Nonlinear Model Predictive Control (NMPC)

NMPC= repeated optimal control

1. State estimate $\hat{x}(t_k)$ at $t_k$

2. Solve OCP

$$\min_{u(\cdot)} \int_{t_k}^{t_k+T} F(x(t), u(t)) \, d\tau + E(x(t_k + T))$$

$$\frac{dx(\tau)}{d\tau} = f(x(\tau), u(\tau)), \quad x(t_k) = \hat{x}(t_k)$$

$x(\tau) \in \mathcal{X}, \quad u(\tau) \in \mathcal{U}$

$x(t_k + T) \in \mathcal{E}$

3. Apply $u^*(\tau)$ for $\tau \in [t_k, t_{k+1}]$

NMPC design

- Stabilization of $x_s$: $F(x, u) = (x - x_s)^T Q(x - x_s) + (u - u_s)^T R(u - u_s) \geq \|x - x_s\|$ + suitable terminal constraints / penalties or reachability conditions

- Tracking of $r(t)$: $F(t, x, u) = (h(x) - r(t))^T Q(h(x) - r(t)) + \ldots \geq \|h(x) - r(t)\|$

**Observation:** control task at hand influences design of OCP.
What is Economic NMPC?

How to improve performance of a continuous process? → Solve OCP with long/infinite horizon $T_\infty$.

$$\begin{align*}
\min_{u(\cdot)} \int_0^{T_\infty} F(x(\tau), u(\tau))d\tau \\
\text{subject to} \\
\frac{dx(\tau)}{d\tau} = f(x(\tau), u(\tau)), \quad x(0) = x_0 \\
u(\tau) \in \mathcal{U} \subset \mathbb{R}^{n_u}, x(\tau) \in \mathcal{X} \subset \mathbb{R}^{n_x}
\end{align*}$$

Challenges:
- Structure of optimal solutions?
- Numerics?
- ...

Solution: receding horizon approximation with shorter horizon $T$.

$$\begin{align*}
\min_{u(\cdot)} \int_{t_k}^{t_k+T} F(x(\tau), u(\tau))d\tau \\
\text{subject to} \\
\frac{dx(\tau)}{d\tau} = f(x(\tau), u(\tau)), \quad x(t_k) = \hat{x}(t_k) \\
u(\tau) \in \mathcal{U} \subset \mathbb{R}^{n_u}, x(\tau) \in \mathcal{X} \subset \mathbb{R}^{n_x}
\end{align*}$$

- $F \approx$ economic criteria: yield, profit, ...
- $F$ can be non-quadratic: $F = a^T x + b^T u$, ...

Economic NMPC

- NMPC with generalized (economic) objectives.
- Approximation of an infinite-horizon OCP by receding-horizon solutions.

[Rawlings & Amrit `09; Würth et al. `11; Angeli et al. `12; Grüne `13; Ellis et al. `14; ...]
## Motivation
- Economic MPC

## Turnpike properties and dissipativity
- Turnpike conditions and converse results

## Asymptotic and practical convergence in EMPC
- Exact and approximate turnpikes

## Recovering asymptotic convergence in EMPC
- Terminal constraints and penalties

## Summary and outlook
How to describe turnpike behavior in OCPs?

Problem setup

$$\min_{u(\cdot)} \int_0^T F(x(\tau), u(\tau)) d\tau$$

subject to

$$\frac{dx(\tau)}{d\tau} = f(x(\tau), u(\tau)), \quad x(0) = x_0 \in X_0$$

$$u(\tau) \in U \subset \mathbb{R}^{nu}, \quad x(\tau) \in X \subset \mathbb{R}^{nx}$$

Conceptual idea of turnpike properties

- Property of OCPs with and without terminal constraints.
- Optimal solutions approach neighborhood of a specific steady state.
- Time spend at turnpike grows with increasing horizon length $T$.
- If turnpike at $\bar{x}$, then for $T = \infty$, we have that $\lim_{t \to \infty} x^*(t) \approx \bar{x}$.
- Different notions for turnpikes: dichotomy in OCPs, hyper-sensitive OCPs, ...

[Dorfman, Samuelson & Solow `58; McKenzie `76; Carlson et al. `91; Damm et al. `14; Trelat & Zuazua `14; ...]
Parametric Optimal Control Problems

Optimal fish harvest

\[
\begin{align*}
\min_{u(\cdot)} \int_{0}^{T} & ax(t) + bu(t) - cx(t)u(t)dt \\
\text{subject to} & \\
\dot{x} &= x(x_S - x - u), \quad x(0) = x_0 \\
u(t) &\in [0, u_{max}], x(t) \in (0, \infty)
\end{align*}
\]

→ Similar behavior for different initial conditions and horizon lengths.
→ Similarity properties of solutions of parametric OCPs.
Parametric Optimal Control Problems

Optimal fish harvest (quadratic objective)

$$\min_{u(\cdot)} \int_0^T \frac{1}{2} q(x(t) - x_C)^2 + \frac{1}{2} r(u(t) - u_C)^2 dt$$

subject to

$$\dot{x} = x(x_S - x - u), \quad x(0) = x_0$$

$$u(t) \in [0, u_{max}], x(t) \in (0, \infty)$$

$$u_{max} = 5, x_S = 5$$

$$q = 10, r = 1, x_C = 4, u_C = 5$$

→ Similar behavior for different initial conditions and horizon lengths.
→ Similarity properties of solutions of parametric OCPs.
Definition (Turnpike property).
Consider the optimal pairs \( z^*(\cdot, x_0) \) and 
\[
\Theta_{\varepsilon, T} := \{ t \in [0, T] : \| z^*(\cdot, x_0) - \bar{z} \| > \varepsilon \}.
\]
The optimal pairs \( z^*(\cdot, x_0) \) of \( \text{OCP}_T(x_0) \) have an input-state turnpike property with respect to \( \bar{z} \) if there exists \( \nu : [0, \infty) \to [0, \infty) \) s. t.
\[
\forall x_0 \in \mathcal{X}_0, \forall T \geq 0, \forall \varepsilon > 0 : \quad \mu[\Theta_{\varepsilon, T}] < \nu(\varepsilon),
\]
where \( \mu[\cdot] \) is the Lebesgue measure on the real line.
The solution pairs \( z^*(\cdot, x_0) \) of \( \text{OCP}_T(x_0) \) are said to have an exact input-state turnpike property if additionally 
\[
\mu[\Theta_{0, T}] < \nu(0) < \infty.
\]
[Carlson et al. ´91, Faulwasser et al. ´14, ´17]
Turnpike Properties of OCPs

Turnpikes are either **approximate** or **exact**.

→ approximate

→ exact
When do turnpikes occur in OCPs?

**Definition** (Strict dissipativity w.r.t. \((\bar{x}, \bar{u})\)). \(\Sigma: \dot{x} = f(x, u)\) is said to be *strictly dissipative with respect to the steady state pair* \((\bar{x}, \bar{u})\) if there exists a bounded non-negative storage function \(S: \mathcal{X} \to \mathbb{R}_0^{+}\) and \(\alpha \in \mathcal{K}\) such that for all admissible pairs \(z(\cdot, x_0), \) all \(x_0 \in \mathcal{X},\) and all horizons \(T > 0\)

\[
S(x(T, x_0)) - S(x_0) \leq \int_0^T -\alpha(\| (x(\tau), u(\tau)) - (\bar{x}, \bar{u}) \|) + F(x(\tau), u(\tau)) - F(\bar{x}, \bar{u}) d\tau.
\]

[Diehl et al. `11; Angeli et al. `12; ...]

**Theorem** (Dissipativity ⇒ turnpike).
Suppose that

- from all \(x_0 \in \mathcal{X}_0\) the optimal steady state \(\bar{x}\) is exponentially reachable,
- \(\Sigma\) is strictly dissipative w.r.t. to \((\bar{x}, \bar{u})\).

Then the optimal pairs \(z^*(\cdot, x_0)\) of OCP\(_T(x_0)\) have a turnpike property with respect to the steady state pair \((\bar{x}, \bar{u})\).

[Grüne `13; Faulwasser at al. `14, `17; Damm et al. `14]
Convergence of NMPC based on Exact Turnpikes

NMPC scheme without terminal constraints and without terminal penalty

\[
\min_{u(\cdot)} \int_{t_k}^{t_k+T} F(x(\tau), u(\tau)) d\tau
\]

subject to

\[
\forall \tau \in [t_k, t_k + T] : \quad \frac{d}{d\tau} x(\tau) = f(x(\tau), u(\tau)), \quad x(t_k) = \hat{x}(t_k)
\]

\[
u(\tau) \in \mathcal{U}, \ x(\tau) \in \mathcal{X}
\]

Main assumptions

- No terminal constraints, no end penalty.
- No structural assumptions on \( F \) \xrightarrow{} economic NMPC.
- **Exact turnpike property** in \( \text{OCP}_T(x_0) \):

![Diagram showing turnpike property](image)
Theorem (Convergence of NMPC based on exact turnpike).
Suppose that

- $\Sigma$ is controlled via $\text{OCP}_T(\hat{x}(t_k))$,
- for all $\hat{x}(t_k) \in \mathcal{X}_0$, $\text{OCP}_T(\hat{x}(t_k))$ has an exact turnpike property at $\bar{z}$,
- $\hat{x}(t_0) \in \mathcal{X}_0$.

Then

- $\text{OCP}_T(\hat{x}(t_k))$ is recursively feasible, and
- there exist a horizon length $T \in (0, \infty)$, a sampling time $\delta > 0$ and a time $\bar{t} \geq t_0$ such that
  \[ \forall t \geq \bar{t} : \quad \hat{x}(t, x_0, u^{\text{mpc}}(\cdot)) = \bar{x}. \]
Stability of NMPC based on Exact Turnpikes

Main steps of the proof:

- If, for all \( x_0 \in \mathcal{X}_0 \), we have an exact turnpike, then \( \mathcal{X}_0 \) is rendered positively invariant by NMPC scheme.

- End pieces of exact turnpike solutions are identical.

- Construction of admissible (optimal) input trajectory

\[
\tilde{u}_{k+1}(t, x(t_{k+1})) = \begin{cases} 
  u^*(t, x(t_k)), & \forall t \in [0, T_1(x(t_k))) + t_{k+1} \\
  \bar{u}^*, & \forall t \in [T_1(x(t_k)), T_1(x(t_k))) + \delta) + t_{k+1} \\
  u^*(t, x(t_k)), & \forall t \in [T_1(x(t_k)) + \delta, T] + t_{k+1} 
\end{cases}
\]
Example: Optimal Fish Harvest

\[
\min_{u(\cdot)} \int_{t_k}^{t_k+T} \left( ax(\tau) + bu(\tau) - cx(\tau)u(\tau) \right) d\tau
\]
subject to
\[
\frac{dx}{d\tau} = x(x_s - x - u), \quad x(t_k) = \hat{x}(t_k)
\]
\[
u(t) \in [0, u_{max}], \quad x(t) \in (0, \infty)
\]

- \(x\) fish density
- \(u\) fishing rate
- \(x_s = 5\) highest sustainable fish density
- \(a = 1, \quad b = c = 2, \quad u_{max} = 5\)
- \(T = 1.2, \quad \delta = 0.1\)

[Cliff & Vincent '73]

Open-loop turnpike solutions

Closed-loop NMPC solutions

Questions
- How to verify turnpikes in OCPs? When are turnpikes exact?
- What if turnpikes are only approximate?
Example – Chemical Reactor

Van de Vusse Reactor \[ A \xrightarrow{k_1} B \xrightarrow{k_2} C, \quad 2A \xrightarrow{k_3} D \]

Dynamics (partial model)
\[
\begin{align*}
\dot{c}_A &= r_A(c_A, \vartheta) + (c_{in} - c_A)u_1 \\
\dot{c}_B &= r_B(c_A, c_B, \vartheta) - c_Bu_1 \\
\dot{\vartheta} &= h(c_A, c_B, \vartheta) + \alpha(u_2 - \vartheta) + (\vartheta_{in} - \vartheta)u_1, \\
r_A(c_A, \vartheta) &= -k_1(\vartheta)c_A - 2k_3(\vartheta)c_A^2 \\
r_B(c_A, c_B, \vartheta) &= k_1(\vartheta)c_A - k_2(\vartheta)c_B \\
h(c_A, c_B, \vartheta) &= -\delta\left(k_1(\vartheta)c_A\Delta H_{AB} + k_2(\vartheta)c_B\Delta H_{BC} + 2k_3(\vartheta)c_A^2\Delta H_{AD}\right) \\
k_i(\vartheta) &= k_{i0}\exp\left(-\frac{E_i}{\vartheta + \vartheta_0}\right), \quad i = 1, 2, 3.
\end{align*}
\]

Constraints
\[
\begin{align*}
c_A &\in [0, 6] \text{ mol} \\
c_B &\in [0, 4] \text{ mol} \\
u_1 &\in [3, 35] \frac{1}{h} \\
u_2 &\in [0, 200] \frac{1}{\circ C} \\
\vartheta &\in [70, 150] \circ C
\end{align*}
\]

Objective = maximize produced amount of B
\[
J_T(x_0, u(\cdot)) = \int_0^T -\beta c_B(t)u_1(t)dt, \quad \beta > 0
\]

[Chen et al. ’95; Rothfuß, Rudolph, Zeitz ’96]
Example – Chemical Reactor

\[ T = 0.01667h \]

Distance to equilibrium

\[ \|x_\infty - x_s\| \]

\[ N \]
Overview – EMPC without Terminal Constraints

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[Grüne `13; Faulwasser & Bonvin `15, `17; ...]
Singular OCPs and Exact Turnpikes

OCP with input box constraints and input affine data

\[ \min_{u(\cdot)} \int_0^T F_0(x(\tau)) + \sum_{i=1}^{n_u} F_1^i(x(\tau)) u_i(\tau) d\tau \]

subject to

\[ \Sigma : \quad \frac{dx(\tau)}{d\tau} = f_0(x(\tau)) + \sum_{i=1}^{n_u} f_1^i(x(\tau)) u_i(\tau), \quad x(0) = x_0 \in X_0 \]

\[ u(\tau) \in [u_1^{min}, u_1^{max}] \times \cdots \times [u_n^{min}, u_n^{max}] \]

(NCP-SING)

Necessary conditions of optimality for OCP-SING

\[ H(\lambda_0, \lambda, x, u) = \lambda_0 \left( F_0(x) + \sum_{i=1}^{n_u} F_1^i(x) u_i \right) + \lambda^\top \left( f_0(x) + \sum_{i=1}^{n_u} f_1^i(x) u_i \right) \]

\[ \frac{dx^*(\tau)}{d\tau} = H_\lambda(\lambda_0^*, \lambda^*(\tau), x^*(\tau), u^*(\tau)), \quad x^*(0) = x_0 \]

\[ \frac{d\lambda^*(\tau)}{d\tau} = -H_x(\lambda_0^*, \lambda^*(\tau), x^*(\tau), u^*(\tau)), \quad \lambda^*(T) = 0 \quad \text{(NCO)} \]

\[ \forall \tau \in [0, T] \text{ and } \forall u \in U \]

\[ H(\lambda_0^*, \lambda^*(\tau), x^*(\tau), u^*(\tau)) \leq H(\lambda_0^*, \lambda^*(\tau), x^*(\tau), u), \]
Singular OCPs and Exact Turnpikes

Necessary conditions of optimality imply

\[ u_i^*(\tau) \in \{u_{i,\text{min}}, u_{i,\text{max}}\} \quad \text{if} \quad s_i(x^*(\tau), \lambda^*(\tau)) \neq 0 \]
\[ u_i^*(\tau) \in [u_{i,\text{min}}, u_{i,\text{max}}] \quad \text{if} \quad s_i(x^*(\tau), \lambda^*(\tau)) = 0 \]

\[ s_i(x, \lambda) = \lambda_0 F_1^i(x) + \lambda^\top f_1^i(x), \quad i = 1, \ldots, n_u. \]

\[ \Rightarrow \text{Either optimal inputs are on the boundary of } \mathcal{U}, \text{ or a singular arc with } s_i(\lambda, x) = 0. \]

**Definition (Steady-state singular OCP).**

OCP-SING is said to be **steady-state singular** if, for any non-vanishing interval \([\tau_0, \tau_1] \subset [0, T]\), the condition

\[ s_i(x^*(\tau), \lambda^*(\tau)) = 0, \forall \tau \in [\tau_0, \tau_1], \forall i = 1, \ldots, n_u \]

implies that

\[ (x^*(\tau), u^*(\tau), \lambda^*(\tau)) = (\bar{x}, \bar{u}, \bar{\lambda}) \]

where \((\bar{x}, \bar{u}, \bar{\lambda})\) specifies a unique steady state of (NCO).

\[ \Rightarrow \text{Only singular arc is a steady state!} \]
Singular OCPs and Exact Turnpikes

**Theorem** (Exactness of turnpikes).
Suppose that OCP-SING

(i) is steady-state singular with respect to $(\bar{\lambda}, \bar{x}, \bar{u})$ such that,
\[ \forall i \in \{1, \ldots, n_u\}, \quad \bar{u}_i \notin \{u_{i,min}, u_{i,max}\}, \text{ and} \]

(ii) the optimal solutions to OCP-SING have a turnpike at $\bar{z} = (\bar{x}, \bar{u})$.

Then, the turnpike at $\bar{z}$ is exact. [Faulwasser & Bonvin `17]

**Remarks**

- Approximate turnpikes can be verified via dissipativity condition.
- Proof uses singular nature of OCP-SING, i.e.,
\[
\begin{align*}
    u_i^*(\tau) \in \{u_{i,min}, u_{i,max}\} & \quad \text{if } s_i(x^*(\tau), \lambda^*(\tau)) \neq 0 \\
    u_i^*(\tau) = \bar{u}_i & \quad \text{if } s_i(x^*(\tau), \lambda^*(\tau)) = 0
\end{align*}
\]
- Optimal solutions cannot stay arbitrarily close to turnpike, without
\[
u^*(\tau) = \bar{u} \implies x^*(\tau) = \bar{x}.
\]
- Verifying steady-state singularity?
Steady-State Singularity of Linear-Quadratic OCPs

Special case of OCP-SING:

- Linear dynamics $\dot{x} = Ax + Bu$, $x(0) = x_0 \in \mathcal{X}_0$.
- Quadratic objective with $F(x,u) = \frac{1}{2}x^\top Qx + x^\top Su + q^\top x + r^\top u$ and convex input constraints $\mathcal{U} \subset \mathbb{R}^{n_u}$.

Necessary conditions of optimality on singular arc $\rightarrow$ linear DAE

$$(\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix}) (\begin{pmatrix} \dot{x} \\ \hat{\lambda} \\ \hat{u} \end{pmatrix}) = (\begin{pmatrix} A & 0 & B \\ -Q & -A^T & -S \\ S^T & B^T & 0 \end{pmatrix}) (\begin{pmatrix} x \\ \lambda \\ u \end{pmatrix}) + (\begin{pmatrix} 0 \\ -q \\ r \end{pmatrix}).$$

Lemma (Steady-state singularity of linear quadratic OCPs).
If for all $s \in \mathbb{C}$

$$\det(s\tilde{E} - \tilde{A}) = p(s) = \text{constant} \neq 0,$$

then OCP-SING is steady-state singular.

$\rightarrow$ Use of properties of nilpotent DAEs
$\rightarrow$ Extension to nonlinear dynamics?
Example – Fuller’s Problem

\[ \min_{u(\cdot)} \int_0^T (x_1(\tau))^2 d\tau \]

subject to

\[ \dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \quad x(0) = x_0 \]

\[ u(t) \in [-1, 1] \text{ a.e.,} \quad u(\cdot) \in \mathcal{L}^\infty \]

\[ \Rightarrow \quad \det(s \tilde{E} - \tilde{A}) = -1 \]
Considered OCP

\[
\min_{u(\tau)} \quad \int_0^T F(x(\tau), u(\tau), \tau) \, d\tau
\]

subject to

\[
\frac{\partial}{\partial \tau} x(\tau) = f(x(\tau), u(\tau)) \quad x(0) = x_0 \quad x(\tau) \in X_0
\]

\[
u(\tau) \in U \subset \mathbb{R}^{n_u}, \quad x(\tau) \in X \subset \mathbb{R}^{n_x}
\]

\[
F_{lq}(x, u) = \frac{1}{2} x^T Q x + x^T Su + \frac{1}{2} u^T R u + q^T x + r^T u
\]

**Assumption.** OCP is regular at turnpike \((\bar{x}, \bar{u})\), i.e. consider

\[
H(x, u, \lambda) = F(x, u) + \lambda^T f(x, u)
\]

such that \(H \in C^2\) and

\[
\det H_{uu}(\bar{x}, \bar{u}) = \det R \neq 0.
\]
Back to Regular OCPs

**Lemma** (No exact turnpikes in regular OCPs).

Let

- the OCP be linear-quadratic and regular,
- let it exhibit a turnpike at \((\bar{x}, \bar{u}) \in \text{int}(\mathcal{X} \times \mathcal{U})\), and
- let \(A, B\) be controllable.

Then the turnpike property is approximate, i.e. it is not exact.

**Sketch of proof:**

- **W.l.o.g.** \((\bar{x}, \bar{u}) = 0\) and \(x_0 = 0\).

- For \(x_0 = 0\), we have \(u^* = -R^{-1}(r + B^\top \lambda^*)\):
  \[ u^* = 0 \iff r = -B^\top \lambda^* \]

- Starting from \(x_0 = 0\):
  \[ \dot{\lambda}^* = -A^\top \lambda^* - q, \quad r = -B^\top \lambda^*, \quad \lambda^*(T) = 0 \]

- \((A, B)\) controllable \(\Rightarrow u^*(\tau) \neq 0\)
Role of Adjoints in Turnpike Properties

NCO for $\mathcal{X} = \mathbb{R}^{n_x}$:

$$\frac{dx^*(\tau)}{d\tau} = H_\lambda(\lambda_0^*, \lambda^*(\tau), x^*(\tau), u^*(\tau)), \quad x^*(0) = x_0$$

$$\frac{d\lambda^*(\tau)}{d\tau} = -H_x(\lambda_0^*, \lambda^*(\tau), x^*(\tau), u^*(\tau)), \quad \lambda^*(T) = 0$$

\(\forall \tau \in [0, T] \text{ and } \forall u \in \mathcal{U}\)

$$H(\lambda_0^*, \lambda^*(\tau), x^*(\tau), u^*(\tau)) \leq H(\lambda_0^*, \lambda^*(\tau), x^*(\tau), u)$$

Turnpike \((\bar{x}, \bar{u}) \in \text{int}(\mathcal{X} \times \mathcal{U})\) corresponds to \(\bar{\lambda}\) such that:

$$0 = H_\lambda(\lambda_0, \bar{\lambda}, \bar{x}, \bar{u})$$

$$0 = -H_x(\lambda_0, \bar{\lambda}, \bar{x}, \bar{u})$$

$$0 = H_u(\lambda_0, \bar{\lambda}, \bar{x}, \bar{u})$$

Observations

- Turnpike \((\bar{x}, \bar{u}) \in \text{int}(\mathcal{X} \times \mathcal{U})\) with \(\bar{\lambda} \neq 0\) has a leaving arc.

- If turnpike is not exact, the leaving arc leads to practical convergence of NMPC.
Recovering Asymptotic Convergence in EMPC

- Add a terminal constraint, e.g. \( x(t_k + T) = \bar{x} \)

- Terminal penalty (Mayer term) \( E(x) = -S(x) \) or rotate \( F \) by storage function:

\[
F(x, u) \rightarrow \tilde{F}(x, u) := F(x, u) - \frac{\partial S}{\partial x} f(x, u) - F(\bar{x}, \bar{u})
\]

\[
\tilde{F}(x, u) \geq \alpha(||(x, u) - (\bar{x}, \bar{u})||)
\]

\( \Rightarrow \) Without terminal constraints open-loop solutions change due to rotation!

Adjoint interpretation of rotation

- Rotated stage costs imply that \( \lambda^*(\tau) \approx 0 \) whenever \( z^*(\tau) \approx (\bar{x}, \bar{u}) \).

- \( \lambda^*(\tau) \approx 0 = \lambda^*(T) \) whenever \( z^*(\tau) \approx (\bar{x}, \bar{u}) \).
Linear Terminal Penalties in EMPC

\[
\begin{align*}
\min_{u(\cdot)} & \quad \int_{t_k}^{t_k+T} F(x(\tau), u(\tau)) d\tau + \bar{\lambda}^T x(t_k + T) \\
\text{subject to} & \quad \frac{d}{d\tau} x(\tau) = f(x(\tau), u(\tau)), \quad x(t_k) = \hat{x}(t_k) \\
& \quad u(\tau) \in \mathcal{U} \subset \mathbb{R}^{n_u}, x(\tau) \in \mathcal{X} \subset \mathbb{R}^{n_x}
\end{align*}
\]

(OCP\textsubscript{T,\bar{\lambda}}(\hat{x}(t_k)))

Second variation at \(\bar{x}, \bar{u}, \bar{\lambda}:\)

\[
A = f_x, B = f_u, Q = H_{xx}, S = H_{xu}, R = H_{uu}, q = F_x, r = F_u
\]

\[
\begin{align*}
\min_{u(\cdot)} & \quad \int_{t_k}^{t_k+T} F_{lq}(x(\tau), u(\tau)) d\tau + \bar{\lambda}^T x(t_k + T) \\
\text{subject to} & \quad \frac{d}{d\tau} x(\tau) = Ax(\tau) + Bu(\tau), \quad x(t_k) = \hat{x}(t_k) \\
& \quad F_{lq}(x, u) = \frac{1}{2} x^T Q x + x^T S u + \frac{1}{2} u^T R u + q^T x + r^T u
\end{align*}
\]

(LQR\textsubscript{T,\bar{\lambda}}(\hat{x}(t_k)))
Linear Terminal Penalties in EMPC

\[ \min_{u(\cdot)} \int_{t_k}^{t_k+T} F(x(\tau), u(\tau)) d\tau + \bar{\lambda}^T x(t_k + T) \]

subject to

\[ \frac{d}{d\tau} x(\tau) = f(x(\tau), u(\tau)), \quad x(t_k) = \hat{x}(t_k) \]

\[ u(\tau) \in \mathcal{U} \subset \mathbb{R}^{n_u}, x(\tau) \in \mathcal{X} \subset \mathbb{R}^{n_x} \]

**Theorem** (Convergence of NMPC with linear end penalty). Suppose that

- \( \Sigma \) is controlled via OCP\(_{T,\bar{\lambda}}(\hat{x}(t_k)) \), \( \Sigma \) is locally controllable at \((\bar{x}, \bar{u})\),
- for all \( \hat{x}(t_k) \in \mathcal{X}_0 \), OCP\(_{T,\bar{\lambda}}(\hat{x}(t_k)) \) is strictly dissipative w.r.t. \((\bar{x}, \bar{u}) \in \text{int}(\mathcal{X} \times \mathcal{U})\),
- for some finite horizon \( T > 0 \), the solution to LQR\(_{T,\bar{\lambda}}(\hat{x}(t_k)) \) is stabilizing.

Then there exists \( T > 0, \delta > 0 \) such that

- OCP\(_{T,\bar{\lambda}}(\hat{x}(t_k)) \) is recursively feasible, and
- \( \lim_{t \to \infty} \hat{x}(t, x_0, u^{\text{mpc}}(\cdot)) = \bar{x} \). \hfill \text{[Zanon \& Faulwasser `17]}
Example – Chemical Reactor

Van de Vusse Reactor \( A \xrightarrow{k_1} B \xrightarrow{k_2} C, \quad 2A \xrightarrow{k_3} D \)

Dynamics (partial model)
\[
\begin{align*}
\dot{c}_A &= r_A(c_A, \vartheta) + (c_{in} - c_A)u_1 \\
\dot{c}_B &= r_B(c_A, c_B, \vartheta) - c_Bu_1 \\
\dot{\vartheta} &= h(c_A, c_B, \vartheta) + \alpha(u_2 - \vartheta) + (\vartheta_{in} - \vartheta)u_1,
\end{align*}
\]
\[
\begin{align*}
r_A(c_A, \vartheta) &= -k_1(\vartheta)c_A - 2k_3(\vartheta)c_A^2 \\
r_B(c_A, c_B, \vartheta) &= k_1(\vartheta)c_A - k_2(\vartheta)c_B \\
h(c_A, c_B, \vartheta) &= -\delta \left( k_1(\vartheta)c_A \Delta H_{AB} + k_2(\vartheta)c_B \Delta H_{BC} + 2k_3(\vartheta)c_A^2 \Delta H_{AD} \right)
\end{align*}
\]
\[
k_i(\vartheta) = k_{i0} \exp \frac{-E_i}{\vartheta + \vartheta_0}, \quad i = 1, 2, 3.
\]

Constraints
\[
\begin{align*}
c_A &\in [0, 6] \text{ mol} \\
c_B &\in [0, 4] \text{ mol} \\
u_1 &\in [3, 35] \text{ \frac{1}{h}} \\
u_2 &\in [0, 200] \text{ \degree C}
\end{align*}
\]

Objective = maximize produced amount of B
\[
J_T(x_0, u(\cdot)) = \int_0^T -\beta c_B(t)u_1(t)dt, \quad \beta > 0
\]

[Chen et al. ’95; Rothfuß, Rudolph, Zeitz ’96]
Example – Chemical Reactor

\[ T = 0.01667 \, h \]

Distance to equilibrium

\[ ||x_\infty - x_s|| \]

[Graph showing concentration profiles and distance to equilibrium over time]
Example – Chemical Reactor

\[ T = 0.01667h \quad \text{or} \quad T = 0.01667h, \quad E(x) = \lambda^\top x \]
Summary and Outlook

Turnpikes and dissipativity

- Suff. conditions for turnpikes via dissipativity (OCPs with or without terminal constraints).
- Suff. conditions for exact turnpikes.

Approximate versus exact turnpikes

- Linear-quadratic singular OCP → exactness of turnpikes via nilpotent DAE
- Linear-quadratic regular OCP → approximate turnpikes (NCO = DAE with index 1)

EMPC with linear end penalty (gradient correction)

- Allows recovering asymptotic convergence/stability

Outlook

- Turnpikes with active constraints?
- Time-varying turnpikes? Classification thereof?
- ...

Thank you! Questions?
References


T. Faulwasser, M. Korda, C.N. Jones, D. Bonvin. Turnpike and dissipativity properties in dynamic real-time optimization and economic MPC. Proc. 53rd IEEE Conf. on Decision and Control (CDC), Los Angeles, California, USA. (2014)


T. Faulwasser and D. Bonvin. On the design of economic NMPC based on approximate turnpike properties. CDC 2015


M. Zanon and T. Faulwasser. Economic MPC without terminal constraints: Gradient-correcting end penalties enforce stability Submitted 2017