On the variational formulation of some stationary second order mean field games systems

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Mean Field Games and related topics - 4,
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Outline

Mean Field Game PDE system

Model introduced by J.-M. Lasry and P.-L. Lions (2006)

$$\begin{cases} -\partial_t v(x,t) - \sigma^2 \Delta v(x,t) + H(x, \nabla v(x,t)) = f(x, m(t)), & \mathbb{R}^d \times (0,T), \\ \partial_t m(x,t) - \sigma^2 \Delta m(x,t) - \operatorname{div}\big(\partial_p H(x, \nabla v(x,t)) m(x,t)\big) = 0, & \mathbb{R}^d \times (0,T), \\ v(x,T) = g(x, m(T)) \quad \text{for } x \in \mathbb{R}^d, \quad m(0) = m_0 \in \mathcal{P}_1. \end{cases}$$

- $H(x,\cdot)$ is convex.
- In the first line we have a Hamilton-Jacobi-Bellman (HJB) equation backward in time.
- In the second line we have a Fokker-Planck equation forward in time.
- In this talk we will focus on coupling terms f which are local, i.e. "$f(x, m(t)) = f(x, m(x,t))$".

The stationary version is given by

$$\begin{cases} -\sigma^2\Delta v(x) + H(x,\nabla v(x)) + \lambda = f(x,m), & \mathbb{R}^d \times (0,T), \\ -\sigma^2\Delta m(x) - \mathrm{div}\big(\partial_p H(x,\nabla v(x))m(x)\big) = 0, & \mathbb{R}^d \times (0,T), \\ m \geq 0, \quad \int_{\mathbb{R}^d} u\mathrm{d}x = 0, \quad \int_{\mathbb{R}^d} m\mathrm{d}x = 1. \end{cases}$$

- The previous system corresponds to the long time average[1] of the time-evolving system.
- In some cases, the time-evolving and the stationary problems correspond to the optimality condition of some associated variational problems.
- Under density constraints, existence of solutions of a variation of the previous system is shown in A. Mészáros and S. '15.

[1] When $H(x,p) = \frac{1}{2}|p|^2$ a rigorous proof is provided in P. Cardaliaguet, J.-M. Lasry, P.-L. Lions, A. Porretta (2013).

Some references

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- P. Cardaliaguet, A. R. Mészáros and F. Santambrogio, "First order Mean Field Games with density constraints: Pressure equals Price", *SICON*, 2016.
- J.-D. Benamou, G. Carlier and F. Santambrogio "Variational Mean Field Games", in *Active Particles, Volume 1: Theory, Models, Applications*, 2016.

The variational problems

- Let $q > 1$ and $q' := q/(q-1)$.
- Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with a smooth boundary.
- We suppose that the Hamiltonian H satisfies
  - $H : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is continuous.
  - $H(x, \cdot)$ is strictly convex and differentiable (and so the same is valid for $H^*(x, \cdot)$)
  - There exist C_1, $C_2 > 0$ such that
  $$\frac{1}{q' C_1}|\xi|^{q'} - C_2 \leq H(x,\xi) \leq \frac{C_1}{q'}|\xi|^{q'} + C_2, \forall\, x \in \Omega, \quad \xi \in \mathbb{R}^d.$$
  This implies that H^* satisfies
  $$\frac{C_1^{1-q}}{q}|\eta|^q - C_2 \leq H^*(x,\eta) \leq \frac{C_1^{q-1}}{q}|\eta|^q + C_2, \quad \forall\, x \in \Omega, \quad \eta \in \mathbb{R}^d.$$
  - There exists a modulus of continuity ω such that
  $$|H(x,\xi) - H(y,\xi)| \leq \omega(|x-y|)(|\xi|^{q'} + 1), \quad \forall\, x, y \in \Omega, \quad \xi \in \mathbb{R}^d.$$

- Define $b_q : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ as

$$b_q(x, m, w) := \begin{cases} mH^*(x, -w/m) & \text{if } m > 0, \\ 0 & \text{if } (m, w) = (0, 0), \\ +\infty & \text{otherwise.} \end{cases}$$

  which is a convex, proper, l.s.c. function (of "perspective type").

- Define $\mathcal{B}_q : W^{1,q}(\Omega) \times L^q(\Omega)^d \to \mathbb{R} \cup \{+\infty\}$ as

$$\mathcal{B}_q(m, w) := \int_\Omega b_q(x, m(x), w(x)) \mathrm{d}x.$$

- Finally, let $\mathcal{F} : W^{1,q}(\Omega) \to \mathbb{R}$.

We consider the following variational problems

$$
\inf \quad \mathcal{B}_q(m,w) + \mathcal{F}(m),
$$

$$
\text{subject to} \quad \begin{array}{rcll} -\Delta m + \mathrm{div}(w) & = & 0 & \text{in } \Omega, \\ (\nabla m + w)\cdot \hat{n} & = & 0 & \text{on } \partial\Omega, \\ \int_\Omega m \mathrm{d}x = 1, & & & \end{array} \qquad (P_1)
$$

and

$$
\inf \quad \mathcal{B}_q(m,w) + \mathcal{F}(m),
$$

$$
\text{subject to} \quad \begin{array}{rcll} -\Delta m + \mathrm{div}(w) & = & 0 & \text{in } \Omega, \\ (\nabla m + w)\cdot \hat{n} & = & 0 & \text{on } \partial\Omega, \\ \int_\Omega m \mathrm{d}x = 1, & 0 \leq m \leq \kappa, & & \end{array} \qquad (P_2)
$$

where $\kappa \in W^{1,q}(\Omega)$ is such that

$$
\underline{\kappa} := \min_{x\in\overline{\Omega}} \kappa(x) > 0 \quad \text{and} \quad \int_\Omega \kappa(x)\mathrm{d}x > 1.
$$

Our main assumptions are the following

(I) $q > d \geq 2$.

(II) $\mathcal{F}$ is weakly lower semicontinuous, Gâteaux differentiable in $W_+^{1,q}$ and

- bounded from below in $W_+^{1,q}(\Omega)$ if problem (P_1) is considered.
- For all $R > 0$ there exists $C_R > 0$ such that that $\mathcal{F}(m) \geq C_R$ if $0 \leq m \leq R$ in Ω, if problem (P_2) is considered.

- Assumption (I) is restrictive on the growth of H, but it is crucial in our analysis because of the embedding $W^{1,q}(\Omega) \hookrightarrow C(\overline{\Omega})$.
- On the other hand, assumption (II) is rather general since no convexity is assumed, and, moreover, dependence on ∇m is allowed.

Existence of solutions

We have the following result:

Theorem

Under the previous assumptions, problems (P_1) and (P_2) admit at least one solution.

Sketch of the proof:

(1) The existence for (P_2) follows easily by standard arguments. The key is that if (m_n, w_n) is a minimizing sequence, the inequality $m \leq \kappa$ and the growth of H^* provide uniform bounds on $\|w_n\|_q$.

(2) In order to prove the existence for problem (P_1), let $\gamma > 1/|\Omega|$ be arbitrary and let (m_γ, w_γ) be a solution of (P_2) with $\kappa \equiv \gamma$.

- Using the PDE we get

$$\|m_\gamma\|_\infty \leq c_0 \|m_\gamma\|_{1,q} \leq c_0 c_1 (1 + \|w_\gamma\|_q) \leq 2c_0 c_1 \max\{1, \|w_\gamma\|_q\}.$$

- Assuming, w.l.o.g., that $\|w_\gamma\|_q \geq 1$, we get that $m \leq 2c_0c_1\|w_\gamma\|_q$ a.e. in Ω.
- Using this fact and that

$$\mathcal{B}_q(m_\gamma, w_\gamma) + \mathcal{F}(m_\gamma) \leq \mathcal{B}_q(1/|\Omega|, 0) + \mathcal{F}(1/|\Omega|),$$

(since $(1/|\Omega|, 0)$ is feasible for (P_2)), the growth condition for H^* implies that

$$\|w_\gamma\|_q \leq qC_1^{q-1}\left(\mathcal{F}(1/|\Omega|) + 2C_2 - C_{\mathcal{F}}\right)(2c_0c_1)^{q-1},$$

where $C_{\mathcal{F}} = \inf_{m \in W_+^{1,q}(\Omega)} \mathcal{F}(m)$. Thus,

$$\|m_\gamma\|_\infty \leq (2c_0c_1)^q qC_1^{q-1}\left(\mathcal{F}(1/|\Omega|) + 2C_2 - C_{\mathcal{F}}\right).$$

The result follows.

Optimality conditions and MFG systems

Now, having the existence of solutions, we want to establish the optimality conditions to obtain the desired MFG system.

- We need to compute $\partial\mathcal{B}_q$. In order to get an idea of the result, for $x \in \Omega$ define

$$A_{q'}(x) := \{(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}^d \; : \; \alpha + H(x, -\beta) \leq 0\}.$$

It easy to check that

$$b_q^*(x, \cdot, \cdot) = \chi_{A_{q'}(x)}(\cdot, \cdot),$$

and

$$\partial_{(m,w)} b_q(x, m, w) = \begin{cases} (-H(x, -\beta_x), \beta_x) & \text{if } \; m > 0, \\ (\alpha, \beta) \in A_{q'}(x) & \text{if } \; (m, w) = (0, 0), \\ \emptyset & \text{otherwise}, \end{cases}$$

where, if $m > 0$, $\beta_x := -\nabla H^*(x, -w/m)$.

- Define

$$\overline{\mathcal{A}_{q'}} := \left\{ (\alpha, \beta) \in \mathcal{M}(\overline{\Omega}) \times L^{q'}(\Omega)^d \ : \ \alpha + H(\cdot, -\beta) \in \mathcal{M}_-(\overline{\Omega}) \right\},$$

or equivalently,

$$\overline{\mathcal{A}_{q'}} := \Big\{ (\alpha, \beta) \in \mathcal{M}(\overline{\Omega}) \times L^{q'}(\Omega)^d \ : \ \alpha^{\mathrm{ac}} + H(\cdot, -\beta) \leq 0, \text{a.e. in } \Omega \text{ and } \ \alpha^{\mathrm{s}} \in \mathcal{M}_-(\overline{\Omega}) \Big\}.$$

Theorem

(i) $\mathcal{B}_r^*(\alpha, \beta) = \chi_{\overline{\mathcal{A}_{r'}}}(\alpha, \beta)$ *for all* $(\alpha, \beta) \in (W^{1,q}(\Omega))^* \times L^{q'}(\Omega)^d$.

(ii) *Suppose that* $\mathcal{B}_q(m, w) < \infty$. *Then, if* $v := (w/m)\mathbb{I}_{\{m>0\}} \notin L^q(\Omega)^d$ *we have that* $\partial \mathcal{B}_q(m, w) = \emptyset$. *Otherwise,* $\partial \mathcal{B}_q(m, w)$ *exists and* [2]

$$\partial \mathcal{B}_q(m, w) = \Big\{ (\alpha, \beta) \in \overline{\mathcal{A}_{q'}} \ ; \quad \alpha \llcorner \{m > 0\} = -H(\cdot, \nabla H^*(\cdot, -v)) \\ \text{and} \quad \beta \llcorner \{m > 0\} = -\nabla H^*(\cdot, -v) \Big\}.$$

[2] See H. Brézis, *Intégrales convexes dans les espaces de Sobolev*, Israel J. Math. '73

As a consequence we obtain

Theorem

There exists $(m, u, \lambda) \in W^{1,q}(\Omega) \times W^{1,q'}(\Omega) \times \mathbb{R}$ *such that*

$$\begin{cases} -\Delta u + H(\cdot, \nabla u) + \lambda = D\mathcal{F}(m), & \text{in } \Omega \\ -\Delta m - \text{div}\,(m \nabla_\xi H(\cdot, \nabla u)) = 0, & \text{in } \Omega, \\ (\nabla m + m \nabla_\xi H(\cdot, \nabla u)) \cdot n = 0, & \text{on } \partial\Omega, \\ \int_\Omega u dx = 0, \quad \int_\Omega m dx = 1, \quad m(x) > 0 & \text{in } \overline{\Omega}, \end{cases}$$

where both PDE are interpreted in a weak sense.

Sketch of the proof:

(1) Define $\hat{\mathcal{B}}_q(m, w) := \mathcal{B}_q(m, w) + \chi_{G^{-1}(0)}(m, w)$, where

$$G(m, w) = (-\Delta m + \text{div}(w), \int_\Omega m dx - 1)$$

(2) If (m, w) is a solution of (P_1), it is possible to prove that

$$(-D\mathcal{F}(m), 0) \in \partial\hat{\mathcal{B}}_q(m, w)$$

(3) It is also possible to find a point $(\hat{m}, \hat{w})$ such that $G(\hat{m}, \hat{w}) = 0$ with $\hat{m} > 0$. Therefore, since $q > d$, $\mathcal{B}_q$ is continuous at $(\hat{m}, \hat{w})$ and so

$$(-D\mathcal{F}(m), 0) \in \partial\hat{\mathcal{B}}_q(m, w) = \partial\mathcal{B}_q(m, w) + \partial\chi_{G^{-1}(0)}(m, w).$$

(4) In particular, $\partial\mathcal{B}_q(m, w) \neq \emptyset$ and so

$$v := (w/m)\mathbb{I}_{\{m>0\}} \in L^q(\Omega)^d.$$

(5) Since m solves

$$-\Delta m + \text{div}(vm) = 0$$

and $v \in L^q(\Omega)^d$, with $q > d$, by the Harnack inequality proved in Trudinger '73, we have that $m > 0$ in Ω. Since $\partial\Omega$ is regular, classical reflection arguments show that $m > 0$ in $\overline{\Omega}$.

(6) The result easily follows from the characterization of $\partial\mathcal{B}_q(m, w)$.

- Similarly, for problem (P_2) we obtain

Theorem

There exists $(m,u,p,\lambda) \in W^{1,q}(\Omega) \times W^{1,q'}(\Omega) \times \mathcal{M}(\overline{\Omega}) \times \mathbb{R}$ *such that*

$$\begin{cases} -\Delta u + H(\cdot, \nabla u) - p + \lambda = D\mathcal{F}(m), & \text{in } \Omega, \\ -\Delta m - \operatorname{div}\left(m \nabla_\xi H(\cdot, \nabla u)\right) = 0, & \text{in } \Omega, \\ \left(\nabla m + m \nabla_\xi H(\cdot, \nabla u)\right) \cdot n = 0, & \text{on } \partial\Omega, \\ \int_\Omega u \mathrm{d}x = 0, \quad \int_\Omega m \mathrm{d}x = 1, \quad 0 < m(x) \leq \kappa(x) & \text{in } \overline{\Omega}, \\ \operatorname{spt}(p) \subseteq \{m = \kappa\}, \quad p \geq 0 & \text{in } \overline{\Omega} \end{cases}$$

where both PDE are interpreted in a weak sense.

- p corresponds to a Lagrange multiplier associated to $m \leq \kappa$.
- This result improves the one in Mészáros-S. '15 (when $q > d$).

Some choices for $\mathcal{F}$

- We consider

$$\mathcal{F}(m) := \int_{\Omega} F\left(x, m(x), \nabla m(x)\right) \mathrm{d}x,$$

where $F : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is a Carathéodory function such that

(i) For a.e. $x \in \Omega$ and all $z \in \mathbb{R}$ the function $F(x, z, \cdot)$ is convex.

(ii) There exists $\gamma \in L^1(\Omega)$ such that for a.e. $x \in \Omega$

$$F(x, z, \xi) \geq \gamma(x) \quad \forall\, z \geq 0, \;\; \xi \in \mathbb{R}^d.$$

(iii) For all $R > 0$ there exists $a_1 \in L^1(\Omega)$, $a_2 \in L^{q'}(\Omega)$ and $b = b(R) \geq 0$ such that for a.e. $x \in \Omega$, $0 \leq z \leq R$ and $\xi \in \mathbb{R}^d$

$$|\partial_z F(x, z, \xi)| \leq a_1(x) + b|\xi|^q,$$
$$|\nabla_\xi F(x, z, \xi)| \leq a_2(x) + b|\xi|^{q-1}.$$

- In the standard case, when F is independent of ∇m, and we denote by $f(x,m) = \partial_m F(x,m)$, we get that $f \in L^1(\Omega)$. Therefore, if we consider (P_1) by the results by Stampacchia '65, we get that $u \in W^{1,s}(\Omega)$ for all $s \in (1, d/(d-1))$.

- If in addition, $x \in \Omega \to f(x, m(x)) \in L^r$ for some $r > d$ and $\nabla_\xi H(x, \cdot)$ is Hölder continuous, uniformly on $x \in \Omega$, we can prove that for some α_0, $\alpha_1 \in (0,1)$

$$u \in C^{1,\alpha_0}_{\text{loc}}(\Omega) \quad \text{and} \quad m \in C^{1,\alpha_1}_{\text{loc}}(\Omega).$$

- Of course, functions depending non-locally on m can also be considered.

- A very simple example of $\mathcal{F}$ depending only on m is

$$\mathcal{F}(m) = \frac{1}{\alpha+1}\int_\Omega m(x)^{\alpha+1}\mathrm{d}x$$

which gives the existence (for α arbitrary) of solutions of

$$\begin{cases} -\Delta u + H(\cdot,\nabla u) + \lambda & = m^\alpha & \text{in } \Omega, \\ \nabla u \cdot n & = 0 & \text{on } \partial\Omega, \\ -\Delta m - \operatorname{div}\left(m\nabla_\xi H(\cdot,\nabla u)\right) & = 0, & \text{in } \Omega, \\ (\nabla m + m\nabla_\xi H(\cdot,\nabla u))\cdot n & = 0, & \text{on } \partial\Omega, \\ \displaystyle\int_\Omega m\mathrm{d}x = 1, & m(x) > 0 & \text{in } \overline{\Omega}. \end{cases}$$

- In the focusing case, for $\alpha > 0$ we get the existence of solutions of

$$\left\{\begin{array}{rll} -\Delta u + H(\cdot, \nabla u) - p + \lambda & = -m^{\alpha} & \text{in } \Omega, \\ \nabla u \cdot n & = 0 & \text{on } \partial\Omega, \\ -\Delta m - \operatorname{div}\left(m \nabla_{\xi} H(\cdot, \nabla u)\right) & = 0, & \text{in } \Omega, \\ \left(\nabla m + m \nabla_{\xi} H(\cdot, \nabla u)\right) \cdot n & = 0, & \text{on } \partial\Omega, \\ \int_{\Omega} m \mathrm{d}x = 1, & 0 < m \leq \kappa & \text{in } \overline{\Omega}, \\ \operatorname{spt}(p) \subseteq \{m = \kappa\}, & p \geq 0 & \text{in } \overline{\Omega} \end{array}\right.$$

- A very simple example of $\mathcal{F}$ depending only on ∇m is

$$\mathcal{F}(m) = \frac{1}{2}\int_{\Omega} |\nabla m(x)|^2 \mathrm{d}x$$

which gives the existence of solutions of

$$\begin{cases} -\Delta u + H(\cdot, \nabla u) + \lambda & = -\Delta m, \quad \text{in } \Omega, \\ \nabla(u - m)\cdot n & = 0, \quad \text{on } \partial\Omega, \\ -\Delta m - \mathrm{div}\,(m \nabla_\xi H(\cdot, \nabla u)) & = 0, \quad \text{in } \Omega, \\ (\nabla m + m\nabla_\xi H(\cdot, \nabla u))\cdot n & = 0, \quad \text{on } \partial\Omega, \\ \int_{\Omega} m \mathrm{d}x = 1, & m(x) > 0 \quad \text{in } \overline{\Omega}. \end{cases}$$

A simple application to multipopulations MFGs

We consider here the system

$$
(MFG)_N \begin{cases} -\Delta u_i + H^i(\cdot, \nabla u_i) + \lambda_i & = f^i(x, (m_i)_{i=1}^N), \quad \text{in } \Omega, \\ \nabla u_i \cdot n & = 0, \quad \text{on } \partial\Omega, \\ -\Delta m_i - \operatorname{div}\left(m_i \nabla_\xi H^i(\cdot, \nabla u_i)\right) & = 0, \quad \text{in } \Omega, \\ (\nabla m_i + m_i \nabla_\xi H^i(\cdot, \nabla u_i)) \cdot n & = 0, \quad \text{on } \partial\Omega, \\ \int_\Omega m_i \mathrm{d}x = 1, & m_i(x) > 0 \quad \text{on } \overline{\Omega}, \\ i = 1, ..., N, & \end{cases}
$$

We assume that

- The Hamiltonians $H^i : \Omega \times \mathbb{R}^d \to \mathbb{R}$ satisfy the growth conditions assumed for H.

- The couplings f^i satisfy

$$(I) \quad \exists\, \gamma_i \in L^1(\Omega) \text{ such that } \int_0^z f^i(x, z_i, (\zeta_j)_{j\neq i}) \mathrm{d}z_i \geq \gamma_i(x)$$

$$\text{for a.e. } x \in \Omega, \quad \forall\, z \geq 0, \quad \forall\, (\zeta_1, ..., \zeta_{i-i}, \zeta_{i+1}, ..., \zeta_N) \in [0, +\infty)^{N-1},$$

$$(II) \quad \forall\, R > 0, \quad \exists\, a_i \in L^1(\Omega) \text{ such that } \quad |f^i(x, z, (\zeta_j)_{j\neq i})| \leq a_i(x),$$

$$\text{for a.e. } x \in \Omega, \quad \forall\, 0 \leq z \leq R, \quad \forall\, (\zeta_1, ..., \zeta_{i-i}, \zeta_{i+1}, ..., \zeta_N) \in [0, +\infty)^{N-1},$$

and

$$(III) \quad \begin{array}{c} \forall\, (\zeta_1, ..., \zeta_{i-i}, \zeta_{i+1}, ..., \zeta_N) \in [0, +\infty)^{N-1} \\ \text{the map } z \in [0, +\infty) \to f^i(x, z, (\zeta_j)_{j\neq i}) \in \mathbb{R} \text{ is non-decreasing.} \end{array}$$

Proposition

Under the previous assumptions system $(MFG)_N$ admits at least one solution $m = (m_1, ..., m_N)$, $u = (u_1, ..., u_N)$ and $\lambda = (\lambda_1, ..., \lambda_N)$, where, for all $i = 1, ..., N$, $m_i \in W^{1,q}(\Omega)$ and $u_i \in W^{1,s}(\Omega)$ (for all $s \in (1, d/(d-1))$.

Alternatively, we can assume that

$$f^i(x, \zeta_i, (\zeta_j)_{j\neq i}) = \partial_{\zeta_i} F(x, \zeta) \quad \text{for a.e. } x \in \Omega, \ \forall\, \zeta = (\zeta_1, ..., \zeta_N) \in \mathbb{R}^N.$$

Proposition

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- The previous assumption is restrictive. On the other hand, it does not require the strong boundedness condition (II) and the monotonicity assumption (III).
- Moreover, this framework allows us to introduce density constraints of the form $m \in \mathcal{K}$, where

$$\mathcal{K} := \left\{ m \in W^{1,q}(\Omega)^N \ ; \ \sum_{i=1}^{N} \alpha_i m_i(x) \leq \kappa(x) \right\}.$$

Suppose that $\kappa \in W^{1,q}(\Omega)$, $\kappa > 0$, $\alpha_i \geq 0$, $\forall\, i = 1, \ldots, N$,

$\exists\, \bar{i} \in \{1, \ldots, N\}$ such that $\alpha_{\bar{i}} > 0$ and $\sum_{i=1}^N \alpha_i < \|\kappa\|_1$.

Proposition

Under the previous assumptions, system

$$
\begin{cases}
-\Delta u_i + H^i(\cdot, \nabla u_i) - \alpha_i p + \lambda_i = f^i(x, (m_i)_{i=1}^N) \quad \text{in } \Omega, \\
\nabla u_i \cdot n = 0 \quad \text{on } \partial\Omega, \\
-\Delta m_i - \text{div}\left(m_i \nabla_\xi H^i(\cdot, \nabla u_i)\right) = 0 \quad \text{in } \Omega, \\
(\nabla m_i + m_i \nabla_\xi H^i(\cdot, \nabla u_i)) \cdot n = 0 \quad \text{on } \partial\Omega, \\
\int_\Omega m_i \mathrm{d}x = 1, m_i(x) > 0 \quad \text{in } \overline{\Omega}, \\
\sum_{i=1}^N \alpha_i m_i(x) \leq \kappa(x) \quad \text{for all } x \in \overline{\Omega}, \\
p \geq 0 \quad \text{and} \quad \text{spt}(p) \subseteq \left\{ x \in \overline{\Omega} \quad \sum_{i=1}^N \alpha_i m_i(x) = \kappa(x) \right\}.
\end{cases}
$$

admits at least one solution m, u, λ and p, where $m_i \in W^{1,q}(\Omega)$, $u_i \in W^{1,s}(\Omega)$ (for all $s \in (1, d/(d-1))$) and $p \in \mathcal{M}(\overline{\Omega})$.

A word on the numerical resolution

- It is possible to construct discrete versions of (P_1) and (P_2) in such a way to obtain in the optimality system the finite difference scheme introduced by Achdou-Capuzzo-Dolcetta '10 in the case of (P_1), and a natural variation in the case of (P_2).
- If $\mathcal{F}$ is convex, then we can applied first order methods in order to solve numerically the problem. See e.g. the application of the augmented Lagrangian algorithm in
  - J.M. Benamou and G. Carlier '14
  - J.M. Benamou, G. Carlier and F. Santambrogio '16.
  - Y. Achdou and M. Laurière '16.
  - R. Andreev '16
- In Briceño, Kalise, S. '16, we study and compare different first order proximal methods for the resolution of stationary MFG systems, which can be of first or second order, with and without density constraints.

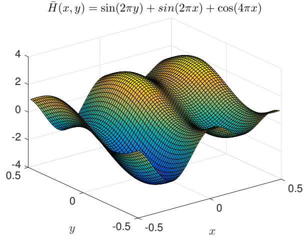
Example

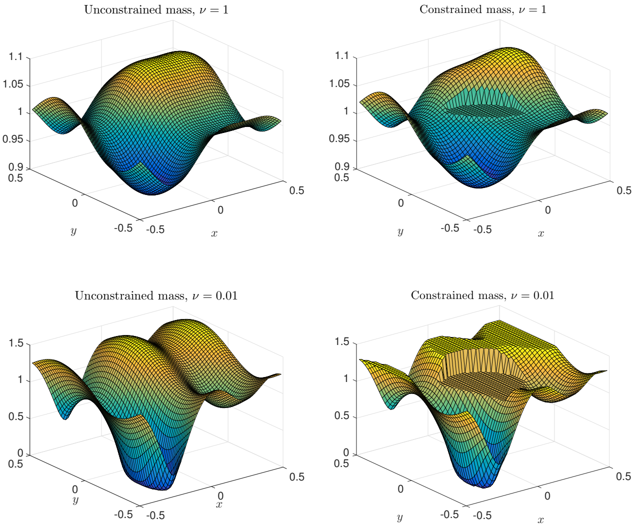
We consider here an example in Achdou-Capuzzo-Dolcetta 10' with an additional density constraint

$$q = 2, \quad f(x,y,m) = m^2 - \sin(2\pi y) - \sin(2\pi x) - \cos(4\pi x),$$
$$m(x,y) \leq \kappa(x,y) := \mathcal{I}_R(x,y) + (1 - \mathcal{I}_R(x,y))\bar{d}$$

où

$$\mathcal{I}_R(x,y) := \begin{cases} 1 & \text{si } x^2 + y^2 \leq R^2 \\ 0 & \text{sinon} \end{cases}, \quad \bar{d} = 1.3, \ \ R = 0.25\,.$$




Unconstrained mass, $\nu = 1$
Constrained mass, $\nu = 1$
Unconstrained mass, $\nu = 0.01$
Constrained mass, $\nu = 0.01$
1.1
1.05
1
0.95
0.9
1.5
0.5
0
-0.5
x
y

Comments and perspectives

- Theoretical study for the time-dependent case and planning problem.

- Numerical study for the time-dependent case. In an ongoing work with L. Briceño, D. Kalise and M. Laurière, we study the discrete problem with and without congestion (the latter corresponds to a Mean Field Type Control problem).

- For problem (P_1), can we get rid of the density constraint when $1 < q \leq d$?
- Numerical analysis when $\mathcal{F}$ is not convex?
- Numerical analysis for variational multipopulation MFGs.