ONE SIDED MINIMAX DIFFERENTIABILITY FOR THE COMPUTATION OF CONTROL, SHAPE, AND TOPOLOGICAL DERIVATIVES

Michel C. Delfour\textsuperscript{1} and Kevin Sturm\textsuperscript{2}

\textsuperscript{1}Centre de recherches mathématiques  
Département de mathématiques et de statistique  
Université de Montréal, Canada

\textsuperscript{2}Johann Radon Institute  
Altenberger Strasse 69  
4040 Linz, Austria

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Simple Illustrative Examples in PDE Control and Shape
- Derivative of PDE Constrained Utility Function with respect to Control
- Shape Derivative
- Construction of Micheletti: complete metric group and its tangent space

Averaged Adjoint for State Constrained Objective Functions
- Some Background
- Abstract Framework
- A New Condition in the Singleton Case with an Extra Term
- Back to the Simple Illustrative Example from PDE Control

Example of a Topological Derivative: Non-zero Extra Term
- Topological Derivative
- A One Dimensional Example

Mutivalued Case
- Two Theorems Without and With the Extra Term
- First Theorem: Mild Generalization
- Second Theorem: General Case
- Second Theorem: A Non-convex Example where $X(0)$ is not a singleton

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5 References
Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, bounded open and $a \in L^2(\Omega)$ be the control variable to which is associated the state $u = u(a) \in H_0^1(\Omega)$ solution of the variational state equation

$$
\int_{\Omega} \nabla u(a) \cdot \nabla \psi - a \psi \, dx = 0, \quad \forall \psi \in H_0^1(\Omega),
$$

(1.1)

where $x \cdot y$ denotes the inner product of $x$ and $y$ in $\mathbb{R}^N$.

Given a target function $g \in L^2(\Omega)$, associate with $u(a)$ the objective function

$$
f(a) \overset{\text{def}}{=} \int_{\Omega} \frac{1}{2} |u(a) - g|^2 \, dx.
$$

(1.2)

By introducing the Lagrangian, we get an unconstrained minimax formulation

$$
G(a, \varphi, \psi) \overset{\text{def}}{=} \int_{\Omega} \frac{1}{2} |\varphi - g|^2 \, dx + \int_{\Omega} \nabla \varphi \cdot \nabla \psi - a \psi \, dx
$$

$$
f(a) = \inf_{\varphi \in H^1(\Omega) \psi \in H_0^1(\Omega)} \sup_{\psi \in H_0^1(\Omega)} G(a, \varphi, \psi).
$$
Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, bounded open and $a \in L^2(\Omega)$ be the control variable to which is associated the state $u = u(a) \in H^1_0(\Omega)$ solution of the variational state equation

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$$

$$
f(a) = \inf_{\varphi \in H^1(\Omega)} \sup_{\psi \in H^1(\Omega)} G(a, \varphi, \psi).
$$
If we are only interested in a descent method, we can obtain the semidifferential of \( f(a) \) by a similar minimax formulation. Given the direction \( b \in L^2(\Omega) \), to compute

\[
df(a; b) = \lim_{t \searrow 0} \frac{f(a + tb) - f(a)}{t},
\]

where the state \( u^t \in H^1_0(\Omega) \) at \( t > 0 \) is solution of

\[
\int_\Omega \nabla u^t \cdot \nabla \psi - (a + tb) \psi \, dx = 0, \quad \forall \psi \in H^1_0(\Omega).
\]  \hspace{1cm} (1.3)

The associated Lagrangian is

\[
L(t, \varphi, \psi) \overset{\text{def}}{=} \int_\Omega \frac{1}{2} |\varphi - g|^2 \, dx + \int_\Omega \nabla \varphi \cdot \nabla \psi - (a + tb) \psi \, dx.
\]

It is readily seen that

\[
g(t) \overset{\text{def}}{=} \inf_{\varphi \in H^1_0(\Omega)} \sup_{\psi \in H^1_0(\Omega)} L(t, \varphi, \psi) = f(a + tb)
\]

\[
dg(0) \overset{\text{def}}{=} \lim_{t \searrow 0} \frac{g(t) - g(0)}{t} = df(a; b).
\]
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5. **References**
Consider the state (1.1) and objective function (1.2). Now perturb the domain $\Omega$ by a family of diffeomorphisms $T_t$ generated by a smooth velocity field $V(t)$:

$$\frac{dx}{dt}(t; X) = V(t, x(t; X)), \quad x(0; X) = X, \quad T_t(X) \overset{\text{def}}{=} x(t; X), \quad t \geq 0,$$

$$\Omega_t \overset{\text{def}}{=} T_t(\Omega).$$

The state equation and objective function at $t > 0$ become

$$\int_{\Omega_t} \nabla u_t \cdot \nabla \psi - a \psi \, dx = 0, \quad \forall \psi \in H^1_0(\Omega_t), \quad f(t) \overset{\text{def}}{=} \int_{\Omega_t} |u_t - g|^2 \, dx. \quad (1.4)$$

Introducing the composition $u^t = u_t \circ T_t$ to work in the fixed space $H^1_0(\Omega)$:

$$\int_{\Omega} \left[ A(t) \nabla u^t \cdot \nabla \psi - a \psi \right] j(t) \, dx = 0, \quad \forall \psi \in H^1_0(\Omega), \quad (1.5)$$

$$A(t) = DT_t^{-1} (DT_t^{-1})^*, \quad j(t) = \det DT_t, \quad DT_t \text{ is the Jacobian matrix},$$

$$\Rightarrow f(t) = \int_{\Omega_t} |u_t - g|^2 \, dx = \int_{\Omega} |u^t - g \circ T_t|^2 j(t) \, dx, \quad (1.7)$$

Lagrangian: $L(t, \varphi, \psi) \overset{\text{def}}{=} \int_{\Omega} \left[ \frac{1}{2} |\varphi - g \circ T_t|^2 + A(t) \nabla \varphi \cdot \nabla \psi - a \psi \right] j(t) \, dx.$

$$\Rightarrow g(t) = \inf_{\varphi \in H^1_0(\Omega)} \sup_{\psi \in H^1_0(\Omega)} L(t, \varphi, \psi), \quad dg(0) = \lim_{t \searrow 0} (g(t) - g(0))/t = df(\Omega; V(0)).$$
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Associate with a real vector space (usually a Banach space) $\Theta$ of mappings $\theta : \mathbb{R}^N \to \mathbb{R}^N$ (Micheletti used the space $\Theta = C^k_0(\mathbb{R}^N, \mathbb{R}^N)$, $k \geq 1$), the following space of transformations (endomorphisms) of $\mathbb{R}^N$:

$$\mathcal{F}(\Theta) \overset{\text{def}}{=} \left\{ F : \mathbb{R}^N \to \mathbb{R}^N \text{ bijective} : F - I \in \Theta, \text{ and } F^{-1} - I \in \Theta \right\},$$

(1.8)

where $x \mapsto I(x) \overset{\text{def}}{=} x : \mathbb{R}^N \to \mathbb{R}^N$ is the identity mapping.

Given a fixed set $\Omega_0 \subset \mathbb{R}^N$ (Micheletti used used a bounded open set of class $C^k$), consider the set of images

$$\mathcal{X}(\Omega_0) \overset{\text{def}}{=} \left\{ F(\Omega_0) : \forall F \in \mathcal{F}(\Theta) \right\}$$

(1.9)

of $\Omega_0$ by the elements of $\mathcal{F}(\Theta)$ and the subgroup

$$\mathcal{G}(\Omega_0) \overset{\text{def}}{=} \left\{ F \in \mathcal{F}(\Theta) : F(\Omega_0) = \Omega_0 \right\}.$$

So there is a bijection between the set of images of $\Omega_0$ and the quotient space

$$\mathcal{X}(\Omega_0) \leftrightarrow \mathcal{F}(\Theta) / \mathcal{G}(\Omega_0).$$
The objective is to construct a metric on $\mathcal{F}(\Theta)/\mathcal{G}(\Omega_0)$ that will serve as a distance between two mages $F_1(\Omega_0)$ and $F_2(\Omega_0)$.

Associate with $F \in \mathcal{F}(\Theta)$ the following candidate for a metric

$$d_0(I, F) \stackrel{\text{def}}{=} \| F - I \|_\Theta + \| F^{-1} - I \|_\Theta, \quad d_0(F, G) \stackrel{\text{def}}{=} d_0(I, G \circ F^{-1}). \quad (1.10)$$

Unfortunately, $d_0$ is only a semi-metric that will not satisfy the triangle inequality.

Consider the following second candidate (called Courant metric by Micheletti)

$$d(I, F) \stackrel{\text{def}}{=} \inf_{F = F_1 \circ \cdots \circ F_n} \sum_{i=1}^{n} \| F_i - I \|_\Theta + \| F_i^{-1} - I \|_\Theta, \quad (1.11)$$

where the infimum is taken over all finite factorizations of $F$ in $\mathcal{F}(\Theta)$ of the form

$$F = F_1 \circ \cdots \circ F_n, \quad F_i \in \mathcal{F}(\Theta).$$

In particular $d(I, F) = d(I, F^{-1})$. Extend this function to all $F$ and $G$ in $\mathcal{F}(\Theta)$

$$d(F, G) \stackrel{\text{def}}{=} d(I, G \circ F^{-1}). \quad (1.12)$$

By definition, $d$ is right-invariant since for all $F$, $G$ and $H$ in $\mathcal{F}(\Theta)$

$$d(F, G) = d(F \circ H, G \circ H).$$
The objective is to construct a metric on $\mathcal{F}(\Theta)/\mathcal{G}(\Omega_0)$ that will serve as a distance between two mages $F_1(\Omega_0)$ and $F_2(\Omega_0)$.

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The objective is to construct a metric on \( \mathcal{F}(\Theta)/\mathcal{G}(\Omega_0) \) that will serve as a distance between two mages \( F_1(\Omega_0) \) and \( F_2(\Omega_0) \).

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By definition, \( d \) is right-invariant since for all \( F, G \) and \( H \) in \( \mathcal{F}(\Theta) \)

\[
d(F, G) = d(F \circ H, G \circ H).
\]
(\mathcal{F}(\Theta), d) \text{ is complete for } \Theta \text{ equal to the Banach spaces}

C^k_0(\mathbb{R}^N, \mathbb{R}^N), \ C^k(\mathbb{R}^N, \mathbb{R}^N) \subset B^k(\mathbb{R}^N, \mathbb{R}^N) \text{ and } C^{k,1}(\mathbb{R}^N, \mathbb{R}^N), \ k \geq 0,

and, through special constructions, for the Fréchet spaces

C^\infty_0(\mathbb{R}^N, \mathbb{R}^N) \subset B(\mathbb{R}^N, \mathbb{R}^N) = \cap_{k \geq 0} B^k(\mathbb{R}^N, \mathbb{R}^N).

For any Banach or Fréchet space \( \Theta \subset C^{0,1}(\mathbb{R}^N, \mathbb{R}^N) \), \( \mathcal{F}(\Theta) \) is an open subset of \( I + \Theta \)
- the tangent space is \( \Theta \) at each point \( F \in \mathcal{F}(\Theta) \)
- and the associated smooth structure is trivial.

The analogue would be the general linear group \( GL(n) \) of invertible linear maps from \( \mathbb{R}^N \) to \( \mathbb{R}^N \) which is an open subset of \( \mathcal{L}(\mathbb{R}^N, \mathbb{R}^N) \). So, the tangent space is \( \mathcal{L}(\mathbb{R}^N, \mathbb{R}^N) \).

Choose \( \Theta = C^k_0(\mathbb{R}^N, \mathbb{R}^N), \ k \geq 1 \), \( \mathcal{F}(\Theta) \), and the set \( \mathcal{X}(\Omega_0) \) of the images of an open crack free set \( \Omega_0 \subset \mathbb{R}^N \). Consider a function \( J : \mathcal{X}(\Omega_0) \rightarrow \mathbb{R} \).

**Theorem**

*Let \( \Omega = F(\Omega_0) \in \mathcal{X}(\Omega_0) \) for some \( F \in \mathcal{F}(\Omega_0) \). Then \( J \) is continuous at \( \Omega \) for the Courant metric if and only if

\[
\lim_{t \downarrow 0} J(T_t(\Omega)) = J(\Omega), \quad \frac{dT_t}{dt} = V(t) \circ T_t, \quad T_0 = F,
\]

for all families of velocity fields \( V \in C^0([0, \tau]; C^k_0(\mathbb{R}^N, \mathbb{R}^N)) \).*
(\mathcal{F}(\Theta), d) is complete for \Theta equal to the Banach spaces
\[ C^k_0(\mathbb{R}^N, \mathbb{R}^N), \quad C^k(\overline{\mathbb{R}^N}, \mathbb{R}^N) \subset B^k(\mathbb{R}^N, \mathbb{R}^N) \quad \text{and} \quad C^{k,1}(\overline{\mathbb{R}^N}, \mathbb{R}^N), \quad k \geq 0, \]
and, through special constructions, for the Fréchet spaces
\[ C_0^\infty(\mathbb{R}^N, \mathbb{R}^N) \subset B(\mathbb{R}^N, \mathbb{R}^N) = \bigcap_{k \geq 0} B^k(\mathbb{R}^N, \mathbb{R}^N). \]

For any Banach or Fréchet space \( \Theta \subset C^{0,1}(\overline{\mathbb{R}^N}, \mathbb{R}^N) \), \( \mathcal{F}(\Theta) \) is an open subset of \( I + \Theta \)
- the tangent space is \( \Theta \) at each point \( F \in \mathcal{F}(\Theta) \)
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The analogue would be the general linear group \( GL(n) \) of invertible linear maps from \( \mathbb{R}^N \) to \( \mathbb{R}^N \) which is an open subset of \( \mathcal{L}(\mathbb{R}^N, \mathbb{R}^N) \). So, the tangent space is \( \mathcal{L}(\mathbb{R}^N, \mathbb{R}^N) \).

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for all families of velocity fields \( V \in C^0([0, \tau]; C^k_0(\mathbb{R}^N, \mathbb{R}^N)) \).
\((\mathcal{F}(\Theta), d)\) is complete for \(\Theta\) equal to the Banach spaces
\[
C_0^k(\mathbb{R}^N, \mathbb{R}^N), \; C^k(\overline{\mathbb{R}^N}, \mathbb{R}^N) \subset B^k(\mathbb{R}^N, \mathbb{R}^N) \text{ and } C^{k,1}(\overline{\mathbb{R}^N}, \mathbb{R}^N), \quad k \geq 0,
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- the tangent space is \(\Theta\) at each point \(F \in \mathcal{F}(\Theta)\)
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The analogue would be the general linear group \(GL(n)\) of invertible linear maps from \(\mathbb{R}^N\) to \(\mathbb{R}^N\) which is an open subset of \(\mathcal{L}(\mathbb{R}^N, \mathbb{R}^N)\). So, the tangent space is \(\mathcal{L}(\mathbb{R}^N, \mathbb{R}^N)\).

Choose \(\Theta = C_0^k(\mathbb{R}^N, \mathbb{R}^N), \; k \geq 1\), \(\mathcal{F}(\Theta)\), and the set \(\mathcal{X}(\Omega_0)\) of the images of an open crack free set \(\Omega_0 \subset \mathbb{R}^N\). Consider a function \(J : \mathcal{X}(\Omega_0) \to \mathbb{R}\).

\[\text{Theorem}\]

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\]
for all families of velocity fields \(V \in C^0([0, \tau]; C_0^k(\mathbb{R}^N, \mathbb{R}^N))\).
**Definition**

Let $\Theta = C^k_0(\mathbb{R}^N, \mathbb{R}^N)$. The function $J : \mathcal{X}(\Omega_0) = \{F(\Omega_0) : F \in \mathcal{F}(\Theta)\} \rightarrow \mathbb{R}$ is Hadamard semidifferentiable at $F(\Omega_0)$, $F \in \mathcal{F}(\Theta)$, if

(i) for all $V \in C^0([0, \tau]; D(\mathbb{R}^N, \mathbb{R}^N))$

$$dJ(F(\Omega_0); V) \overset{\text{def}}{=} \lim_{t \searrow 0} \frac{J(T_t(V)(F(\Omega_0))) - J(F(\Omega_0))}{t} \quad \text{exists,} \quad \frac{dT_t}{dt} = V(t) \circ T_t, \ T_0 = F,$$

(ii) and there exists a function $dJ(F(\Omega_0)) : D(\mathbb{R}^N, \mathbb{R}^N) \rightarrow \mathbb{R}$ such that for all $V \in C^0([0, \tau]; D(\mathbb{R}^N, \mathbb{R}^N))$

$$dJ(F(\Omega_0); V) = dJ(F(\Omega_0))(V(0)).$$

**Definition**

$J : \mathcal{X}(\Omega_0) \rightarrow \mathbb{R}$ is Hadamard differentiable at $F(\Omega_0)$, $F \in \mathcal{F}(\Theta)$, if

- it is Hadamard semidifferentiable at $F(\Omega_0)$
- and the function $dJ(F(\Omega_0)) : D(\mathbb{R}^N, \mathbb{R}^N) \rightarrow \mathbb{R}$ is linear and continuous.
**Definition**

Let $\Theta = C^k_0(\mathbb{R}^N, \mathbb{R}^N)$. The function $J : \mathcal{X}(\Omega_0) = \{ F(\Omega_0) : F \in \mathcal{F}(\Theta) \} \to \mathbb{R}$ is Hadamard semidifferentiable at $F(\Omega_0)$, $F \in \mathcal{F}(\Theta)$, if

1. for all $V \in C^0([0, \tau]; D(\mathbb{R}^N, \mathbb{R}^N))$

   \[
   dJ(F(\Omega_0); V) \overset{\text{def}}{=} \lim_{t \searrow 0} \frac{J(T_t(V)(F(\Omega_0))) - J(F(\Omega_0))}{t} \quad \text{exists}, \quad \frac{dT_t}{dt} = V(t) \circ T_t, \ T_0 = F,
   \]


2. and there exists a function $dJ(F(\Omega_0)) : D(\mathbb{R}^N, \mathbb{R}^N) \to \mathbb{R}$ such that for all $V \in C^0([0, \tau]; D(\mathbb{R}^N, \mathbb{R}^N))$

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**Definition**

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5. References
AVERAGED ADJOINT METHOD
SOME BACKGROUND

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In this paper, a \textit{Lagrangian} is a function of the form
\[(t, x, y) \mapsto G(t, x, y) : [0, \tau] \times X \times Y \to \mathbb{R}, \quad \tau > 0,\]
where $Y$ is a \textit{vector space}, $X$ is a subset of a vector space, and $y \mapsto G(t, x, y)$ is \textit{affine}. Associate with the \textit{parameter} $t \geq 0$ the \textit{parametrized minimax function}
\[t \mapsto g(t) \overset{\text{def}}{=} \inf_{x \in X} \sup_{y \in Y} G(t, x, y) : [0, \tau] \to \mathbb{R}. \quad (2.1)\]

When the limits exist we shall use the following compact notation:
\[
\begin{align*}
dg(0) & \overset{\text{def}}{=} \lim_{t \searrow 0} \frac{g(t) - g(0)}{t} = \begin{cases} 
\frac{dg(0)}{def} \overset{\text{def}}{=} \lim_{t \searrow 0} \inf (g(t) - g(0)) / t \\
\frac{dg(0)}{def} \overset{\text{def}}{=} \lim_{t \searrow 0} \sup (g(t) - g(0)) / t \\
\frac{dg(0)}{def} \overset{\text{def}}{=} \lim_{t \searrow 0} (G(t, x, y) - G(0, x, y)) / t \\
\phi \in X, \quad d_x G(t, x, y; \phi) \overset{\text{def}}{=} \lim_{\theta \searrow 0} \frac{G(t, x + \theta \phi, y) - G(t, x, y)}{\theta} \\
\psi \in Y, \quad d_y G(t, x, y; \psi) \overset{\text{def}}{=} \lim_{\theta \searrow 0} \frac{G(t, x, y + \theta \psi) - G(t, x, y)}{\theta}.
\end{cases}
\end{align*}
\]

The notation $t \searrow 0$ and $\theta \searrow 0$ means that $t$ and $\theta$ go to 0 by strictly positive values.
Since $G(t, x, y)$ is affine in $y$, for all $(t, x) \in [0, \tau] \times X$,

$$\forall y, \psi \in Y, \quad d_y G(t, x, y; \psi) = G(t, x, \psi) - G(t, x, 0) = d_y G(t, x, 0; \psi).$$

The state equation at $t \geq 0$:

\[
\boxed{\text{to find } x^t \in X \text{ such that for all } \psi \in Y, \ d_y G(t, x^t, 0; \psi) = 0.}
\]

The set of solutions (states) $x^t$ at $t \geq 0$ is denoted

$$E(t) \overset{\text{def}}{=} \left\{ x^t \in X : \forall \varphi \in Y, \ d_y G(t, x^t, 0; \varphi) = 0 \right\}$$

The standard adjoint state equation at $t \geq 0$:

\[
\boxed{\text{to find } p^t \in Y \text{ such that } \forall \varphi \in X, \ d_x G(t, x^t, p^t; \varphi) = 0, \quad Y(t, x^t) \overset{\text{def}}{=} \text{set of solutions}.}
\]

Under appropriate conditions and uniqueness of the pair $(x^t, p^t)$,

$$dg(0) = d_t G(0, x^0, p^0),$$

where $(x^0, p^0)$ is the solution of the coupled state-adjoint state equations at $t = 0$. 
states: \( E(t) \overset{\text{def}}{=} \left\{ x^t \in X : \forall \varphi \in Y, \; d_y G(t, x^t, 0; \varphi) = 0 \right\} \)

minimizers: \( X(t) \overset{\text{def}}{=} \left\{ x^t \in X : g(t) = \inf_{x \in X} \sup_{y \in Y} G(t, x, y) = \sup_{y \in Y} G(t, x^t, y) \right\} \).

**Lemma (Constrained Infimum and Minimax)**

(i) \( \inf_{x \in X} \sup_{y \in Y} G(t, x, y) = \inf_{x \in E(t)} G(t, x, 0). \)

(ii) The minimax \( g(t) = +\infty \) if and only if \( E(t) = \emptyset \). In that case \( X(t) = X \).

(iii) If \( E(t) \neq \emptyset \), then

\[
X(t) = \left\{ x^t \in E(t) : G(t, x^t, 0) = \inf_{x \in E(t)} G(t, x, 0) \right\} \subset E(t) \quad (2.2)
\]

and \( g(t) < +\infty \).
**Hypothesis** (H0). Let $X$ be a vector space.

(i) For all $t \in [0, \tau]$, $x^0 \in X(0)$, $x^t \in X(t)$, and $y \in Y$, the function

$$s \mapsto G(t, x^0 + s(x^t - x^0), y) : [0, 1] \to \mathbb{R} \quad (2.3)$$

is absolutely continuous. This implies that, for almost all $s$, the derivative exists and is equal to $d_x G(t, x^0 + s(x^t - x^0), y; x^t - x^0)$ and that it is the integral of its derivative. In particular,

$$G(t, x^t, y) = G(t, x^0, y) + \int_0^1 d_x G(t, x^0 + s(x^t - x^0), y; x^t - x^0) \, ds. \quad (2.4)$$

(ii) For all $t \in [0, \tau]$, $x^0 \in X(0)$, $x^t \in X(t)$, $y \in Y$, $\varphi \in X$, and almost all $s \in (0, 1)$, $d_x G(t, x^0 + s(x^t - x^0), y; \varphi)$ exists and the function $s \mapsto d_x G(t, x^0 + s(x^t - x^0), y; \varphi)$ belongs to $L^1(0, 1)$. 
Standard adjoint at \( t \geq 0 \): to find \( p^t \in Y \) such that \( \forall \varphi \in X, \quad d_x G(t, x^t, p^t; \varphi) = 0. \)

**Definition (K. Sturm)**

Given \( x^0 \in X(0) \) and \( x^t \in X(t) \), the averaged adjoint state equation:

\[
\text{to find } y^t \in Y \text{ such that } \forall \varphi \in X, \quad \int_0^1 d_x G(t, x^0 + s(x^t - x^0), y^t; \varphi) \, ds = 0.
\] (2.5)

The set of solutions will be denoted \( Y(t, x^0, x^t) \).

At \( t = 0 \), \( Y(0, x^0, x^0) \) reduces to the set of standard adjoint states

\[
Y(0, x^0) \overset{\text{def}}{=} \left\{ p^0 \in Y : \forall \varphi \in X, \quad d_x G(0, x^0, p^0; \varphi) = 0 \right\}.
\] (2.6)

An important consequence of the introduction of the averaged adjoint state is the following identity: for all \( x^0 \in X(0) \), \( x^t \in X(t) \), and \( y^t \in Y(t, x^0, x^t) \),

\[
g(t) = G(t, x^t, 0) = G(t, x^t, y^t) = G(t, x^0, y^t).
\] (2.7)
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Standard adjoint at $t \geq 0$: to find $p^t \in Y$ such that $\forall \varphi \in X$, $d_x G(t, x^t, p^t; \varphi) = 0$.

**Definition (K. Sturm)**

Given $x^0 \in X(0)$ and $x^t \in X(t)$, the **averaged adjoint state equation**:

| to find $y^t \in Y$ such that $\forall \varphi \in X$, $\int_0^1 d_x G(t, x^0 + s(x^t - x^0), y^t; \varphi) \, ds = 0$. |

(2.5)

The set of solutions will be denoted $Y(t, x^0, x^t)$.

At $t = 0$, $Y(0, x^0, x^0)$ reduces to the set of **standard adjoint states**

| $Y(0, x^0) \overset{\text{def}}{=} \{ p^0 \in Y : \forall \varphi \in X, \ d_x G(0, x^0, p^0; \varphi) = 0 \}$ |

(2.6)

An **important consequence** of the introduction of the averaged adjoint state is the following identity: for all $x^0 \in X(0)$, $x^t \in X(t)$, and $y^t \in Y(t, x^0, x^t)$,

| $g(t) = G(t, x^t, 0) = G(t, x^t, y^t) = G(t, x^0, y^t)$. |

(2.7)
An *important consequence* of the introduction of the averaged adjoint state is the following identity: for all $x^0 \in X(0)$, $x^t \in X(t)$, and $y^t \in Y(t, x^0, x^t)$,

$$g(t) = G(t, x^t, 0) = G(t, x^t, y^t) = G(t, x^0, y^t)$$  \hspace{1cm} (2.8)

$$g(0) = G(0, x^0, 0) = G(0, x^0, y^0).$$  \hspace{1cm} (2.9)

As a result

$$g(t) - g(0) = G(t, x^0, y^t) - G(0, x^0, y^0)$$

$$dg(0) = \lim_{t \searrow 0} \frac{g(t) - g(0)}{t} = \lim_{t \searrow 0} \frac{G(t, x^0, y^t) - G(0, x^0, y^0)}{t}.$$
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5 References
Consider the Lagrangian functional

\[(t, x, y) \mapsto G(t, x, y) : [0, \tau] \times X \times Y \to \mathbb{R}, \quad \tau > 0,\]

where \(X\) and \(Y\) are vector spaces and the function \(y \mapsto G(t, x, y)\) is affine. Let \((H0)\) and the following hypotheses be satisfied:

\( (H1) \) for all \( t \in [0, \tau] \), \( g(t) \) is finite, \( X(t) = \{x^t\} \) and \( Y(t, x^0, x^t) = \{y^t\} \) are singletons;

\( (H2) \) \( d_t G(t, x^0, y) \) exists for all \( t \in [0, \tau] \) and all \( y \in Y \);

\( (H3) \) the following limit exists

\[
\lim_{s \searrow 0, t \searrow 0} d_t G(s, x^0, y^t) = d_t G(0, x^0, y^0). \tag{2.10}
\]

Then, \(dg(0)\) exists and

\[dg(0) = d_t G(0, x^0, y^0).\]

Condition \((H3)\) is similar and typical of what can be found in the literature. See, for instance, [Correa-Seeger (1985)].
**Proof.**

From Hypothesis (H2), \( d_t G(t, x^0, y) \) exists for all \( t \in [0, \tau] \) and \( y \in Y \). Hence, there exists \( \theta_t \in (0, 1) \) such that

\[
G(t, x^0, y^t) - G(0, x^0, y^0) = G(0, x^0, y^t) + t d_t G(\theta_t t, x^0, y^t) - G(0, x^0, y^0)
\]

\[
= d_y G(0, x^0, 0; y^t - y^0) + t d_t G(\theta_t t, x^0, y^t) = t d_t G(\theta_t t, x^0, y^t)
\]

\[
\Rightarrow \frac{G(t, x^0, y^t) - G(0, x^0, y^0)}{t} = d_t G(\theta_t t, x^0, y^t)
\]

since \( d_y G(0, x^0, 0; y^t - y^0) = 0 \). From hypothesis (H3)

\[
\lim_{s \searrow 0, t \searrow 0} d_t G(s, x^0, y^t) = d_t G(0, x^0, y^0). \tag{2.11}
\]

\[
\Rightarrow dG(0) = \lim_{t \searrow 0} \frac{G(t, x^0, y^t) - G(0, x^0, y^0)}{t} = d_t G(0, x^0, y^0). \tag{2.12}
\]
This is an extension of [Sturm (2014)] [Sturm (2015), Thm. 3.1] with only a local differentiability condition at \( t = 0 \). To our best knowledge, the extra term \( R(0, x^0, y^0) \) is new. An example of a topological derivative will be given later.

**Theorem (Singleton Case, [Delfour-Sturm (2017), Delfour-Sturm (2016)])**

Consider the Lagrangian functional

\[
(t, x, y) \mapsto G(t, x, y) : [0, \tau] \times X \times Y \to \mathbb{R}, \quad \tau > 0,
\]

where \( X \) and \( Y \) are vector spaces and the function \( y \mapsto G(t, x, y) \) is affine. Let (H0) and the following hypotheses be satisfied:

(H1) for all \( t \in [0, \tau] \), \( g(t) \) is finite, \( X(t) = \{ x^t \} \) and \( Y(t, x^0, x^t) = \{ y^t \} \) are singletons;

(H2) \( d_1 G(0, x^0, y^0) \) exists;

(H3) the following limit exists

\[
R(0, x^0, y^0) \equiv \lim_{t \downarrow 0} d_y G \left( t, x^0, 0; \frac{y^t - y^0}{t} \right).
\]  

(2.13)

Then, \( d g(0) \) exists and

\[
d g(0) = d_1 G(0, x^0, y^0) + R(0, x^0, y^0).
\]
**Proof.**

Recalling that \( g(t) = G(t, x^t, y^t) = G(t, x^0, y^t) \),

\[
g(t) - g(0) = G(t, x^0, y^t) - G(0, x^0, y^0) = G(t, x^0, y^0) + d_y G(t, x^0, 0; y^t - y^0) - G(0, x^0, y^0)
\]

\[
\Rightarrow \quad \frac{g(t) - g(0)}{t} = d_y G \left( t, x^0, 0; \frac{y^t - y^0}{t} \right) + \frac{G(t, x^0, y^0) - G(0, x^0, y^0)}{t}
\]

\[
\Rightarrow \quad dg(0) = \lim_{t \searrow 0} d_y G \left( t, x^0, 0; \frac{y^t - y^0}{t} \right) + d_t G(0, x^0, y^0)
\]

from hypotheses (H2) and (H3).

Condition (H3) is optimal since under hypotheses (H1)

\[
dg(0) \text{ exists } \iff \lim_{t \searrow 0} d_y G \left( t, x^0, 0; \frac{y^t - y^0}{t} \right) \text{ exists}
\]
Hypotheses (H2) and (H3) are weaker and more general than (H2) and (H3). 

(H2) It is only assumed that $d_t G(0, x^0, y^0)$ exists. 
Hypothesis (H2) assumes that $d_t G(t, x^0, y)$ exists for all $t \in [0, \tau]$ and $y \in Y$. 

(H3) Hypothesis (H3) assumes that 

$$\lim_{s \to 0, \ t \to 0} d_t G(s, x^0, y^t) = d_t G(0, x^0, y^0).$$

(2.14) 

which implies 

$$R(0, x^0, y^0) = \lim_{t \to 0} d_y G \left( t, x^0, 0; \frac{y^t - y^0}{t} \right) = 0.$$ 

(2.15) 

Hence, condition (H3) with $R(0, x^0, y^0) = 0$ is weaker and potentially more general (when the limit is not zero) than (H3). 

All this is possible since $G(t, x, y)$ is a Lagrangian. For zero-sum games, condition (H3) and a similar condition for the max min would not be as interesting.
Recalling that $g(t) = G(t, x^t, y)$ and $g(0) = G(0, x^0, y)$ for any $y \in Y$, then for the standard adjoint state $p^0$ at $t = 0$

$$g(t) - g(0) = G(t, x^t, p^0) - G(t, x^0, p^0) + \left( G(t, x^0, p^0) - G(0, x^0, p^0) \right).$$

Dividing by $t > 0$

$$\frac{g(t) - g(0)}{t} = \frac{G(t, x^t, p^0) - G(t, x^0, p^0)}{t} + \frac{G(t, x^0, p^0) - G(0, x^0, p^0)}{t}
= \int_0^1 d_x G \left( t, (1 - \theta)x^0 + \theta x^t, p^0; \frac{x^t - x^0}{t} \right) d\theta + \frac{G(t, x^0, p^0) - G(0, x^0, p^0)}{t}.$$

Therefore, in view of hypothesis (H2), the limit $dg(0)$ exists if and only if the limit of the first term exists

$$\Rightarrow \quad dg(0) = \lim_{t \searrow 0} \int_0^1 d_x G \left( t, (1 - \theta)x^0 + \theta x^t, p^0; \frac{x^t - x^0}{t} \right) d\theta + d_t G(0, x^0, p^0) \quad \text{and the existence of the limit of the first term can replace hypothesis (H3). As a result, we have two ways of expression hypothesis (H3) since}
\lim_{t \searrow 0} \int_0^1 d_x G \left( t, (1 - \theta)x^0 + \theta x^t, p^0; \frac{x^t - x^0}{t} \right) d\theta = \lim_{t \searrow 0} d_y G \left( t, x^0, 0; \frac{y^t - y^0}{t} \right).
Since $d_x G$ and $d_x d_y G$ both exist, Hypothesis (H3) can be rewritten as follows:

$$
d_y G \left( t, x^0, 0; \frac{y^t - y^0}{t} \right) = d_y G \left( t, x^0, 0; \frac{y^t - y^0}{t} \right) - d_y G \left( t, x^t, 0; \frac{y^t - y^0}{t} \right)
$$

$$
= \int_0^1 d_x d_y G \left( t, \theta x^0 + (1 - \theta) x^t, 0; \frac{y^t - y^0}{t^{\alpha}}; \frac{x^0 - x^t}{t^{1-\alpha}} \right) \, d\theta,
$$

for some $\alpha \in [0, 1]$. For instance with $\alpha = 1/2$, it would be sufficient to find bounds on the differential quotients

$$
\frac{y^t - y^0}{t^{1/2}} \quad \text{and} \quad \frac{x^t - x^0}{t^{1/2}}
$$

which is less demanding than finding a bound on $(x^t - x^0)/t$ or $(y^t - y^0)/t$.

When the integral can be taken inside

$$
d_y G \left( t, x^0, 0; \frac{y^t - y^0}{t} \right) = d_x d_y G \left( t, \frac{x^0 + x^t}{2}, 0; \frac{y^t - y^0}{t^{\alpha}}; \frac{x^0 - x^t}{t^{1-\alpha}} \right)
$$

$$
\lim_{t \downarrow 0} d_y G \left( t, x^0, 0; \frac{y^t - y^0}{t} \right) = \lim_{t \downarrow 0} d_x d_y G \left( t, \frac{x^0 + x^t}{2}, 0; \frac{y^t - y^0}{t^{\alpha}}; \frac{x^0 - x^t}{t^{1-\alpha}} \right).
$$
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5 References
If we are only interested in a descent method, we can obtain the semidifferential of $f(a)$ by a similar minimax formulation.

Given the direction $b \in L^2(\Omega)$, we want to compute

$$df(a; b) = \lim_{t \downarrow 0} \frac{f(a + tb) - f(a)}{t}.$$  

The state $u^t \in H_0^1(\Omega)$ at $t > 0$ is solution of

$$\int_{\Omega} \nabla u^t \cdot \nabla \psi - (a + tb) \psi \, dx = 0, \quad \forall \psi \in H_0^1(\Omega),$$  

and the associated Lagrangian is

$$L(t, \varphi, \psi) \overset{\text{def}}{=} \int_{\Omega} \frac{1}{2} |\varphi - g|^2 \, dx + \int_{\Omega} \nabla \varphi \cdot \nabla \psi - (a + tb) \psi \, dx.$$  

It is readily seen that

$$g(t) \overset{\text{def}}{=} f(a + tb) = \inf_{\varphi \in H_0^1(\Omega)} \sup_{\psi \in H_0^1(\Omega)} L(t, \varphi, \psi)$$  

$$dg(0) \overset{\text{def}}{=} \lim_{t \downarrow 0} \frac{g(t) - g(0)}{t} = df(a; b).$$
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Recall
\[ L(t, \varphi, \psi) \overset{\text{def}}{=} \int_{\Omega} \frac{1}{2} |\varphi - g|^2 + \nabla \varphi \cdot \nabla \psi - (a + tb) \psi \, dx. \]

It is readily seen that
\[ dy L(t, \varphi, \psi; \psi') = \int_{\Omega} \nabla \varphi \cdot \nabla \psi' - (a + tb) \psi' \, dx \]
\[ dx L(t, \varphi, \psi; \varphi') = \int_{\Omega} (\varphi - g) \varphi' + \nabla \varphi' \cdot \nabla \psi \, dx, \quad dt L(t, \varphi, \psi) = - \int_{\Omega} b \psi \, dx. \]

Observe that the derivative of the state \( \dot{u} \in H^1_0(\Omega) \) exists:
\[ \int_{\Omega} \nabla \left( \frac{u^t - u^0}{t} \right) \cdot \nabla \psi - b \psi \, dx = 0, \quad \forall \psi \in H^1_0(\Omega), \quad (2.17) \]

implies that \( (u^t - u^0)/t = \dot{u} \in H^1_0(\Omega) \) solution of
\[ \int_{\Omega} \nabla \dot{u} \cdot \nabla \psi - b \psi \, dx = 0, \quad \psi \in H^1_0(\Omega). \quad (2.18) \]

The averaged adjoint \( y^t \in H^1_0(\Omega) \) is solution of
\[ \int_{\Omega} \left( \frac{u^t + u^0}{2} \right) \varphi + \nabla y^t \cdot \nabla \varphi \, dx = 0, \quad \forall \varphi \in H^1_0(\Omega). \]
\[\int_{\Omega} \left( \frac{u^t + u^0}{2} \right) \varphi + \nabla y^t \cdot \nabla \varphi \, dx = 0, \quad \forall \varphi \in H^1_0(\Omega).\]

adjoint at \( t = 0 \) :
\[\int_{\Omega} u^0 \varphi + \nabla y^0 \cdot \nabla \varphi \, dx = 0, \quad \forall \varphi \in H^1_0(\Omega),\]

\[\Rightarrow \int_{\Omega} \frac{1}{2} \left( \frac{u^t - u^0}{t} \right) \varphi + \nabla \left( \frac{y^t - y^0}{t} \right) \cdot \nabla \varphi \, dx = 0, \quad \forall \varphi \in H^1_0(\Omega). \quad (2.19)\]

It remains to check that the limit in (2.13) exists:
\[d_y G(t, x^0, 0; (y^t - y^0)/t) \to 0\]

\[\int_{\Omega} \nabla u^0 \cdot \nabla \left( \frac{y^t - y^0}{t} \right) - (a + tb) \left( \frac{y^t - y^0}{t} \right) \, dx = -t \int_{\Omega} b \left( \frac{y^t - y^0}{t} \right) \, dx \]

\[= -t \int_{\Omega} \nabla \left( \frac{u^t - u^0}{t} \right) \cdot \nabla \left( \frac{y^t - y^0}{t} \right) \, dx = \frac{t}{2} \int_{\Omega} \left| \frac{u^t - u^0}{t} \right|^2 \, dx = \frac{t}{2} \int_{\Omega} |\dot{u}|^2 \, dx \rightarrow 0\]
as \( t \to 0 \) using (2.18) and (2.19). Therefore, by Theorem 9,
\[df(a; b) = -\int_{\Omega} b y^0 \, dx, \quad y^0 \in H^1_0(\Omega), \quad (2.20)\]

\[\int_{\Omega} (u - g)\varphi + \nabla y^0 \cdot \nabla \varphi \, dx = 0, \quad \forall \varphi \in H^1_0(\Omega). \quad (2.21)\]
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5 References
The topological derivative rigorously introduced by [Sokołowski-Zochowski (1999)] induces topological changes.

For instance, let $f$ be an objective function defined on a family of open subsets of $\mathbb{R}^N$. Given a point $a$ in the open set $\Omega$, let $\overline{B}_r(a)$ be a closed ball of radius $r$ and center $a$ such that $\overline{B}_r(a) \subset \Omega$.

Consider the perturbed domain $\Omega_r \overset{\text{def}}{=} \Omega \setminus \overline{B}_r(a)$: $\Omega$ minus the hole $\overline{B}_r(a)$. In this simple case the topological derivative is defined as

$$df(0) \overset{\text{def}}{=} \lim_{r \searrow 0} \frac{f(\Omega_r) - f(\Omega)}{|\overline{B}_r(a)|}, \quad (3.1)$$

where $|\overline{B}_r(a)|$ is the volume of $\overline{B}_r(a)$ in $\mathbb{R}^N$.

When $f$ is of the form $f(\Omega) = \int_\Omega \varphi \, dx$, the application of the Lebesgue differentiation theorem gives $df(0) = -\varphi(a)$. Of course, many other types of topological perturbations can be considered (see the recent IFIP paper of [Delfour (2017)]).
1 Simple Illustrative Examples in PDE Control and Shape
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5 References
Given $\varepsilon$, $0 < \varepsilon < 1$, $a > 0$, and the domain $\Omega = (-a, a)$, consider the problem: to find $u \in W^{1,2-\varepsilon}(-a, a)$ such that

$$\forall \varphi \in W^{1,2-\varepsilon}(-a, a) \quad \int_{-a}^{a} \frac{du}{dx} \frac{d\varphi}{dx} + u \varphi \, dx = \int_{-a}^{a} \sqrt{|x|} \frac{d\varphi}{dx} + \sqrt{|x|} \varphi \, dx.$$  (3.2)

Here, $X = W^{1,2-\varepsilon}(-a, a)$ and $Y = W^{1,\frac{2-\varepsilon}{1-\varepsilon}}(-a, a)$ are reflexive Banach spaces since $2 - \varepsilon > 1$ and $\frac{2-\varepsilon}{1-\varepsilon} > 1$. The elements of $X$ will be denoted $u$ and $x \in (-a, a)$ will be the space variable. There exists a unique\(^1\) solution $u(x) = \sqrt{|x|}$, $-a \leq x \leq a$, and the injections $W^{1,2-\varepsilon}(-a, a) \to C^0[-a, a]$ and $W^{1,\frac{2-\varepsilon}{1-\varepsilon}}(-a, a) \to C^0[-a, a]$ are continuous and the following objective function is well-defined:

$$f(\Omega) \overset{\text{def}}{=} |u(a)|^2 + |u(-a)|^2 - 2 |u(0)|^2.$$

---

\(^1\)Given measurable functions $k_1, k_2 : [-a, a] \to \mathbb{R}$ such that $\alpha \leq k_i(x) \leq \beta$ for some constants $\alpha > 0$ and $\beta > 0$, and real numbers $1 < p < \infty$, $p^{-1} + q^{-1} = 1$, associate with the continuous bilinear mapping

$$\varphi, \psi \mapsto b(\varphi, \psi) \overset{\text{def}}{=} \int_{-a}^{a} k_1(x) \frac{d\varphi}{dx} \frac{d\psi}{dx} + k_2(x) \varphi \psi \, dx : W^{1,p}(-a, a) \times W^{1,q}(-a, a) \to \mathbb{R},$$

the continuous linear operator $A : W^{1,p}(-a, a) \to W^{1,q}(-a, a)'$ which is a topological isomorphism for all $p \in (1, \infty)$ ([Auscher-Tchamitchian (1998)]). Here, $p = 2 - \varepsilon$ and $q = (2 - \varepsilon)/(1 - \varepsilon)$.
Given $\varepsilon, 0 < \varepsilon < 1$, $a > 0$, and the domain $\Omega = (-a, a)$, consider the problem:
to find $u \in W^{1,2-\varepsilon}(-a, a)$ such that
\[
\forall \varphi \in W^{1,2-\varepsilon}(-a, a) \quad \int_{-a}^{a} \frac{du}{dx} \frac{d\varphi}{dx} + u \varphi dx = \int_{-a}^{a} \frac{d}{dx} \sqrt{|x|} \frac{d\varphi}{dx} + \sqrt{|x|} \varphi dx.
\] (3.2)

Here, $X = W^{1,2-\varepsilon}(-a, a)$ and $Y = W^{1,2-\varepsilon}(-a, a)$ are reflexive Banach spaces since
$2 - \varepsilon > 1$ and $\frac{2-\varepsilon}{1-\varepsilon} > 1$. The elements of $X$ will be denoted $u$ and $x \in (-a, a)$ will be
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injections $W^{1,2-\varepsilon}(-a, a) \to C^0[-a, a]$ and $W^{1,2-\varepsilon}(-a, a) \to C^0[-a, a]$ are continuous
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f(\Omega) \overset{\text{def}}{=} |u(a)|^2 + |u(-a)|^2 - 2 |u(0)|^2.
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\[1\]Given measurable functions $k_1, k_2 : [-a, a] \to \mathbb{R}$ such that $\alpha \leq k_i(x) \leq \beta$ for some constants $\alpha > 0$ and $\beta > 0$, and real numbers $1 < p < \infty$, $p^{-1} + q^{-1} = 1$, associate with the continuous bilinear mapping
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\]
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$p \in (1, \infty)$ ([Auscher-Tchamitchian (1998)]). Here, $p = 2 - \varepsilon$ and $q = (2 - \varepsilon)/(1 - \varepsilon)$. 

---
\( \overline{B}_r(0) \subset \mathbb{R} \) be the closed ball of radius \( r \) in \( 0 \). Volume: \( t = |\overline{B}_r(0)| = 2r \). Perturbed domain is \( \Omega_r = \Omega \setminus \overline{B}_r(0) = (-a, -r) \cup (r, a) \) has 2 connected components and it is not possible to construct a bijection between \( \Omega \) and \( \Omega_r \).

**Perturbed problems** parametrized by \( r, 0 < r < a/2 \): to find \( u_r \in W^{1,2-\varepsilon}(\Omega_r) \) s. t.

\[
\forall \varphi \in W^{1,\frac{2-\varepsilon}{1-\varepsilon}}(\Omega_r), \quad \int_{\Omega_r} \frac{d u_t}{dx} \frac{d \varphi}{dx} + u_t \varphi \, dx = \int_{\Omega_r} \frac{d \sqrt{|x|}}{dx} \frac{d \varphi}{dx} + \sqrt{|x|} \varphi \, dx
\]

with the objective function

\[
j(r) \overset{\text{def}}{=} |u_r(a)|^2 - |u_r(r)|^2 + |u_r(-a)|^2 - |u_r(-r)|^2.
\]

The function \( u_r(x) = \sqrt{|x|} \) is the unique solution and

\[
j(r) = 2a - 2r \quad \Rightarrow \quad dj(0) \overset{\text{def}}{=} \lim_{r \searrow 0} \frac{1}{2r} (j(r) - j(0)) = -1.
\]

By construction, \( \Omega_r = T_r(\Omega \setminus \{0\}) \), where

\[
\text{Bijection } x \mapsto T_r(x) \overset{\text{def}}{=} \begin{cases} 
  x - r \left(1 + \frac{x}{a}\right), & x \in (-a, 0) \\
  x + r \left(1 - \frac{x}{a}\right), & x \in (0, a)
\end{cases}
\begin{align*}
: \Omega \setminus \{0\} & \to \Omega_r = \Omega \setminus \overline{B}_r(0) \\
\end{align*}
\]

and notice that \( T_r(0^-) = -r, T_r(0^+) = r, T_r(a^-) = a, \) and \( T_r(-a^+) = -a \).
\( \overline{B}_r(0) \subset \mathbb{R} \) be the closed ball of radius \( r \) in \( 0 \). Volume: \( t = |\overline{B}_r(0)| = 2r \). Perturbed domain is \( \Omega_r = \Omega \setminus \overline{B}_r(0) = (-a, -r) \cup (r, a) \) has 2 connected components and it is not possible to construct a bijection between \( \Omega \) and \( \Omega_r \).

**Perturbed problems** parametrized by \( r, 0 < r < a/2 \): to find \( u_r \in W^{1,2-\varepsilon}(\Omega_r) \) s. t.

\[
\forall \varphi \in W^{1,\frac{2-\varepsilon}{1-\varepsilon}}(\Omega_r), \quad \int_{\Omega_r} \frac{du_t}{dx} \frac{d\varphi}{dx} + u_t \varphi \, dx = \int_{\Omega_r} \frac{d\sqrt{|x|}}{dx} \frac{d\varphi}{dx} + \sqrt{|x|} \varphi \, dx
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with the objective function

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  x + r \left( 1 - \frac{x}{a} \right), & x \in (0, a) 
\end{cases} : \Omega \setminus \{0\} \to \Omega_r = \Omega \setminus \overline{B}_r(0)
\]

and notice that \( T_r(0^-) = -r \), \( T_r(0^+) = r \), \( T_r(a^-) = a \), and \( T_r(-a^+) = -a \).
for $a = 1$

the function $u$ has a cusp at the point 0
Prior to proceeding, it is advantageous to simplify the computations by observing that
the function $u^r(x) = \sqrt{|T_r(x)|}$ is symmetrical with respect to $x = 0$, that is,
$u^r(-x) = u^r(x)$ and

$$j(r) = 2 \left[ u^r(a)^2 - u^r(0^+)^2 \right]. \quad (3.3)$$

As a result

$$dj(0) = \lim_{r \searrow 0} \frac{j(r) - j(0)}{2r} = \lim_{r \searrow 0} \frac{u^r(a)^2 - u^r(0^+)^2}{r}$$

By changing the variable $r$ to $t$, it is sufficient to apply Theorem 9 to the following
problem on $(0, a)$: to find $u^t \in W^{1, 2-\varepsilon}(0, a)$ such that for all $\varphi \in W^{1, 2-\varepsilon}(0, a)$

$$\int_0^a \frac{a}{a-t} \frac{du^t}{dx} \frac{d\varphi}{dx} + \frac{a-t}{a} u^t \varphi \, dx = \int_0^a \frac{1}{2\sqrt{T_t(x)}} \frac{d\varphi}{dx} + \frac{a-t}{a} \sqrt{T_t(x)} \varphi \, dx \quad (3.4)$$

with the objective function

$$j^+(t) \overset{\text{def}}{=} u^t(a)^2 - u^t(0^+)^2, \quad dj^+(0) = \lim_{t \searrow 0} (j^+(t) - j^+(0))/t.$$
From Theorem 9, the Lagrangian associated with the perturbed problems is

\[
G(t, \varphi, \psi) \overset{\text{def}}{=} |\varphi(a)|^2 - |\varphi(0)|^2 \\
+ \int_0^a \left( \frac{a}{a-t} \right) \frac{d\varphi}{dx} \frac{d\psi}{dx} + \left( \frac{a-t}{a} \right) \varphi \psi \, dx \\
- \int_0^a \frac{1}{2 \sqrt{T_t(x)}} \frac{d\psi}{dx} + \left( \frac{a-t}{a} \right) \sqrt{T_t(x)} \psi \, dx.
\]

(3.5)

It is non-convex in the \( \varphi \) variable in view of the presence of the term \(-|\varphi(0)|^2\). The standard adjoint \( p^t \) is solution of the adjoint equation

\[
\forall \varphi \in W^{1,2-\varepsilon}(0, a), \quad \left\{ \begin{array}{l}
2u^t(a) \varphi(a) - 2u^t(0) \varphi(0) \\
+ \int_0^a \left( \frac{a}{a-t} \right) \frac{d\varphi}{dx} \frac{dp^t}{dx} + \left( \frac{a-t}{a} \right) \varphi p^t \, dx = 0.
\end{array} \right.
\]

(3.6)

In particular this is true for all \( \varphi \in H^1(0, a) = W^{1,2}(0, a) \subset W^{1,2-\varepsilon}(0, a) \). Since the differential operator is uniformly coercive for \( 0 \leq t \leq a/2 \), there exist a unique \( p^t \in H^1(0, a) \).
But, in view of the fact that for $0 \leq t \leq a/2$, $u^t$ is finite for all $x$, we get more regularity: $p_t \in H^2(0, a) \cap C^\infty(0, a)$ is solution of

$$
-a a^{-2} p_t + \frac{a - t}{a} p^t = 0 \text{ in } (0, a)
$$

$$
\frac{a}{a-t} \frac{dp^t}{dx}(a) = -2 u^t(a), \quad \frac{a}{a-t} \frac{dp^t}{dx}(0) = -2 u^t(0).
$$

The explicit solution for $u^t(0) = \sqrt{t}$ and $u^t(a) = \sqrt{a}$ is

$$
p^t(x) = \frac{a}{a-t} \frac{2}{e^{a-t} - e^{-(a-t)}} \left[ \sqrt{t} \left( e^{\frac{a-t}{a}(a-x)} + e^{-\frac{a-t}{a}(a-x)} \right) - \sqrt{a} \left( e^{\frac{a-t}{a}x} + e^{-\frac{a-t}{a}x} \right) \right].
$$

At $t = 0$,

$$
p^0(x) = -2\sqrt{a} \frac{e^x + e^{-x}}{e^a - e^{-a}}.
$$
The right-hand side \( t \)-derivative is

\[
d_t G(t, \varphi, \psi) = \int_0^a \frac{a}{(a-t)^2} \frac{d\varphi}{dx} \frac{d\psi}{dx} - \frac{1}{a} \varphi \psi \, dx + \frac{1}{a} \int_0^a \sqrt{T_t} \psi \, dx \\
- \frac{1}{2} \int_0^a \left[ \frac{-1}{2(T_t)^{3/2}} \frac{d\psi}{dx} + \left( \frac{a-t}{a} \right) \frac{1}{\sqrt{T_t}} \psi \right] \, dx \\
+ \frac{1}{2a} \int_0^a \left[ \frac{-x}{2(T_t)^{3/2}} \frac{d\psi}{dx} + \left( \frac{a-t}{a} \right) \frac{x}{\sqrt{T_t}} \psi \right] \, dx.
\]

At \( t = 0 \), substitute \( u^0(x) = \sqrt{x} \) and \( p^0 \) and Integrate by parts

\[
d_t G(0, u^0, p^0) = \int_0^a \frac{1}{a} \frac{d\sqrt{x}}{dx} \frac{dp^0}{dx} dx - \frac{1}{a} \sqrt{x} p^0 \, dx + \frac{1}{a} \int_0^a \sqrt{x} p^0 \, dx \\
- \frac{1}{2} \int_0^a \left[ \frac{-1}{2x^{3/2}} \frac{dp^0}{dx} + \frac{1}{\sqrt{x}} p^0 \right] \, dx + \frac{1}{2a} \int_0^a \left[ \frac{-1}{2\sqrt{x}} \frac{dp^0}{dx} + \sqrt{x}p^0 \right] \, dx \\
= \frac{1}{2a} \int_0^a \frac{d\sqrt{x}}{dx} \frac{dp^0}{dx} \, dx + \sqrt{x}p^0 \, dx - \frac{1}{2} \int_0^a \left[ \frac{d}{dx} \frac{1}{\sqrt{x}} \frac{dp^0}{dx} + \frac{1}{\sqrt{x}} p^0 \right] \, dx = 0.
\]
Go back to the Lagrangian (3.5) of the perturbed problem and compute

$$d_x G(t, \bar{\varphi}, \psi; \varphi) = 2 \bar{\varphi}(a) \varphi(a) - 2 \bar{\varphi}(0) \varphi(0) + \int_0^a \left( \frac{a}{a-t} \right) \frac{d\varphi}{dx} \frac{d\psi}{dx} + \left( \frac{a-t}{a} \right) \varphi \psi \, dx.$$ 

The averaged adjoint state equation for $y^t$ must satisfy the equation: for all $\varphi \in W^{1,2-\varepsilon}(0, a)$

$$0 = \int_0^1 d_x G(t, u^0 + s(u^t - u^0), y^t; \varphi)$$

$$= (u^t(a) + u^0(a)) \varphi(a) - (u^t(0) + u^0(0)) \varphi(0) + \int_0^a \left( \frac{a}{a-t} \right) \frac{d\varphi}{dx} \frac{dy^t}{dx} + \left( \frac{a-t}{a} \right) \varphi \, y^t \, dx.$$ 

Its solution $y^t \in H^2(0, a) \cap C^\infty(0, a)$ satisfies the following equations

$$- \left( \frac{a}{a-t} \right) \frac{d^2 y^t}{dx^2} + \left( \frac{a-t}{a} \right) y^t = 0, \quad \text{in } (0, a)$$

$$\left( \frac{a}{a-t} \right) \frac{dy^t}{dx}(0) = -(u^t(0) + u^0(0)) \text{ at } x = 0$$

$$\left( \frac{a}{a-t} \right) \frac{dy^t}{dx}(a) = -(u^t(a) + u^0(a)) \text{ at } x = a.$$
Its explicit expression with $u^t(0) = \sqrt{t}$ and $u^t(a) = \sqrt{a}$ is

$$y^t(x) = \frac{a}{a-t} \left[ \frac{1}{e^{a-t} - e^{-(a-t)}} \right] \left[ \sqrt{t} \left( e^{\frac{a-t}{a}(a-x)} + e^{-\frac{a-t}{a}(a-x)} \right) - 2\sqrt{a} \left( e^{\frac{a-t}{a}x} + e^{-\frac{a-t}{a}x} \right) \right].$$

The condition to be checked is the existence of the limit (the extra term)

$$\lim_{t \to 0} dyG \left( t, u^0, 0; \frac{y^t - y^0}{t} \right).$$

So, for $\psi = (y^t - y^0)/t \in H^2(0, a)$,

$$dyG(t, u^0, 0; \psi) = \int_0^a \left( \frac{a}{a-t} \right) \frac{du^0}{dx} \frac{d\psi}{dx} + \left( \frac{a-t}{a} \right) u^0 \psi dx - \int_0^a \frac{1}{2\sqrt{T_t(x)}} \frac{d\psi}{dx} + \left( \frac{a-t}{a} \right) \sqrt{T_t(x)} \psi dx$$

$$= \int_0^a \left( \frac{a}{a-t} \right) \frac{du^0}{dx} \frac{d\psi}{dx} + \left( \frac{a-t}{a} \right) u^0 \psi dx - \int_0^a \frac{a}{a-t} \frac{d\sqrt{T_t(x)}}{dx} \frac{d\psi}{dx} + \frac{a-t}{a} \sqrt{T_t(x)} \psi dx$$

$$= \int_0^a \left( \frac{a}{a-t} \right) \frac{d(u^0 - u^t)}{dx} \frac{d\psi}{dx} + \left( \frac{a-t}{a} \right) (u^0 - u^t) \psi dx$$

$$= (u^0 - u^t) \left( \frac{a}{a-t} \right) \frac{d}{dx} \left( \frac{y^t - y^0}{t} \right) \bigg|_{x=0}^a \to -1.$$
Its explicit expression with $u^t(0) = \sqrt{t}$ and $u^t(a) = \sqrt{a}$ is

$$y^t(x) = \frac{a}{a-t} \frac{1}{e^{a-t} - e^{-(a-t)}} \left[ \sqrt{t} \left( e^{\frac{a-t}{a}(a-x)} + e^{-\frac{a-t}{a}(a-x)} \right) - 2\sqrt{a} \left( e^{\frac{a-t}{a}x} + e^{-\frac{a-t}{a}x} \right) \right].$$

The condition to be checked is the existence of the limit (the extra term)

$$\lim_{t \searrow 0} d_y G \left( t, u^0, 0; \frac{y^t - y^0}{t} \right).$$

So, for $\psi = (y^t - y^0) / t \in H^2(0, a)$,

$$d_y G(t, u^0, 0; \psi)$$

$$= \int_0^a \left( \frac{a}{a-t} \right) \frac{du^0}{dx} \frac{d\psi}{dx} + \left( \frac{a-t}{a} \right) u^0 \psi dx - \int_0^a \frac{1}{2\sqrt{T_t(x)}} \frac{d\psi}{dx} + \left( \frac{a-t}{a} \right) \sqrt{T_t(x)} \psi dx$$

$$= \int_0^a \left( \frac{a}{a-t} \right) \frac{d(u^0 - u^t)}{dx} \frac{d\psi}{dx} + \left( \frac{a-t}{a} \right) (u^0 - u^t) \psi dx$$

$$= (u^0 - u^t) \left( \frac{a}{a-t} \right) \frac{d}{dx} \left( \frac{y^t - y^0}{t} \right) \bigg|_{x=0} \rightarrow -1.$$
Theorem

(i) For $0 < \varepsilon < 1$ and $t \in [0, a/2]$,

$$\|u^t\|_{W^{1,2-\varepsilon}(0,a)} \leq c(\varepsilon, a), \quad \|u^t - u^0\|_{C^0[0,a]} \leq \sqrt{t}, \quad u^t \rightharpoonup u^0 \text{ in } W^{1,2-\varepsilon}(0,a)$$

weak and this rate of convergence is sharp.

(ii) For $x \in (0,a)$, the material derivative is given by

$$\dot{u}(x) \overset{\text{def}}{=} \lim_{t \searrow 0} \frac{u^t(x) - u^0(x)}{t} = \frac{1}{2} \left( \frac{1}{\sqrt{|x|}} - \frac{\sqrt{|x|}}{a} \right) \geq 0,$$

$$\dot{u} \in L^{2-\varepsilon}(0,a) \text{ for } 0 < \varepsilon \leq 1, \text{ but } \dot{u} \notin L^2(0,a).$$

Moreover, as $t \to 0$,

$$\left\| \frac{(u^t - u^0)}{t} - \dot{u} \right\|_{L^{2-\varepsilon}(0,a)} \to 0. \quad (3.7)$$

(iii) As for the derivative of $\dot{u}$,

$$\frac{d\dot{u}}{dx}(x) = -\frac{1}{4\sqrt{|x|}} \begin{cases} 1/x + 1/a, & x \in (0,a) \\ 1/x - 1/a, & x \in (-a,0) \end{cases} \quad \frac{d\dot{u}}{dx}(0^+) = -\infty.$$
Therefore,

\[ \frac{d\dot{u}}{dx} \notin L^1(0, a), \text{ and, a fortiori, } \frac{d\dot{u}}{dx} \notin L^{2-\varepsilon}(0, a). \]

From part (iii) we cannot apply the chain rule to get \(dj(0^+)\) since the expression is undetermined:

\[ 2u^0(a)\dot{u}(a) - 2u^0(0)\dot{u}(0) = 2u^0(a)\dot{u}(a) - 2[0(\infty)]! \]
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5. **References**
We give two theorems for the existence and expressions of $dg(0)$ in the multivalued case where only a right-hand side derivative of $g$ is expected.

- New conditions and quadratic examples were given in [Delfour-Sturm (2017)] without the extra term.

- Complete conditions including the extra term were published in [Delfour-Sturm (2016)] at an IFAC meeting in 2016 prior to the publication of [Delfour-Sturm (2017)] due to longer publication delays in the Journal of Convex Analysis.

Here, we give the latest version from [Delfour-Sturm (2016)].

The first theorem is a mild generalization of the singleton case. Yet, it can be applied to PDE problems with non-homogeneous Dirichlet boundary conditions where non-unique extensions are used (cf. [Delfour-Zolésio (2011)]).

A new non-convex multivalued example will be given for the second more general theorem.
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5 References
Theorem (A first extension)

Given $X$, $Y$, and $G$, let (H0) and the following hypotheses be satisfied:

(H1) for all $t$ in $[0, \tau]$, $X(t) \neq \emptyset$ and $g(t)$ is finite, and for all $x^t \in X(t)$ and $x^0 \in X(0)$, $Y(t, x^0, x^t) \neq \emptyset$;

(H2) for all $x \in X(0)$ and $y \in Y(0, x)$, $d_t G(0, x, y)$ exists;

(H3) there exist $\hat{x}^0 \in X(0)$, $\hat{y}^0 \in Y(0, \hat{x}^0)$, and $R(0, \hat{x}^0, \hat{y}^0)$ such that for each sequence $t_n \to 0$, $0 < t_n \leq \tau$, there exist a subsequence $\{t_{n_k}\}$ of $\{t_n\}$, $x^{t_{n_k}} \in X(t_{n_k})$, and $y^{t_{n_k}} \in Y(t_{n_k}, \hat{x}^0, x^{t_{n_k}})$ such that

$$
\lim_{k \to \infty} d_y G\left(t_{n_k}, \hat{x}^0, 0; (y^{t_{n_k}} - \hat{y}^0)/t_{n_k}\right) = R(0, \hat{x}^0, \hat{y}^0).
$$

Then, $dg(0)$ exists and there exist $\hat{x}^0 \in X(0)$ and $\hat{y}^0 \in Y(0, \hat{x}^0)$ such that

$$
dg(0) = d_t G(0, \hat{x}^0, \hat{y}^0) + R(0, \hat{x}^0, \hat{y}^0).
$$

When $X(0) = \{x^0\}$ and $Y(0, x^0) = \{y^0\}$ are singletons, the above hypotheses are equivalent to the ones of Thm. 9.
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**Theorem (General Case)**

Given $X$, $Y$, and $G$, let (H0) and the following hypotheses be satisfied:

(H1) $\forall t \in [0, \tau], X(t) \neq \emptyset$, $g(t)$ is finite, and $\forall x^t \in X(t)$ and $x^0 \in X(0)$, $Y(t, x^0, x^t) \neq \emptyset$;

(H2) for all $x \in X(0)$ and $y \in Y(0, x)$, $d_t G(0, x, y)$ exists and, for each $x \in X(0)$, there exists a function $y \mapsto R(0, x, y) : Y(0, x) \to \mathbb{R}$ satisfying (H3) and (H4) below;

(H3) for each sequence $t_n \to 0$, $0 < t_n \leq \tau$, $\exists x^0 \in X(0)$ such that for all $y^0 \in Y(0, x^0)$, $\exists a$ subsequence $\{t_{n_k}\}$ of $\{t_n\}$, $x^{t_{n_k}} \in X(t_{n_k})$, and $y^{t_{n_k}} \in Y(t_{n_k}, x^0, x^{t_{n_k}})$ such that

$$\liminf_{k \to \infty} d_y G \left( t_{n_k}, x^0, 0; (y^{t_{n_k}} - y^0)/t_{n_k} \right) \geq R(0, x^0, y^0);$$

(H4) for each sequence $t_n \to 0$, $0 < t_n \leq \tau$ and all $x^0 \in X(0)$, there exist $y^0 \in Y(0, x^0)$, a subsequence $\{t_{n_k}\}$ of $\{t_n\}$, $x^{t_{n_k}} \in X(t_{n_k})$, and $y^{t_{n_k}} \in Y(t_{n_k}, x^0, x^{t_{n_k}})$ such that

$$\limsup_{k \to \infty} d_y G \left( t, x^0, 0; (y^{t_{n_k}} - y^0)/t_{n_k} \right) \leq R(0, x^0, y^0).$$

Then, $dg(0)$ exists and there exists $\hat{x}^0 \in X(0)$ and $\hat{y}^0 \in Y(0, \hat{x}^0)$ such that

$$dg(0) = d_t G(0, \hat{x}^0, \hat{y}^0) + R(0, \hat{x}^0, \hat{y}^0)$$

$$= \sup_{y \in Y(0, \hat{x}^0)} d_t G(0, \hat{x}^0, y) + R(0, \hat{x}^0, y) = \inf_{x \in X(0)} \sup_{y \in Y(0, x)} d_t G(0, x, y) + R(0, x, y).$$
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Consider the objective function and the constraint set
\[ f(x) \overset{\text{def}}{=} Qx \cdot x, \quad U \overset{\text{def}}{=} \{ x \in \mathbb{R}^n : Ax \cdot x = 1 \}, \quad \inf f(U) \] (4.1)

Where \( Q \) is an arbitrary symmetrical \( n \times n \) matrix and \( A > 0 \) is a symmetrical \( n \times n \) positive definite matrix. \( U \neq \emptyset \) is compact and the function \( f \) is not necessarily convex.

The minimization problem is equivalent to the generalized eigenvalue problem
\[ \lambda(Q, A) \overset{\text{def}}{=} \inf_{x \neq 0} \frac{Qx \cdot x}{Ax \cdot x} \] (4.2)

where the minimizer \( \hat{x} \) is solution of the problem
\[ [Q - \lambda(Q, A)A] \hat{x} = 0, \quad A\hat{x} \cdot \hat{x} = 1. \] (4.3)

The semidifferential of \( \lambda(Q, A) \) with respect to \( Q \) in a direction \( Q' \) and \( A \) in the direction \( A' \) can be found in [Delfour 2011, pp. 166–168] for symmetrical matrices:
\[ d\lambda(Q, A; Q', A') = \inf_{x \in X(0)} Q' x \cdot x (Ax \cdot x) - (Qx \cdot x) A' x \cdot x \]
\[ = \inf_{x \in X(0)} Q' x \cdot x - \lambda(Q, A) A' x \cdot x, \] (4.4)

minimizers \( X(0) \overset{\text{def}}{=} \{ x \in \mathbb{R}^n : [Q - \lambda(Q, A)A]x = 0 \text{ and } Ax \cdot x = 1 \} \) (4.5)

states \( E(0) \overset{\text{def}}{=} \{ x \in \mathbb{R}^n : Ax \cdot x = 1 \} \).
For $t \geq 0$, $x \in \mathbb{R}^n$, and $y \in \mathbb{R}$, introduce the Lagrangian

$$G(t, x, y) \overset{\text{def}}{=} (Q + tQ')x \cdot x + y [(A + tA')x \cdot x - 1]$$

$$g(t) \overset{\text{def}}{=} \inf_{x \in \mathbb{R}^n} \sup_{y \in \mathbb{R}} G(t, x, y), \quad dg(0) \overset{\text{def}}{=} \frac{g(t) - g(0)}{t}.$$  

where $A'$ and $Q'$ are symmetrical matrices. Set $Q(t) = Q + tQ'$ and $A(t) = A + tA'$. It is easy to check that

$$d_t G(t, x, y) = Q'x \cdot x + y A'x \cdot x$$

$$d_x G(t, x, y; x') = 2 [Q(t) + y A(t)] x \cdot x'$$

$$d_y G(t, x, y; y') = y' [A(t)x \cdot x - 1].$$

Since $A$ is positive definite, there exists $\alpha > 0$ such that for all $x \in \mathbb{R}^n$, $Ax \cdot x \geq \alpha \|x\|^2$. Hence, there exists $\tau > 0$ such that for all $0 \leq t \leq \tau$

$$\forall t, 0 \leq t \leq \tau, \forall x \in \mathbb{R}^n, \quad A(t)x \cdot x \geq \frac{\alpha}{2} \|x\|^2$$

and for such $t$, the set of constraints $E(t) \overset{\text{def}}{=} \{x : A(t)x \cdot x = 1\} \neq \emptyset$ is compact. So there exist minimizers $x^t \in \mathbb{R}^n$ and $X(t)$ is not empty for $0 \leq t \leq \tau$

$$\lambda^t \overset{\text{def}}{=} \inf_{A(t)x \cdot x = 1} Q(t)x \cdot x = Q(t)x^t \cdot x^t.$$
To summarize,

\[ d_t G(t, x, y) = Q'x \cdot x + y A'x \cdot x \] (4.13)

\[ \left[ Q(t) + y^t A(t) \right] \frac{x^t + x^0}{2} = 0 \text{ (average adjoint equation)} \] (4.14)

\[ \forall y', \quad d_y G(t, x^t, 0; y') = y' [A(t)x^t \cdot x^t - 1] = 0 \text{ (state equation)} \] (4.15)

\[ d_y G \left( t, x^0, 0; \frac{y^t - y^0}{t} \right) = \frac{y^t - y^0}{t} [A(t)x^0 \cdot x^0 - 1]. \] (4.16)

From the Lagrange Multiplier rule, the standard adjoint is solution of

\[ \left[ Q(t) + p^t A(t) \right] x^t = 0 \quad \Rightarrow \quad p^t = -Q(t)x^t \cdot x^t = -\lambda^t. \] (4.17)

The set of minimizers is given by the expression

\[ X(t) = \left\{ x \in \mathbb{R}^n : [Q(t) + p^t A(t)]x = 0 \text{ and } A(t)x \cdot x = 1 \right\}. \] (4.18)

For all \( x^t \in X(t), x^t \neq 0 \) and \( -x^t \in X(t) \). So \( X(t) \) is not a singleton. However,

\[ \forall x^t \in X(t), \quad Y(t, x^t) = \{-\lambda^t\} \]

and \( Y(t, x^t) \) is a singleton independent of the choice of the minimizer \( x^t \in X(t) \).
Mutivalued Case
A Non-convex Example where $X(0)$ is not a singleton

Given $x^0 \in X(0)$ and $x^t \in X(t)$, the *averaged adjoint* is solution of the equation:

$$\forall x', \quad 0 = \int_0^1 dx G(t, x^0 + s(x^t - x^0), y^t; x') ds$$

$$= 2 \int_0^1 \left[ Q(t) + y^t A(t) \right] (x^0 + s(x^t - x^0)) \cdot x' ds$$

$$= 2 \left[ Q(t) + y^t A(t) \right] \frac{x^t + x^0}{2} \cdot x'$$

$$\Rightarrow \left[ Q(t) + y^t A(t) \right] \frac{x^t + x^0}{2} = 0. \quad (4.19)$$

$$\Rightarrow Y(t, x^0, x^t) = \begin{cases} 
- \frac{Q(t) x^t + x^0}{2} \cdot \frac{x^t + x^0}{2}, & \text{if } x^t + x^0 \neq 0 \\
0, & \text{if } x^t + x^0 = 0
\end{cases} \quad (4.20)$$

Therefore, $Y(t, x^0, x^t) \neq \emptyset$. 
A preliminary lemma.

(i) For all \( t \), \( 0 \leq t \leq \tau \),

\[
\forall x^t \in X(t), \quad Y(t, x^t, x^t) = \{-\lambda^t\}
\]  

(4.21)

where \( \lambda^t \) is the minimum of the objective function \( Q(t)x \cdot x \) with respect to \( E(t) = \{x \in \mathbb{R}^n : A(t)x \cdot x = 1\} \) as seen in (4.12).

(ii) For each sequence \( \{t_n : 0 < t_n \leq \tau\} \), there exist \( \bar{x} \in X(0) \), \( x^{t_n} \in X(t_n) \), and \( y^{t_n} \in Y(t_n, \bar{x}, x^{t_n}) \) such that

\[
x^{t_n} \to \bar{x}, \quad \lambda^{t_n} \to \lambda^0, \quad \text{and} \quad y^{t_n} \to y^0 = -\lambda^0,
\]  

(4.22)

and the set of averaged adjoint states \( Y(t_n, \bar{x}, x^{t_n}) = \{y^{t_n}\} \) is a singleton.

(iii) As \( t \searrow 0 \), the quotient

\[
\frac{\lambda^t - \lambda^0}{t}
\]  

(4.23)

is bounded.

(iv) For the sequences of part (ii), the quotients

\[
\frac{\lambda^{t_n} - \lambda^0}{t_n} \quad \text{and} \quad \frac{y^{t_n} - y^0}{t_n}
\]  

(4.24)

are bounded.
Given symmetrical $n \times n$ matrices $A, A', Q,$ and $Q'$ such that $A$ is positive definite, there exists at least one $x^0$ such that $Ax^0 \cdot x^0 = 1$ and

$$\lambda(Q, A) = \inf_{Ax \cdot x = 1} Qx \cdot x = Qx^0 \cdot x^0. \tag{4.25}$$

Moreover

$$d\lambda(Q, A; Q', A') \stackrel{\text{def}}{=} \lim_{t \to 0} \frac{\lambda(Q + tQ', A + tA') - \lambda(Q, A)}{t} = \inf_{x^0 \in X(0)} \left[ Q' - \lambda(Q, A) A' \right] x^0 \cdot x^0, \tag{4.26}$$

$$X(0) \stackrel{\text{def}}{=} \{ x \in \mathbb{R}^n : Ax \cdot x = 1 \text{ and } [Q - \lambda(Q, A)A]x = 0 \}. \tag{4.27}$$

If $X(0)$ is not simple the dimension of the space $X(0)$ is greater or equal to 2 and we only have a semi-différential.
Proof.

(i) **Hypothesis (H1).** We have seen that for all $0 \leq t \leq \tau$, $X(t) \neq \emptyset$ and that, for all $x^t \in X(t)$, $Y(t, x^t) = \{-\lambda^t\}$. For the averaged adjoint $y^t$

\[
\Rightarrow Y(t, x^0, x^t) = \begin{cases} 
\left\{ -\frac{Q(t)\frac{x^t+x^0}{2} \cdot \frac{x^t+x^0}{2}}{A(t)\frac{x^t+x^0}{2} \cdot \frac{x^t+x^0}{2}} \right\}, & \text{if } x^t + x^0 \neq 0 \\
\mathbb{R}, & \text{if } x^t + x^0 = 0
\end{cases}
\]

(ii) **Hypothesis (H2).** We have seen that $d_t G(t, x, y) = Q'x \cdot x + y A'x \cdot x$. So for all $x^0 \in X(0)$ and the singleton $Y(0, x^0) = \{-\lambda^0\}$

\[
d_t G(t, x^0, y^0) = Q'x^0 \cdot x^0 - \lambda^0 A'x^0 \cdot x^0.
\]

(iii) **Hypothesis (H3).** For each sequence $t_n \to 0$, $0 < t_n \leq \tau$, choose the sequence $\{x^{t_n}\}$ and its limit $\bar{x} \in X(0)$ from the Lemma (ii) and use the fact that the corresponding sequence \( \frac{y^{t_n} - y^0}{t_n} \) is bounded by some constant $c$ from the Lemma (iv):

\[
\left| d_y G \left( t_n, \bar{x}, 0; \frac{y^{t_n} - y^0}{t_n} \right) \right| = \left| \frac{y^{t_n} - y^0}{t} \right| [A(t_n)\bar{x} \cdot \bar{x} - 1] \\
\leq \left| \frac{y^{t_n} - y^0}{t_n} \right| |A(t_n)\bar{x} \cdot \bar{x} - 1| \leq c \left| A(t_n)\bar{x} \cdot \bar{x} - 1 \right| \to c \left| A(0)\bar{x} \cdot \bar{x} - 1 \right| = 0
\]
(iv) *Hypothesis (H4).* For all \( x^0 \in X(0) \) \( Y(0, x^0) = \{-\lambda^0\} \) is a singleton independent of \( x^0 \in X(0) \). As in (iii), for each sequence \( t_n \to 0 \), \( 0 < t_n \leq \tau \), choose the sequence \( \{x^{t_n}\} \) and its limit \( \bar{x} \in X(0) \) from the Lemma (ii) and use the fact that the corresponding sequence \( \frac{y^{t_n} - y^0}{t_n} \) is bounded by some constant \( c \) from the Lemma (iv):

\[
\left| d_y G \left( t_n, x^0, 0; \frac{y^{t_n} - y^0}{t_n} \right) \right| = \left| \frac{y^{t_n} - y^0}{t} \left[ A(t_n)x^0 \cdot x^0 - 1 \right] \right|
\leq \left| \frac{y^{t_n} - y^0}{t_n} \right| \left| A(t_n)x^0 \cdot x^0 - 1 \right|
\leq c \left| A(t_n)x^0 \cdot x^0 - 1 \right| \to c \left| A(0)x^0 \cdot x^0 - 1 \right| = 0.
\]

(v) The conclusion follows from Theorem 12 where the \( \sup \) disappears since \( Y(0, x^0) = \{-\lambda^0\} = \{-\lambda(Q, A)\} \) is a singleton independent of \( x^0 \in X(0) \).
Thank you for your attention


[Lemaire (1970)] Lemaire B., Problèmes min-max et applications au contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles linéaires, Thèse de doctorat d’état, Univ. de Montpellier, France 1970.


