_0 = 1,\ s_1 = 1,\ s_2 = 1, \quad V[m] = m^2$



M

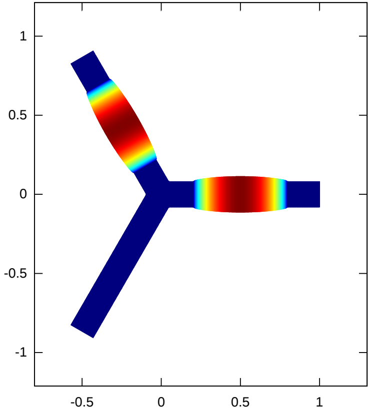


M (blu) and U (red)

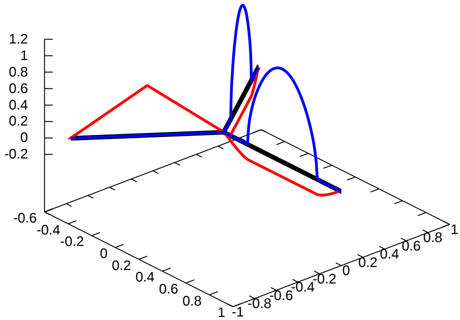
Computed $\rho = -1.116603$

Small viscosity, cost active on two edges

$\nu = 10^{-4}$, $\quad s_0 = 1,\ s_1 = 1,\ s_2 = 0,\quad V[m] = m^2$



M

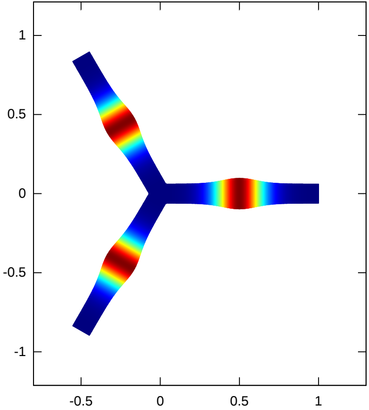


M (blu) and U (red)

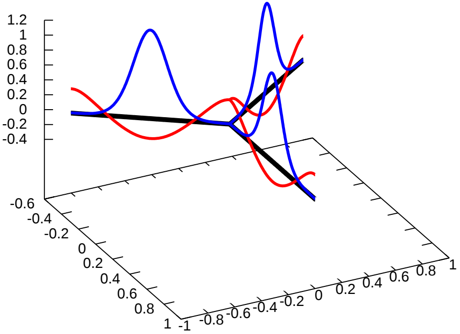
Computed $\rho = -0.725463$

Numerical experiments

$\nu = 0.1, \qquad s_0 = 1,\ s_1 = 1,\ s_2 = 1, \qquad V[m] = 1 - \frac{4}{\pi} arctan(m)$



M

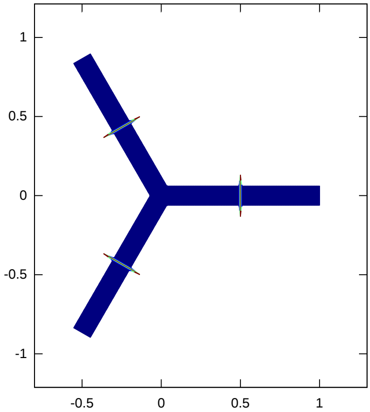


M and U

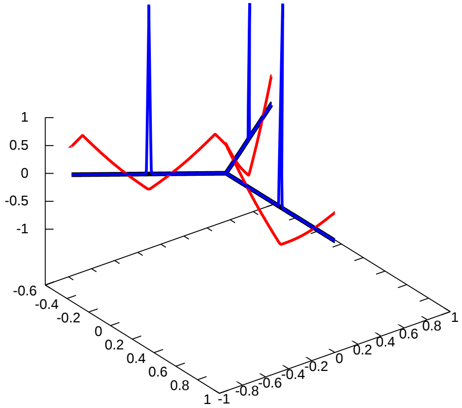
Computed $\Lambda = -1.219979$

Numerical experiments

$\nu = 10^{-3}$, $\quad s_0 = 1,\ s_1 = 1,\ s_2 = 1,\quad V[m] = 1 - \frac{4}{\pi} arctan(m)$



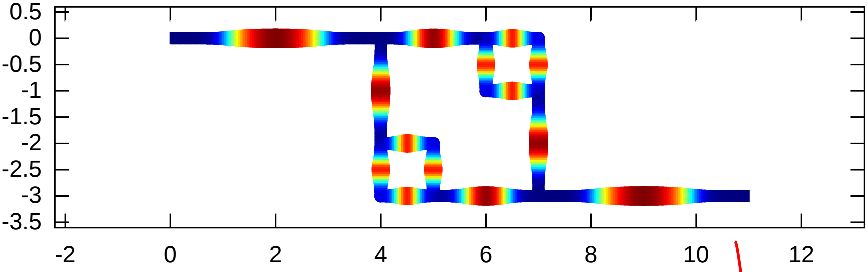
M

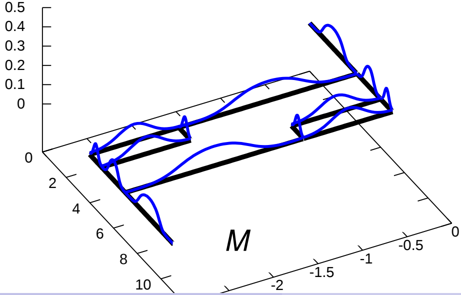


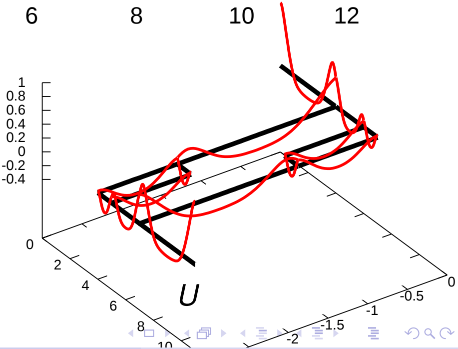
M and *U*

Computed $\Lambda = -2.832411$

More complex networks







Comments and Remarks:

- rigorous derivation of the system starting from the game with N players.
- more general transitions conditions (arbitrary weights for the edges, controlled weights...) and/or lack of continuity condition.
- first order MFG systems on networks.

Thank You!